Invariant Theory in Differential Geometry

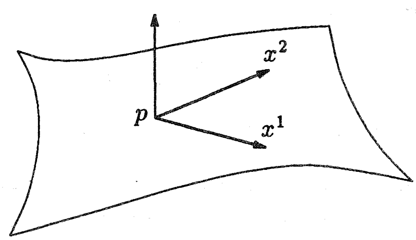
Michael Eastwood, University of Adelaide

This talk is an exposition of background material for some recent developments in invariant theory carried out in joint work with Toby Bailey and Robin Graham [1]. These developments are based on work of Rod Gover [6]. The programme was initiated by Charles Fefferman [4].

Motivating Example (Gauss)

Suppose S is a smooth oriented surface in $\mathbb{R}^n$. We can ask for "invariants" at $p \in S$. The are two steps in the construction of such invariants:

1. Geometric. Choose coördinates:



to put S into *normal form*, namely as the graph of

$$f(x^1, x^2) = Q_{ij}x^i x^j + \cdots$$

for some quadratic form Q. The remaining freedom is then easy to describe—it is SO(2) acting on the (x^1, x^2) coördinates.

2. Algebraic. We can form $\operatorname{tr} Q = Q^i{}_i$ and $\operatorname{tr} Q^2 = Q^i{}_j Q^j{}_i$ which are *clearly* invariant. Weyl's classical invariant theory implies that *all* polynomial invariants of Q may be written as linear combinations of such *complete contractions.*

Remarks

- Usually, one considers instead the particular linear combinations

$$\tfrac{1}{2}\operatorname{tr} Q = H \equiv \textit{mean curvature}$$

$$\tfrac{1}{2}((\operatorname{tr} Q)^2 - \operatorname{tr} Q^2) = \det Q = K \equiv \textit{Gauss curvature.}$$

- We could diagonalise Q to reduce the symmetry group still further from SO(2) to $\mathcal{S}_2$. If λ_1 and λ_2 are the two eigenvalues of Q, then $\lambda_1 + \lambda_2$ and $\lambda_1\lambda_2$ are the *elementary symmetric polynomials* (and freely generate the ring of all invariants).

- Instead, we can use *Cayley-Hamilton*

$$Q^2 - 2HQ + K = 0$$

  and Weyl's result to show that H and K generate.

- Cayley-Hamilton is clear as follows:

$$Q^i{}_{[i}Q^j{}_j\delta^l{}_{k]} = 0$$

  where square brackets mean take the skew part.

- We can drop the requirement that S be oriented. Instead, we allow "invariants" to *scale*, i.e. transform by a *character* of O(2).

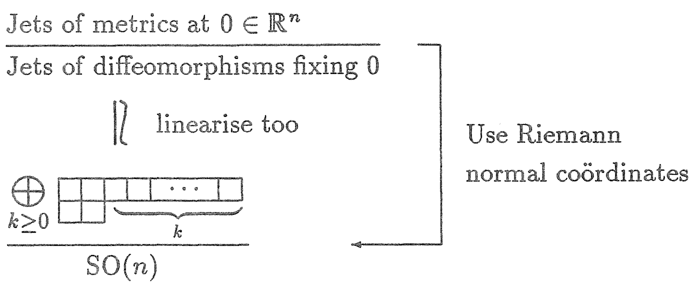
Another Motivating Example (Riemann/Weyl)

There is a similar problem of finding local invariants on a Riemannian manifold. The general theme is the same, namely a geometric step to put the problem into a tractable form followed by an algebraic construction and theorem to say that all invariants have been captured. In this case the two steps are:

1. Use Riemman normal coördinates (and linearise).

2. Use Weyl's classical invariant theory (of SO(n)).

The scheme for step 1 is:



The result is that every local polynomial invariant is a linear combination of complete contractions of $\nabla_i \cdots \nabla_j R_{klmn}$, the covariant derivatives of the Riemannian curvature.

Application: Heat kernel. It is not at all straightforward to calculate the asymptotic expansion of the heat kernel on a Riemmannian manifold. The general form is (Minakshisundaram 1953):

$$K(t,x,x,) \sim (4\pi t)^{-d/2} \sum_{n=0}^{\infty} a_n(x) t^n \quad \text{as } t \downarrow 0.$$

The normalisation is chosen so that $a_0 = 1$. It was soon well-known (e.g. to Arnold, Berger, M$^{\text{C}}$Kean,...) that $a_1 = R/6$ where R is the scalar curvature. To make further progress, since the coefficients are local invariants, it suffices to calculate the expansion of the heat kernel on special manifolds (for example, homogeneous spaces). With conventions

$$(\nabla_i\nabla_j - \nabla_j\nabla_i)X^i \equiv R_{ij}{}^k{}_l X^l \qquad R_{ij} \equiv R_i{}^k{}_{jk} \qquad R \equiv R^i{}_i$$

Berger showed in 1966 that

$$a_2 = \frac{1}{360}\left[12\nabla^i\nabla_i R + 5R^2 - 2R^{ij}R_{ij} + 2R^{ijkl}R_{ijkl}\right]$$

and Sakai in 1971 (more generally Gilkey in 1975 and 1979) showed that

$$a_3 = \frac{1}{45360}\left[\begin{array}{l} 162\nabla^i\nabla_i\nabla^j\nabla_j R + 153(\nabla^i R)(\nabla_i R) - 18(\nabla^i R^{jk})(\nabla_i R_{jk}) \\ - 36(\nabla^i R^{jk})(\nabla_j R_{ik}) + 81(\nabla^i R^{jklm})(\nabla_i R_{jklm}) \\ + 252R\nabla^i\nabla_i R - 72R^{ij}\nabla^k\nabla_k R_{ij} + 216R^{ij}\nabla^k\nabla_i R_{jk} \\ + 108R^{ijkl}\nabla^m\nabla_m R_{ijkl} + 35R^3 - 42RR^{ij}R_{ij} + 42RR^{ijkl}R_{ijkl} \\ - 208R^{ij}R^k{}_i R_{jk} + 192R^{ij}R^{kl}R_{ikjl} - 48R^{ij}R_i{}^{klm}R_{jklm} \\ + 44R^{ijkl}R_{ij}{}^{mn}R_{klmn} + 80R^{ijkl}R_i{}^m{}_k{}^n R_{jmln} \end{array}\right]$$

Other Geometries

CR geometry. It suffices to consider the embedded case. Any real analytic hypersurface can be put into Moser normal form, i.e. there are coördinates $(z^i, x + iy)$ such that

$$2x + h_{i\bar{j}}z^i z^{\bar{j}} = \sum_{\substack{|\alpha|,|\beta|\geq 2 \\ l\geq 0}} A^l_{\alpha\bar{\beta}} z^\alpha z^{\bar{\beta}} y^l$$

is the equation of the surface. (There are various trace relation between the coefficients. The Hermitian form $h_{i\bar{j}}$ is the Levi form of the surface.) This is not unique and there is a group P acting on the remaining freedom. The problem is to list the P-invariant polynomials in $A^l_{\alpha\bar{\beta}}$ (i.e. merely scaling under P). The prospective application is to determining the coefficients in the asymptotic expansion of the Bergman kernel:

$$K(z,z) \sim u^{-n-2}\sum_{k=0}^{n+1} a_k u^k + O(\log u)$$

for u a suitable defining function of the boundary.

Conformal geometry. A conformal metric is an equivalence class of Riemannian metrics where two such metrics g_{ij} and $\widehat{g}_{ij}$ are said to be equivalent if and only if $\widehat{g}_{ij} = \kappa g_{ij}$ for some smooth positive function κ. Thus, one can measure angles but not lengths. A conformal manifold in two dimensions is a Riemann surface.

Theorem (Graham): *Formally*, given a metric g_{ij} near $0 \in \mathbb{R}^n$, a scalar $\lambda > 0$, and $r_j \in \mathbb{R}^n$, there is a unique function $\kappa = \lambda + r_j x^j + \cdots$ such that, for $\widehat{g}_{ij} = \kappa g_{ij}$, we have

$$\widehat{\nabla}_{(i}\widehat{\nabla}_j \cdots \widehat{\nabla}_k \widehat{R}_{lm)}(0) = 0. \qquad \square$$

Riemann normal coördinates for such metrics are called *Graham normal coördinates.*

Geometry for the conformal case. The scheme is

Jets of metrics at $0 \in \mathbb{R}^n$

Jets of diffeomorphisms fixing 0
and jets of conformal rescalings

$\wr\!\wr$ linearise too

Use Graham
normal coördinates

Suitable representation

P

We also need the *Fefferman-Graham ambient metric construction* [5] or some intrinsic equivalent (essentially due to T.Y. Thomas in the 1920's) to organise the resulting representation. Anyway, P is matrices of the form

$$\begin{pmatrix} \lambda & r_j & t \\ 0 & m^i{}_j & s^i \\ 0 & 0 & \lambda^{-1} \end{pmatrix} \qquad \begin{array}{ll} \text{where } \lambda > 0, & m^i{}_j \in \mathrm{SO}(n) \\ s^i = -\frac{1}{\lambda} m^{ij} r_j & t = -\frac{1}{2\lambda} r_j r^j. \end{array}$$

This is to be expected! The reason is that this group appears in the:

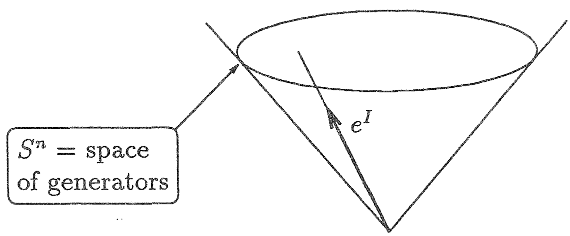
Flat model. Let

$$g_{IJ} = \begin{array}{c} \\ n\left\{ \vphantom{\begin{array}{c} 1 \\ 1 \\ 1 \end{array}} \right. \\ \\ \end{array} \begin{array}{c} \overbrace{\hphantom{000000}}^{n} \\ \left(\begin{array}{c|c|c} 0 & 0 & 1 \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 0 \end{array}\right) \end{array}$$

and G be the identity connected component of the group of $(n+2) \times (n+2)$ matrices preserving g_{IJ}. Then, $e^I = [1, 0, \cdots, 0]^T$ is a null vector and

$$P = \{p \in G \text{ s.t. } pe^I = \lambda e^I \text{ for some } \lambda > 0\}.$$

So G/P is the sphere S^n with its standard flat conformal structure:



G is the group of conformal motions of S^n. The Fefferman-Graham ambient metric contruction provides a "curved" version of the flat model. It gives a thickening of the bundle of metrics by a Lorentzian manifold equipped with

- an Euler field such that
- the Lorentzian metric is
  - homogeneous
  - induces the given conformal metric
  - is Ricci flat.

This works formally to infinite order in the odd-dimensional case and to order n in the $2n$-dimensional case. In any case, linear combinations of complete contractions of ambient curvature and its covariant derivatives provide invariants and we can ask "is this all of them"? (In the even dimensional case we should ask this question only for invariants of "weight" less than n.)

Algebra for the conformal case. This is purely algebraic problem which has now been completely solved in the projective case [6] and the CR and conformal cases [1]. Here is a *model* such problem: Let $W \equiv \mathbb{R}^{n+2}$ as a representation of G by left matrix multiplication, hence a P-module by restriction. There are other "tensor" modules such as $\bigodot_\circ^2 W$ (symmetric trace-free). Weyl's classical theory for G gives results for P (see, [4] or [3])—every P-invariant is a linear combination of complete contractions, e.g.

$$e^I e^J T_{IJ} T^J{}_K \quad \text{for } T \in \bigodot_\circ^2 W$$

using g_{IJ}, e^I, g^{IJ}, and $\epsilon^{IJ\cdots K}$ (the volume form). However, the P-modules whish arise are not quite like this! Typically,

$$H = \left\{T_{IJ} \in \bigodot_\circ^2 W \text{ s.t. } e^I T_{IJ} = 0\right\}.$$

One can still form complete contractions but it is naïve to hope that all invariants will be obtained in this way. For example,

$$T_{IJ} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & & \\ 0 & & X \end{pmatrix} \mapsto \det X$$

is a P-invariant but not a linear combination of compete contractions. Notice that this invariant is homogeneous of degree $n+1$. More generally, let

$$H^l_k = \left\{T \in \bigodot_\circ^l W \text{ s.t. } (e \lrcorner)^{l-k} T = 0\right\}$$

for $0 \le k < l$. Any invariant may be split into its homogeneous parts each of which will be invariant. An invariant is said to be even if it is unchanged by orientation reversal.

Theorem A: Every even P-invariant of degree $\le n$ on H^l_k is a linear combination of complete contractions. (There are no odd P-invariants of degree $< n$.) □

For high degree (i.e. $\ge n+1$) there is a replacement—there is an infinite-dimensional P-module $\mathcal{H}_k$ (roughly $\lim_{l\to\infty} H^l_k$) with a natural surjection $\mathcal{H}_k \twoheadrightarrow H^l_k$ and now

Theorem B: Every linear P-invariant of degree $\ge n+1$ on $\mathcal{H}_k$ is a linear combination of complete contractions.

Remarks on proofs. Theorem A uses "second main theorems of classical invariant theory." Roughly, these say that the only way that tensor identities can arise in dimension n is by skewing over $n+1$ indicies (cf. the proof of Cayley-Hamilton given early in this article). Theorem B uses a geometric interpretation [3] of $\mathcal{H}_k$ as jets on G/P and invariants as differential operators invariant under G. The procedure is then based on ideas of Gover in the projective case.

Outlook. It remains to sort out invariants of high weight in the even-dimensional conformal case and in the CR case (where there is a similar obstruction [4] to the ambient metric construction). Rod Gover has a promising scheme for avoiding the ambient metric contruction (or Thomas's equivalent intrinsic procedure (see [2])) and has implemented this scheme in various special cases.

References

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