

## Travelling Wave Solutions for a Drying Problem

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**Abstract:** *A simple model for drying consisting of an infiltration type of partial differential equation is considered. Whereas the infiltration problem admits plane wave solutions with uniquely determined wave speed, the drying problem exhibits a much more complicated behaviour involving breaking up observed in the pendular state of drying.*

### 1. Introduction and Formulation:

A simple model governing the drying of a porous material at a constant wet-bulb temperature is formulated by Ilić & Turner [1] in nondimensional form (with slight modification) as follows:

$$\frac{\partial S}{\partial t} = \frac{\partial}{\partial x} \left\{ K_S(S) \frac{\partial S}{\partial x} - K_g(S) \right\}, \quad (1)$$

$$K_S(S) = \begin{cases} \alpha S^3 \{ f(S) + \frac{5}{S^2} \}, & S > 0, \\ 0, & S \leq 0, \end{cases}$$

$$K_g(S) = \begin{cases} \beta S^3, & S > 0, \\ 0, & S \leq 0, \end{cases}$$

where  $S$  is the moisture content (volume saturation:  $0 \leq S \leq S_0 < 1$ ) and  $\alpha$  and  $\beta$  are nondimensional constants which are determined by the properties of the porous material and the drying conditions [1];  $f(S) = \text{const.} + \text{const.} e^{-40(1-S)}$ , these constants being positive. Here  $x$  is the vertical axis with positive direction downward and  $t$  is time.

Equation (1) has the appearance of other convection-diffusion problems and in particular resembles the infiltration equation. A brief discussion of such equations and their nonnegative solutions with compact support can be found in Taylor [2]. Taylor discusses both the weak solution and the classical similarity solution.

In this study we ask whether equation (1) admits travelling plane wave solutions on  $-\infty < x < \infty$  i.e. we seek solutions of the form

$$S(x, t) = S(\xi), \quad \xi = x - \lambda t, \quad \lambda > 0, \quad (2)$$

which represents a plane wave travelling from left to right i.e. vertically downwards or

$$S(x, t) = S(\xi), \quad \xi = x + \lambda t, \quad \lambda > 0, \quad (3)$$

which represents a plane wave travelling from right to left i.e. vertically upwards.

The boundary conditions appropriate to a drying problem are

$$S(-\infty) = 0, \quad S(\infty) = S_0, \quad (4)$$

and for the infiltration problem

$$S(-\infty) = S_0, \quad S(\infty) = 0, \quad (5)$$

where  $0 < S_0 < 1$ . In particular we are interested in the initial profiles of the form

$$S(x, 0) = \begin{cases} 0, & -\infty < x \leq 0, \\ g(x), & x > 0, \end{cases} \quad (6)$$

for the drying problem and

$$S(x, 0) = \begin{cases} g(x), & -\infty < x \leq 0, \\ 0, & x > 0, \end{cases} \quad (7)$$

for the infiltration problem. Naturally  $g(x)$  in (6) and (7) cannot be arbitrarily specified but becomes a part of the solution.

Note that these conditions and the equation are satisfied by the discontinuous constant solution  $g(x) = S_0$ .

## 2. Plane Waves Downwards:

Consider the solution of (1) of the form (2). With a prime denoting differentiation with respect to  $\xi$ , equation (1) becomes

$$-\lambda S' = (K_S S' - K_g)'$$

which integrates to

$$-\lambda S = K_S S' - K_g + c_1.$$

The zero boundary condition at  $\xi = -\infty$  (for drying) or  $\xi = \infty$  (for infiltration) gives  $c_1 = 0$ . Thus

$$\frac{dS}{d\xi} = \frac{K_g(S) - \lambda S}{K_S(S)}.$$

This is a first order separable differential equation whose solution is

$$\int^S \frac{K_S(s) ds}{K_g(s) - \lambda s} = \xi + c_2. \quad (8)$$

If we take the initial profile in (6) or (7) we have

$$\int_0^S \frac{K_S(s) ds}{K_g(s) - \lambda s} = \xi. \quad (9)$$

Define

$$\begin{aligned} F(S) &= \int_0^S \frac{K_S(s) ds}{K_g(s) - \lambda s} \\ &= -\frac{\alpha}{\lambda} \int_0^S \frac{s^2 \left\{ \frac{5}{s^2} + f(s) \right\} ds}{1 - \frac{\beta}{\lambda} s^2}. \end{aligned}$$

The numerator of the integrand is positive. The denominator vanishes when  $s = \sqrt{\frac{\lambda}{\beta}}$ . Thus the integrand is positive for  $0 \leq s < \sqrt{\frac{\lambda}{\beta}}$  and hence  $F(S)$  is monotone decreasing such that  $F(S) \rightarrow -\infty$  as  $s \rightarrow \sqrt{\frac{\lambda}{\beta}}$ . To fit the boundary condition (4) we must have  $\lim_{S \rightarrow S_0} F(S) = \infty$  which is impossible i.e. the drying problem does not admit plane wave solutions of the form (2). The boundary condition (5) is satisfied if we set  $S_0 = \sqrt{\frac{\lambda}{\beta}}$  i.e.

$$\lambda = \beta S_0^2,$$

which is the plane wave speed.  $F$  has an inverse  $F^{-1}$  and the plane wave solution for the infiltration problem is

$$S = \begin{cases} F^{-1}(\xi) = F^{-1}(x - \lambda t), & \xi = x - \lambda t < 0 \\ 0, & \xi = x - \lambda t > 0. \end{cases} \quad (10)$$

This solution is differentiable everywhere except  $\xi = 0$  i.e.  $S = 0$  which is a singular point and hence a possible branching point.

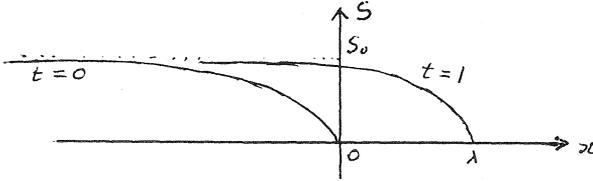
To see the behaviour of these solutions, consider the case when  $f(s) = 0$ . In this case the explicit solution for  $S$  is

$$S = \sqrt{\frac{\lambda}{\beta}} \tanh\left\{ \frac{\sqrt{\lambda\beta}}{5\alpha} \xi \right\} \quad (11)$$

which gives

$$S = \begin{cases} S_0 \tanh\left\{ \frac{\beta S_0}{5\alpha} (\lambda t - x) \right\}, & x - \lambda t < 0 \\ 0, & x - \lambda t \geq 0. \end{cases}$$

These profiles are shown in the following figure.



3. Plane Waves Upwards:

Now consider the solution of (1) of the form (3). The only difference to the previous analysis is that (9) becomes

$$\int_0^S \frac{K_S(s) ds}{K_g(s) + \lambda s} = \xi.$$

Define

$$G(S) = \frac{\alpha}{\lambda} \int_0^S \frac{s^2 \{ \frac{5}{s^2} + f(s) \} ds}{1 + \frac{\beta}{\lambda} s^2}.$$

$G(S)$  is monotone increasing but finite for  $S$  finite being infinite only when  $S \rightarrow \infty$ . The inverse  $G^{-1}$  exists so that

$$S = G^{-1}(\xi) = G^{-1}(x + \lambda t).$$

Since the physical range of  $S$  is  $0 \leq S \leq S_0 < 1$ , there exists  $\xi_0$  such that

$$G(S_0) = \xi_0$$

and the solution is

$$S = \begin{cases} 0, & \xi = x + \lambda t \leq 0, \\ G^{-1}(\xi) = G^{-1}(x + \lambda t), & 0 \leq \xi = x + \lambda t \leq \xi_0 \\ S_0, & \xi = x + \lambda t > \xi_0. \end{cases} \quad (12)$$

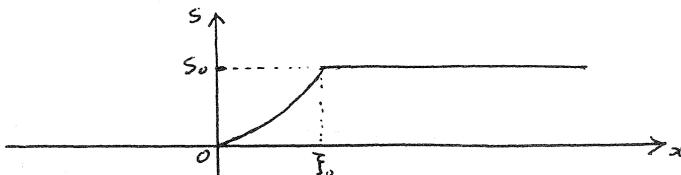
To see the behaviour of these solutions, consider the case when  $f(s) = 0$ . In this case the explicit solution for  $S$  is

$$S = \sqrt{\frac{\lambda}{\beta}} \tan\left\{ \frac{\sqrt{\lambda\beta}}{5\alpha} \xi \right\} \quad (13)$$

which gives

$$S = \begin{cases} \sqrt{\frac{\lambda}{\beta}} \tan\left\{ \frac{\sqrt{\lambda\beta}}{5\alpha} (x + \lambda t) \right\}, & 0 \leq x + \lambda t \leq \xi_0 \\ S_0, & x + \lambda t > \xi_0. \end{cases}$$

The initial profile is plotted in the following diagram:

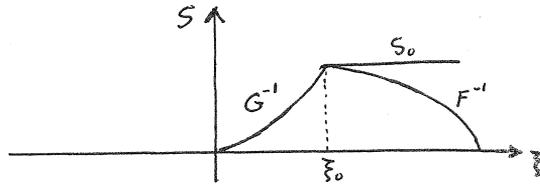


A distinguishing feature of the solution (12) is that it is more fragile than the solution (10) in the following sense. The solution (12) is differentiable

everywhere with a singular point  $\xi = 0$  i.e.  $S = 0$  same as (10) but has an additional singular point when  $\xi = \xi_0$  corresponding to  $S = S_0 =$  local maximum of  $S$ . As the drying proceeds the maximum value decreases i.e. the singular point is not isolated but varies continuously. As is well known singularities act as points for branching i.e. jumping from one solution to a different one. For example at  $\xi = \xi_0$  we have three possible solutions:

$$S = G^{-1}(\xi) \text{ from (12), } S = S_0, \quad S = F^{-1}(\xi) \text{ from (10)}$$

as shown in the following diagram



4. Combined Effect:

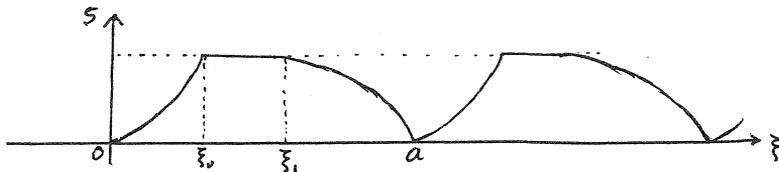
The constant  $c_2$  in equation (8) was determined by the condition  $S(0) = 0$ . Suppose there is another point  $a$  such that  $S(a) = 0$ . We can imagine the region  $\xi < a$  as experiencing infiltration and  $\xi > a$  as experiencing drying. A solution consistent with (11) and (13) is:

$$S = \begin{cases} \sqrt{\frac{\lambda_2}{\beta}} \tan\left\{\frac{\sqrt{\lambda_2\beta}}{5\alpha}\xi\right\}, & 0 \leq \xi \leq \xi_0, \\ S_0, & \xi_0 \leq \xi \leq \xi_1, \\ \sqrt{\frac{\lambda_1}{\beta}} \tanh\left\{\frac{\sqrt{\lambda_1\beta}}{5\alpha}(a-\xi)\right\}, & \xi_1 \leq \xi \leq a, \end{cases}$$

where

$$S_0 = \sqrt{\frac{\lambda_2}{\beta}} \tan\left\{\frac{\sqrt{\lambda_2\beta}}{5\alpha}\xi_0\right\}, \quad \xi_1 = a - \frac{5\alpha}{\sqrt{\lambda_1\beta}} \tanh^{-1}\left\{\sqrt{\frac{\beta}{\lambda_1}}S_0\right\}.$$

In particular  $\xi_1 = \xi_0$  determines  $a$ . This solution can be extended periodically by defining  $S(\xi + a) = S(\xi)$  which has the graph



Of course there are many other possibilities; for example, one can have the above solution plus a finite interval containing  $a$  on which  $S = 0$ .

Both  $\xi_0$  and  $\xi_1$  are singular points and therefore possible branching points. Sooner or later the combined effect of the upward and downward wave motion will break the moisture pocket  $0 < \xi < a$  into two disconnected parts. This bizzare behaviour may seem physically unacceptable, existing only in the mathematician's imagination until one realizes that in the drying stage known as the pendular state, the moisture breaks up into disconnected regions.

### 5. Conclusion:

The simple model formulated in this study fits well our physical intuition for an infiltration problem. If the wetting front is defined as that point where  $S(x-, t) > 0$  but  $S(x+, t) = 0$ , the speed of the front is given uniquely as  $\lambda = \beta S_0^2$ . Corresponding to this speed there is a unique initial profile which moves as a plane wave downwards. It was shown that such solutions do not exist for the drying problem.

For the drying problem it seems that the appropriate plane wave is upwards. For such problems the present simple model provides no relation to determine the wave speed  $\lambda$  uniquely. This is to be expected as the upward motion of moisture comes from the temperature and pressure gradients. (In actual situation drying is accomplished by heating the face  $x = 0$ .) One needs to consider the full drying problem involving a system of quasilinear parabolic partial differential equations for the moisture  $S$ , the temperature  $T$  and pressure  $P$ .

The main characteristic of these upward plane wave solutions is that they introduce a singular point where branching can occur continuously, allowing the combined upward and downward motions to break up the continuous moisture distribution into disconnected parts observed in the pendular state.

### References

1. Ilić, M., & Turner, I.W.: Convective drying of a consolidated slab of wet porous material, *Int. J. Heat Mass Transfer*, 32, pp. 2351-2362, 1989.
2. Tayler, A.B.: *Mathematical Models in Applied Mechanics*, Clarendon Press, Oxford, 1986.