Hopf Bifurcation of a Class of PDEs

Xiangjian He and Zhuhan Jiang
Department of Mathematics, Statistics and Computing Science
University of New England
Armidale, NSW 2351
Australia
e-mail:{shawn,zhuhan}@neumann.une.edu.au

Abstract

The Hopf Bifurcation Theorem is the simplest result which guarantees the bifurcation of a family of time periodic solutions of an evolution equation from a family of equilibrium solutions. In this paper, we apply the theorem to a class of partial differential equations (PDEs). The usual assumption of differentiability of nonlinear terms is dropped by the employment of topological considerations. The method of parameter functionalization plays a very important role.

§1. Introduction

The bifurcation of periodic orbits from certain critical points of a real, n-dimensional ($n \geq 2$), first-order system of autonomous ordinary differential equations was treated by E.Hopf [7] in 1942. To explain briefly Hopf's work, let the differential equation be denoted by

$$\frac{dx}{dt} = F(\lambda, x), \quad x \in R^n, \tag{1.1}$$

where λ is a real parameter, and let x^λ be a critical point at λ. Let it be assumed that F is analytic in a neighborhood of $(\lambda, x) = (0, x^0)$ and let the matrix $F_x(0, x^0)$ have exactly two, non-zero, purely imaginary eigenvalues, say $\pm i\omega_0$, and have no eigenvalues of the form $0, \pm 2i\omega_0, \pm 3i\omega_0, \ldots$. Hopf proved that a non-constant periodic orbit bifurcates from $(\lambda, x) = (0, x^0)$ under the sole additional assumption that $\alpha'(0) \neq 0$ if $\alpha(\lambda) + i\omega(\lambda)$ denotes that eigenvalue of $F_x(\lambda, x^\lambda)$ which is a continuous extension of $+i\omega_0$. Some authors, following Hopf, have approached the bifurcation problem by trying to vary the initial conditions and parameters so as to produce a nontrivial time periodic solution (see, for example, [1] and [13]). Others have introduced the unknown period explicitly as a new parameter in the equations and attempted to find solutions having a known period (see, for example, [2] and [8]). It is difficult to compare those papers since they are set in different technical frameworks and have related but differing hypotheses. But no matter what they did, they all required that the nonlinearities be of C^1 class. In [4], [6] and [11], employment of topological considerations made it possible to throw aside the usual assumptions of differentiability of nonlinear terms, but there Hopf bifurcation theorem are still restricted in an n dimensional space.

In this paper, we discuss Hopf bifurcation of a class of PDEs with nondifferentiable nonlinearities. This needs the Hopf bifurcation theorem in general Banach spaces [5]. Possible applications of this theorem are also discussed in [9].

§2. Preliminaries

Let X be a Banach space and $C([0,\kappa],X)$ be a family of all continuous functions from $[0,\kappa]$ into X, where κ is a positive number in R. $C([0,\kappa],X)$ is a Banach space with norm $\|\cdot\|_{C([0,\kappa],X)}$ defined by

$$\| x \|_{C([0,\kappa],X)} \triangleq \sup_{t\in[0,\kappa]} \| x(t) \|$$

for each $x \in C([0,\kappa],X)$.

Similar to Ascoli's Theorem ([3] p.122) we have the following result.

Lemma 2.1. *Suppose that a sequence $\{x_n\}$ of $C([0,\kappa],X)$ is bounded and equicontinuous on $[0,\kappa]$. If $\{x_n(t)\}$ is compact for each $t \in (0,\kappa]$, then $\{x_n\}$ is also compact, i.e., it has a convergent subsequence.*

Let

$$X_c = X \oplus iX \triangleq \{x_1 + ix_2 | x_1, x_2 \in X, i^2 = -1\}.$$

Consider a linear operator A from its domain in X, denoted by $D(A)$, into X. Then A will also denote its extension to a linear operator from its domain in X_c, denoted by $D(A)_c$, into X_c.

Lemma 2.2. *Suppose that A is the infinitesimal generator of a semigroup of the linear operators, $\{T(t)\}$, on X. Then*

1). for every x in $D(A)$,

$$\frac{dT(t)x}{dt} = AT(t)x = T(t)Ax; \tag{2.1}$$

2). $T(t)$ is continuous in the uniform operator topology for $t \geq 0$ if $T(t)$ is analytic at every $t \geq 0$.

The proof of this lemma is evident.

§3. Elementary Assumptions and Properties

Let us consider the following a class of partial differential equations

$$\frac{\partial u}{\partial t} = a_k(\lambda)\frac{\partial^k u}{\partial x^k} + a_{k-1}(\lambda)\frac{\partial^{k-1} u}{\partial x^{k-1}} + \cdots + a_1(\lambda)\frac{\partial u}{\partial x} + a_0(\lambda)u + a(\lambda, x) \ (t \geq 0,\ \ell_1 \leq x \leq \ell_2), \tag{3.1}$$

where λ is a parameter in R, k is an integer, $a_i(\lambda)$ $(i = 0, 1, \cdots, k)$ is a function of λ and $a(\lambda, x)$ is a function of λ and x. Let X be the set of continuous, periodic functions defined on $[\ell_1, \ell_2]$ with period $\ell_2 - \ell_1$. Then X is a Banach space with the norm

$$\| f \|_X = [\int_{\ell_1}^{\ell_2} |f(x)|^2 dx]^{\frac{1}{2}}$$

for $f \in X$, and, X is a subspace of $L^2[\ell_1, \ell_2]$. Define the linear operator $A(\lambda)$ by

$$A(\lambda)\phi = a_k(\lambda)\frac{d^k\phi}{dx^k} + a_{k-1}(\lambda)\frac{d^{k-1}\phi}{dx^{k-1}} + \cdots + a_1(\lambda)\frac{d\phi}{dx} + a_0(\lambda)\phi$$

from a subset of X to X. Then (3.1) is equivalent to a differential equation in X

$$\frac{dx}{dt} = A(\lambda)x + a(\lambda, x). \tag{3.2}$$

Assumption (HA).

(i). *There exists a* $\lambda_0 \in R$ *such that* $A(\lambda_0) \triangleq A$ *is the infinitesimal generator of a semigroup of linear operators,* $\{T(t)\}$*, on* X*.* $T(t)$ *is analytic at every* $t \geq 0$*. Furthermore, for each* $y \in X$*, there exists an* $x \in D(A)$ *such that* $Ax = y$*.*

Lemma 3.1. $D(A) \subset X$ *is a Banach space with norm* $\|\cdot\|_A$ *defined by*

$$\| x \|_A = \| Ax \|$$

for each $x \in D(A)$*.*

Assumption (HA).

(ii). *A has a purely imaginary eigenvalue* $i\omega_0$ $(\omega_0 \neq 0)$ *which is simple, i.e.,*

$$dim N_c(A - i\omega_0 I) = 1$$

where $N_c(\cdot)$ *denotes the null subspace in* $D(A)_c$ *and* I *is the identity operator from* X_c *onto* X_c*;*

(iii). $in\omega_0$*, for* $n = 0, \pm 2, \pm 3, \cdots$*, are not in* $\sigma(A)$*, the spectrum of* A*;*

(iv). $A(\lambda) - A$ *is a bounded linear operator from* $D(A)$ *into* $D(A)$ *for each* λ *and* $A(\lambda)$ *is continuous with respect to* λ *in the sense of the norms of the linear operators;*

(v). $i\omega_0$ *is an isolated eigenvalue of* A*, i.e., there does not exist any other eigenvalue of* A *in a neighbourhood of* $i\omega_0$*;*

(vi). *For each* λ*,* $A(\lambda)$ *commutes with* A*;*

(vii). $(\mu I - A)^{-1}$ *is compact for* μ *in the resolvent set of* A*.*

By $(HA)(ii)$, (iv) and (v), we have (see, for example, [10] p.213) that there are a continuous $D(A)_c$-valued function $x(\lambda)$ and a continuous complex valued $m(\lambda)$ defined in a neighbourhood of λ_0 such that

$$A(\lambda)x(\lambda) = m(\lambda)x(\lambda)$$

and

$$x(\lambda_0) = x_0 \neq 0, \quad m(\lambda_0) = i\omega_0.$$

Following Hopf, another assumption is

Assumption (Hm). *The real part of* $m(\lambda)$*,* $Re(m(\lambda))$*, takes values of opposite signs in every neighbourhood of* λ_0*.*

Let $B(\lambda_0)$ be a closed neighbourhood of λ_0. The requirements on a in (3.2) are stipulated by the following assumption.

Assumption (Ha).

(i). *a is a continuous function from* $R \times D(A)$ *into* $D(A)$*;*

(ii).

$$\lim_{\| x \|_A \to 0} \frac{\| a(\lambda, x) \|_A}{\| x \|_A} = 0 \tag{3.3}$$

uniformly with respect to $\lambda \in B(\lambda_0)$*;*

(iii). $\| a(\lambda, x) \|_A$ *is bounded for all* x *in a bounded set of* $D(A)$ *and* $\lambda \in B(\lambda_0)$*.*

One may prove that

Lemma 3.2. *If Assumption* (HA) *is true, then*

$$N_c(A - i\omega_0 I) \oplus N_c(A + i\omega_0 I) = N_c(T(\frac{2\pi}{\omega_0}) - I),$$

where $N_c(\cdot)$ *denotes the null space in* $D(A)_c$.

§4. An Existence Result

Let E be a Banach space with norm $\|\cdot\|_E$. Consider an operator $U(\mu, x)$ which is defined on a neighborhood of a point $\{\mu_0, 0\} \in R^2 \times E$ and takes values in E. Let $U(\mu, x)$ admit a representation of the form

$$U(\mu, x) = V(\mu)x + v(\mu, x),$$

where $V(\mu)$ is a bounded linear operator from E into E for each μ and where the remainder term $v(\mu, x)$ satisfies the condition

$$\lim_{\|x\|_E \to 0} \frac{\| v(\mu, x) \|_E}{\| x \|_E} = 0$$

uniformly with respect to μ in a neighborhood of μ_0. We also suppose $V(\mu)x$ and $v(\mu, x)$ are completely continuous on μ and x. Furthermore, we denote the identity operator from E onto E by I_E and the null space of a linear operator L from E into E by $N_E(L)$. Then Kozjakin and Krasnosel'skii [11] prove that

Lemma 4.1. *Let* 1 *be an eigenvalue of the linear operator* $V(\mu_0)$ *with*

$$dim N_E(V(\mu_0) - I_E) = 2$$

and let this eigenvalue be of simple structure. Suppose that there exists a sequence of Jordan curves $\{\mathcal{L}_n\}$ *in* R^2 *converging to* μ_0. *Consider* $1 - m(\mu)$, *where* $m(\mu)$ *is the continuous branch of eigenvalues passing through the eigenvalue 1, be defined and not equal to zero on each curve* $\mathcal{L}_n$. *Then there exist* $\mu_n \to \mu_0$ *and* $x_n \to 0$ $(x_n \neq 0)$ *such that* $x_n = U(\mu_n, x_n)$.

§5. Hopf Bifurcation

Definition 5.1. *We shall say that for* $\lambda = \lambda_0$ **generation of small periodic solutions** *of the system (3.2) with periods close to* T_0 *takes place, if for every* $\epsilon > 0$ *there exists a* λ_ϵ *in the interval* $(\lambda_0 - \epsilon, \lambda_0 + \epsilon)$ *for which the system (3.2) has a nonzero* T_ϵ*-periodic solution* $x_\epsilon(t)$ $(|T_\epsilon - T_0| < \epsilon)$ *such that*

$$\| x_\epsilon(t) \|_{C([0,T_\epsilon],D(A))} = \sup_{0 \le t \le T_\epsilon} \| x_\epsilon(t) \|_A < \epsilon,$$

where $\|\cdot\|_A$ *is the norm of* $D(A)$.

Theorem 5.1. *Suppose Assumptions* (HA), (Hm) *and* (Ha) *hold. Then for* $\lambda = \lambda_0$, *generation of small periodic solutions of the system (3.2) with periods close to* $\frac{2\pi}{\omega_0}$ *takes place.*

Proof. Let $E \triangleq C([0, \kappa], D(A))$ denote the Banach space of continuous $D(A)$-valued functions defined on the interval $[0, \kappa]$, where $\kappa = \frac{2\pi}{\omega_0} + 1$, with the topology of uniform convergence, i.e., for each $x(t) \in E$,

$$\| x(t) \|_E \triangleq \sup_{t \in [0,\kappa]} \| x(t) \|_A$$

where $\|\cdot\|_A$ is the norm of $D(A)$. Consider an operator, from E into E, of the form

$$U(t_0, \lambda; x)(t) = T(t)x(t_0) + \int_0^t T(t-s)[A(\lambda) - A]x(s)ds$$
$$+ \int_0^t T(t-s)a[\lambda, x(s)]ds$$

which depends on two parameters t_0 and λ.

Direct verification shows that $x \in C([0,\kappa], D(A))$ is a fixed point of the operator $U(t_0,\lambda;\cdot)$ if and only if $x(t)$ is a solution of the system (3.2) that satisfies the condition $x(0) = x(t_0)$ (see, for example, [14] pp.146-147). Hence we may extend $x(t)$ to a periodic solution of (3.2) with period t_0.

Let us verify that the operator $U(t_0,\lambda;x)$ satisfies the conditions in Lemma 4.1.

Step 1. Represent $U(t_0,\lambda;x)$ in the form of a sum

$$U(t_0,\lambda;x) = V(t_0,\lambda)x + v(\lambda;x),$$

where

$$V(t_0,\lambda)x(t) = T(t)x(t_0) + \int_0^t T(t-s)[A(\lambda) - A]x(s)ds$$

and

$$v(\lambda;x)(t) = \int_0^t T(t-s)a[\lambda, x(s)]ds.$$

Now, for any given $\varepsilon > 0$, by Assumption $(Ha)(ii)$, there exists a $\delta > 0$ (independent of $\lambda \in B(\lambda_0)$) such that

$$\| a(\lambda,x) \|_A < \varepsilon \| x \|_A \quad \text{whenever} \quad \| x \|_A < \delta \ \text{and} \ x \in D(A).$$

And, since $\{T(t)\}$ is a C_0 semigroup, there exists a constant $\omega \geq 0$ and $C \geq 1$ such that

$$\| T(t) \| \leq Ce^{\omega t} \tag{5.1}$$

for all $\lambda \in B(\lambda_0)$.

Hence, when $\lambda \in B(\lambda_0)$ and $\| x(t) \|_E < \delta$, there exist $\bar{t}, \bar{s} \in [0,\kappa]$ such that $\bar{t} \geq \bar{s}$ and

$$\begin{aligned}
\| v(\lambda;x) \|_E &= \| \int_0^{\bar{t}} T(\bar{t}-s)a[\lambda, x(s)]ds \|_A \\
&\leq \int_0^{\bar{t}} \sup_{0\leq s\leq \bar{t}} \| T(\bar{t}-s)a[\lambda, x(s)] \|_A \, ds \qquad (see[12]p.5) \\
&\leq \kappa \| T(\bar{t}-\bar{s})a[\lambda, x(\bar{s})] \|_A \\
&\leq \kappa \| T(\bar{t}-\bar{s}) \| \cdot \| a[\lambda, x(\bar{s})] \|_A \\
&\leq \kappa C e^{\omega(\bar{t}-\bar{s})}\varepsilon \| x(\bar{s}) \|_A \\
&\leq \kappa C e^{\omega\kappa}\varepsilon \| x(t) \|_E \, .
\end{aligned} \tag{5.2}$$

This proves that

$$\lim_{\|x\|_E \to 0} \frac{\| v(\lambda;x) \|_E}{\| x \|_E} = 0$$

uniformly with respect to λ in the neighbourhood of λ_0, $B(\lambda_0)$.

Step 2. we would like to prove that $V(t_0,\lambda)x$ and $v(\lambda;x)$ are completely continuous with respect to $t_0 \in [0,\kappa]$, $\lambda \in B(\lambda_0)$ and $x \in E$. Let $\{(t_n,\lambda_n,x_n)\}$ be a bounded sequence of $[0,\kappa] \times B(\lambda_0) \times E$. Then, we only need to prove that $\{V(t_n,\lambda_n)x_n\}$ and $\{v(\lambda_n;x_n)\}$ are compact sets. We prove this below.

1). Since $\{\lambda_n\}$ is bounded, it has a convergent subsequence. Without loss of the generality, one may assume that $\{\lambda_n\}$ itself is convergent, i.e. there exists $\lambda^* \in B(\lambda_0)$ such that

$$\lambda_n \to \lambda^*$$

as $n \to \infty$. So, without loss of the generality, one may assume that $A(\lambda_n) - A$ is bounded in the sense of the norms of operators. Furthermore, since x_n is also bounded in E, there exists an $M > 0$ such that

$$\| a(\lambda, x_n(t)) \|_E \leq M$$

for all $\lambda \in B(\lambda_0)$ and $n = 1, 2, 3, \cdots$ by Assumption (Ha)(iii).

2). It is evident that $\{x_n(t_n)\}$ is a bounded sequence in $D(A)$. So, $\{T(t)x_n(t_n)\} \subset C([0,\kappa], D(A))$ is bounded by (5.1). Note that, by Lemma 2.2, $\{T(t)x_n(t_n)\}$ is equicontinuous on $[0,\kappa]$. Furthermore, we have that $T(t)$ is a compact operator (see [14] p.48) for each $t \in (0,\kappa]$ by Assumption (HA)(vii). Then, by Lemma 2.1, $\{T(t)x_n(t_n)\}$ is compact subset of $C([0,\kappa], D(A))$.

3). Similar to the derivation of (5.2), there exist $\bar{t}$ and $\bar{s}$ in $[0,\kappa]$ such that $\bar{t} \geq \bar{s}$ and

$$\begin{aligned} &\| \int_0^t T(t-s)[A(\lambda_n) - A]x_n(s)ds \|_E \\ &\leq \kappa \| T(\bar{t} - \bar{s}) \| \cdot \| [A(\lambda_n) - A] \| \cdot \| x_n(\bar{s}) \|_A . \end{aligned} \tag{5.3}$$

Hence,

$$V_1(\lambda_n; x_n)(t) \triangleq \int_0^t T(t-s)[A(\lambda_n) - A]x_n(s)ds \tag{5.4}$$

is bounded. Furthermore, let

$$B(\lambda) = A(\lambda) - A$$

for each λ. Then, for any $\tau_1, \tau_2 \in [0,\kappa]$ and $\tau_1 \leq \tau_2$, there exist $s_1, s_2 \in [0,\kappa]$ such that

$$\begin{aligned} &\| V_1(\lambda_n; x_n)(\tau_2) - V_1(\lambda_n; x_n)(\tau_1) \|_A \\ &= \| \int_0^{\tau_2} T(\tau_2 - s)B(\lambda_n)x_n(s)ds - \int_0^{\tau_1} T(\tau_1 - s)B(\lambda_n)x_n(s)ds \|_A \\ &\leq \| \int_0^{\tau_2} T(\tau_2 - s)B(\lambda_n)x_n(s)ds - \int_0^{\tau_1} T(\tau_2 - s)B(\lambda_n)x_n(s)ds \|_A \\ &+ \| \int_0^{\tau_1} T(\tau_2 - s)B(\lambda_n)x_n(s)ds - \int_0^{\tau_1} T(\tau_1 - s)B(\lambda_n)x_n(s)ds \|_A \\ &\leq \int_{\tau_1}^{\tau_2} \| T(\tau_2 - s_1) \| \cdot \| B(\lambda_n)x_n(s_1) \|_A \, ds \\ &+ \int_0^{\tau_1} \| T(\tau_2 - s_2) - T(\tau_1 - s_2) \| \cdot \| B(\lambda_n)x_n(s_2) \|_A \, ds \\ &\leq (\tau_2 - \tau_1)Ce^{\omega(\tau_2 - s_1)} \| B(\lambda_n) \| \cdot \| x_n(s_1) \|_A \\ &+ \tau_1 \| T(\tau_2 - s_2) - T(\tau_1 - s_2) \| \cdot \| B(\lambda_n) \| \cdot \| x_n(s_2) \|_A \end{aligned} \tag{5.5}$$

This means that $\{V_1(\lambda_n; x_n)\}$ is equicontinuous by Lemma 2.2. Hence, by Lemma 2.1, $\{V_1(\lambda_n; x_n)\}$ is compact.

4). By the derivation of (5.2), we have that $\{v(\lambda_n; x_n)\}$ is a bounded sequence of E.

5). Similar to the proof of (5.5), we may claim that $\{v(\lambda_n; x_n)\}$ has a Cauchy subsequence.

Step 3. By the Lemma 3.2, we have

$$\begin{aligned} 2 &= \dim N_c(A - i\omega_0 I) + \dim N_c(A + i\omega_0 I) \\ &= \dim N_c(T(\lambda_0, \frac{2\pi}{\omega_0}) - I) \\ &= \dim N(T(\frac{2\pi}{\omega_0}) - I), \end{aligned}$$

and hence, 1 is the eigenvalue of the linear operator $T(T_0)$, where $N(\cdot)$ denotes the null space in $D(A)$ and $T_0 = \frac{2\pi}{\omega_0}$.

Step 4. By Lemma 2.2, one can see that $x(t) \in N_E(V(T_0, \lambda_0) - I_E)$, i.e.,

$$x(t) = V(T_0, \lambda_0)x(t) = T(t)x(T_0) \tag{5.6}$$

for all $t \in [0, \kappa]$ if and only if $x(t)$ is the solution of

$$\frac{dx}{dt} = Ax \tag{5.7}$$

with $x(0) = x(T_0) \neq 0$.

Let $\{x_1, x_2\}$ be a base of $N(T(\lambda_0, T_0) - I)$ and $x_i(t)$ $(i = 1, 2)$ be corresponding solutions of (5.7). Then, for $x(t) \in N_E(V(T_0, \lambda_0) - I_E)$, we have that

$$x(0) = x(T_0) = T(T_0)x(T_0)$$

by (5.6).

Hence $x(0) = x(T_0) \in N(T(T_0) - I)$. So, there exist complex constants d_1 and d_2 such that

$$x(0) = d_1x_1 + d_2x_2.$$

Thus, by the uniqueness of the solutions of (5.7),

$$x(t) = d_1x_1(t) + d_2x_2(t). \tag{5.8}$$

Furthermore, for $i = 1, 2$, $x_i(t) = T(t)x_i$ since $x_i(t)$ is a solution of (5.7). So, $x_i(T_0) = T(T_0)x_i = x_i$. Hence

$$x_i(t) = T(t)x_i = x_i(t) = T(t)x_i(T_0) = V(T_0, \lambda_0)x_i(t)$$

for $i = 1, 2$. This means that $x_i(t) \in N_E(V(T_0, \lambda_0) - I_E)$. $x_1(t)$ and $x_2(t)$ are evidently independent. So, by (5.8),

$$\dim N_E(V(T_0, \lambda_0) - I_E) = 2$$

and 1 is the eigenvalue of $V(T_0, \lambda_0)$.

It is evident that the eigenvalue 1 is of simple structure by the continuity of $A(\lambda) - A$ and Assumption (Hm).

Step 5. By Assumption (Hm), $Re(m(\lambda))$ changes sign in every neighbourhood of λ_0, so we can construct the required Jordan curves.

Hence, we have completed the verification of Lemma 4.1's conditions. Thus, Theorem 5.1 is proved. □

§6. An Example

In order to show the applicability of the Hopf bifurcation theorem (Theorem 5.1), let us consider the following partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + \lambda u + a(u) \ (t \geq 0, \ 0 \leq x \leq 2), \tag{6.1}$$

where λ is a parameter and $a(u)$ will be defined below. Let X be the set of continuous, periodic functions defined on $[0, 2]$ with period 2. Define the linear operator $A(\lambda)$ by

$$A(\lambda)\phi = \frac{d^2\phi}{dx^2} + \frac{d\phi}{dx} + \lambda\phi,$$

from a subset of X to X. Then the domain of $A(\lambda)$, for each λ, is the set of second order continuously differentiable, periodic solutions defined on $[0,2]$. By Theorem 8.2 in [12] p.163, we know that, $A \triangleq A(\pi^2)$ is the infinitesimal generator of a semigroup of linear operators, $\{T(t)\}$, on X and $T(t)$ is analytic for every $t \geq 0$, and for each $\phi \in X$,

$$T(t)\phi = u(\phi, t),$$

where $u(\phi,t)$ is the solution of

$$\frac{du}{dt} = Au \quad u(\phi, 0) = \phi. \tag{6.2}$$

It is evident that for each $x \in X$ there exists a $y \in D(A)$ such that $Ay = x$. Let

$$\phi_k(x) = exp(ik\pi x) \quad (k = 0, \pm 1, \pm 2, \cdots). \tag{6.3}$$

Then $\phi_k \in D(A)$ for each k. Now, for each $\phi \in X$, $\phi(x)$ can be written in the form of

$$\phi(x) = \sum_{k=0}^{\infty}[a_k \cos(k\pi x) + b_k \sin(k\pi x)].$$

Let us define a function from $D(A)$ into $D(A)$ by

$$a(\phi) = |a_1|^3 \cos(\pi x).$$

Then a is a operator, from $D(A)$ into $D(A)$, satisfying Assumption (Ha). Hence, equation (6.1) can be written as a differential equation

$$\frac{du}{dt} = A(\lambda)u + a(u) \tag{6.4}$$

in the Banach space $D(A)$.

Now let us check that Assumptions (HA) and (Hm) are satisfied.

Step 1. It is easy to see that

$$B(\lambda) \triangleq A(\lambda) - A$$

is bounded for each λ and $A(\lambda)$ is continuous with respect to λ in the sense of the norms of the linear operators. Furthermore, it is also obvious that $A(\lambda)$ commutes with A for any $\lambda \in R$. So, Assumption (HA)(iv) and (vi) are satisfied.

Step 2. Note that the spectrum $\delta(A(\lambda))$ of $A(\lambda)$ consists of the simple eigenvalues

$$m_k(\lambda) = (\lambda - k^2\pi^2) + ik\pi, \tag{6.5}$$

$k = 0, \pm 1, \pm 2, \cdots$. with corresponding eigenfunctions $\phi_k(x)$. That is

$$A(\lambda)\phi_k = m_k(\lambda)\phi_k.$$

Hence, A has a purely imaginary eigenvalue

$$i\pi \triangleq i\omega_0$$

which is simple and A has no eigenvalues of the forms 0, $\pm 2i\omega_0$, $\pm 3i\omega_0, \cdots$. This means that Assumptions (HA) (ii) and (iii) are satisfied. Assumption (HA)(v) is satisfied evidently. Refer to [15] (pp.93-94) Theorem 3.1 and its example, we have that $(\mu I - A)^{-1}$ is compact for each μ in the resolvent set of A. This verifies Assumption (HA)(vii).

Step 3. Let

$$m(\lambda) = (\lambda - \pi^2) + i\pi. \tag{6.6}$$

Then the real part of $m(\lambda)$ takes values of opposite signs in every neighbourhood of π^2 evidently. So Assumption (Hm) is satisfied.

Thus, the conditions in Theorem 5.1 are satisfied so that Theorem 5.1 is applicable to this example. In other words, system (6.1) bifurcates infinitely many times when λ is close to the value π^2.

REFERENCES

[1]. Allwright, D.J., "Harmonic Balance and the Hopf bifurcation", Math. Proc. Camb. Phil. Soc., Vol.87, pp.453-467, 1977.

[2]. Fife,P.C., "Branching Phenomena in Fluid Dynamics and Chemical Reaction diffusion Theory", Proc. Symp. Eigenvalues of Nonlinear Problems, Edizioni Cremonese Rome, pp.23-83, 1974.

[3]. Graves, L.M., "The Theory of Functions of Real Variables", McGraw-Hill Book Company, Inc., New York, 1946.

[4]. He,X., "Hopf Bifurcation at Infinity with Discontinuous Nonlinearities", J. Austral. Math. Soc. Ser. B., Vol. 33, pp.133-148, 1991.

[5]. He, X., "Hopf Bifurcation Theorem in Banach Spaces with C^0 nonlinearities", Research Report, University of New England, 1993.

[6]. He, X. and Huilgol, R.R., "Application of Hopf Bifurcation at Infinity to Hunting Vibrations of Rail Vehicle trucks", Proceedings of 12th International Association for Vehicle System Dynamics Symposium, France, pp.240-253, 1992.

[7]. Hopf,E., "Abzweigung Einer Periodischen Lösung Von Einer Stationären Lösung Eines Differentialsystems", Ber. Sächs. Akad. Wiss. Leipzig. Math. Phys. kl., Vol.95, pp.3-22, 1942.

[8]. Iooss,G., "Existence et Stabilité de la Solution périodique Secondaire Intervenant dans les Problèmes D'evolution du Type Navier-Stokes", Arch. Rational Mech. Anal., Vol.47, pp.301-329, 1972.

[9]. Jiang, Z. and He, X., "NLS Type Equations in 2+1 Dimensions: New Type of Solutions and Non-isospectral Problems", In Preparation.

[10]. Kato, T., "Perturbation Theory for Linear Operators", 2nd Edition, Springer-Verlag, New York, 1984.

[11]. Kozjakin,V.S. and Krasnosel'skii,M.A., "The Method of Parameter Functionalization in the Hopf Bifurcation Problem", Nonlinear Analysis, Theory, Methods and Applications, Vol.11, pp.149-161, 1987.

[12]. Krein,S.G., "Linear Differential Equations in Banach Space", (translation of Mathematical Monographs, Vol. 29), Amer. Math. Soc., Providence, Rhode Island, 1970.

[13]. Mees, A.I. and Chua, L.O., "The Hopf Bifurcation Theorem and Its Applications to Nonlinear Oscillations in Circuits and Systems", IEEE Transactions on Circuits and Systems, CAS-26, pp.235-254, 1979.

[14]. Pazy, A., "Semigroups of Linear Operators and Applications to Partial Differential Equations", Springer-Verlag, New York, 1983.

[15]. Schechter, M., "Principles of Functional Analysis", Academic Press, New York, 1971.