Degenerate Holomorphic Mappings of Nondegenerate CR-manifolds[1]

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Abstract. We announce the following result: the Jacobian of a locally defined holomorphic mapping between two real-analytic CR-manifolds M_1 and M_2 in $\mathbb{C}^N$, where M_1 has a nondegenerate Levi form and satisfies a certain rigidity condition, is identically zero, if it is zero at one point on M_1. We also give a description of the image of the mapping in terms of the geometry of M_2.

In this note we consider real-analytic CR-manifolds in a complex space $\mathbb{C}^N$ with nondegenerate Levi forms. Let us start with definitions.

Definition 1. Let M be a real submanifold of $\mathbb{C}^N$, $p \in M$, and $T_p(M)$ – the tangent space to M at p. *The complex tangent space $T_p^c(M)$ to M at the point p* is the maximal complex subspace of $T_p(M)$, i.e.

$$T_p^c(M) = T_p(M) \cap iT_p(M).$$

Definition 2. A real submanifold M in $\mathbb{C}^N$ is called *a CR-manifold* if $\dim_{\mathbb{C}} T_p^c(M)$ is constant on M. The dimension $\dim_{\mathbb{C}} T_p^c(M)$ is *the CR-dimension of M* and is denoted by $CR \dim M$.

Definition 3. Let M be a CR-manifold in $\mathbb{C}^N$, and $r(z_1, \overline{z_1}, \ldots, z_N, \overline{z_N})$ – its defining function, i.e.

$$M = \{r = 0\}, \quad \operatorname{grad} r \neq 0 \quad \text{on} \quad M.$$

The Levi form $\mathcal{L}_M(p)$ of M at $p \in M$ is the restriction of the Hermitian form

$$\sum_{j,l=1}^{N} \frac{\partial^2 r}{\partial z_j \, \partial \overline{z_l}}(p) dz_j \, d\overline{z_l}$$

to the complex tangent space $T_p^c(M)$.

Suppose now that M is real-analytic and passes through the origin. Choose local holomorphic coordinates $(z_1, \ldots, z_n, w_1 = u_1 + iv_1, \ldots, w_k = u_k + iv_k)$ near the origin such that M is given by the equations

$$v = F(z, \overline{z}, u). \tag{1}$$

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Here $n = CR\dim M$, $k = \operatorname{codim}_{\mathbb{R}} M$, $n+k=N$, $z=(z_1,\dots,z_n)$, $u=(u_1,\dots,u_k)$, $v=(v_1,\dots,v_k)$, $F(z,\overline{z},u)$ is a real-analytic vector-function defined in a neighbourhood of the origin. Clearly, we can assume that $\operatorname{grad} F(0)=0$, and hence $T_0^c(M)=\{w=0\}$, $w=u+iv$. Therefore the Levi form of M at the origin is given by the vector with matrix components

$$\mathcal{L}_M(0)=\left(\frac{\partial^2 F}{\partial z_j\,\partial \overline{z_l}}(0)\right).$$

We will consider manifolds with nondegenerate Levi forms. The nondegeneracy of a vector-valued Hermitian form is given by the following definition.

Definition 4. Let $H(z,z')=(<z,z'>^1,\dots,<z,z'>^k)$, $z,z'\in\mathbb{C}^n$ be a vector-valued Hermitian form in $\mathbb{C}^n$. Then H is said to be *nondegenerate* if the following two conditions are satisfied.

(*) The Hermitian forms $<z,z'>^1,\dots,<z,z'>^k$ are linearly independent over $\mathbb{R}$.

(**) If for some $z'\in\mathbb{C}^n$ $H(z,z')=0$ for all $z\in\mathbb{C}^n$ then $z'=0$.

For $k=1$ (*) trivially follows from (**), and the definition coincides with the usual definition of nondegeneracy. For $k>1$ generally speaking neither of the above conditions implies the other. It should be also noted that the definition does not imply the existence of a nondegenerate linear combination of the components of H.

Example 1. Consider the Hermitian form in $\mathbb{C}^3$ $H(z,z')=(z_1\overline{z_2'}+z_2\overline{z_1'},z_1\overline{z_3'}+z_3\overline{z_1'})$. It is given by two 3×3-matrices

$$\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}.$$

The form H is nondegenerate, but obviously every linear combination of the above matrices is degenerate.

Suppose now that two manifolds M_1 and M_2 are given in the form (1), and in some neighbourhood U of the origin a holomorphic mapping f is defined. Let $f(M_1)\subset M_2$, $f(0)=0$. Assume further that the Jacobian $J_f(0)$ of f at the origin is zero. The problem we consider here is to describe the zero set of J_f in U. A conjecture due to Vitushkin (1985) says that for the case of hypersurfaces (i.e. $k=1$) with nondegenerate Levi forms $J_f\equiv 0$ in U. The conjecture turned out to be true and was proved in [I]. An interesting fact is that the Levi form of M_2 is allowed to be degenerate.

To formulate the results we note first that every real-analytic hypersurface M (with possibly degenerate $\mathcal{L}_M(0)$) after a suitable holomorphic change of coordinates near the origin can be written as

$$v=<z,z>+\sum_{j\ge 1,l\ge 1}F_{jl}(z,\overline{z},u), \tag{2}$$

where $< z, z >$ is a Hermitian form representing $\mathcal{L}_M(0)$, $F_{jl}(z, \overline{z}, u)$ is a polynomial of order j in z and l in $\overline{z}$ with coefficients depending on u, $\frac{\partial^2 F_{11}}{\partial z_j \, \partial \overline{z}_l}(0) = 0$, $j, l = 1, \ldots, n$. Further, if $\mathcal{L}_M(0)$ is nondegenerate then the equation (2) can be reduced to

$$v = < z, z > + \sum_{j \geq 2, l \geq 2} F_{jl}(z, \overline{z}, u) \tag{3}$$

(see [**CM**] for details).

Suppose now that two hypersurfaces M_1 and M_2 with M_1 having a nondegenerate Levi form are given in the forms (3) and (2) respectively. Then as it is shown in [**I**] the following is true:

(i) $J_f \equiv 0$ in U, and moreover

(ii) $f(U) \subset M_2 \cap \{w = 0\}$.

In particular, if f is written as

$$f: \quad z \mapsto g(z, w), \quad w \mapsto h(z, w),$$

then $h \equiv 0$.

If $\mathcal{L}_{M_2}(0)$ is nondegenerate, the second statement immediately gives the following estimate: $\dim_{\mathbf{C}} f(U) \leq \chi$, where χ is the signature of $\mathcal{L}_{M_2}(0)$ (the minimum of the numbers of positive and negative eigenvalues). In particular, if M_2 is strictly pseudoconvex near the origin ($\chi = 0$), then $f \equiv 0$. This last fact is known since 1975 [**P**] and also follows from [**V**].

The proof of the above result is very technical and involves detailed analysis of the power series defining f and M_1, M_2. In particular, it heavily relies on the representation (3) for M_1. However, there is a short geometric proof of (i) due to Kruzhilin (unpublished) valid even for smooth hypersurfaces based on the technique of chains (special curves introduced in [**CM**]). Namely, it follows from [**K**] that a chain decreasing its angle with the complex tangent space near a point can not have finite length.

There is a number of results of Baouendi, Bell and Rothschild for more general hypersurfaces and mappings (see [**BBR**], [**BR1**], [**BR2**], [**BR3**], [**BR4**]). In particular, if M_1, M_2 are given in the form (2) and M_1 is essentially finite, then for any holomorphic mapping f either (i) and (ii) above are true, or

$$\frac{\partial h}{\partial w}(0) \neq 0, \tag{4}$$

and f is of finite multiplicity. For M_1 having a nondegenerate Levi form (4) implies that $J_f(0) \neq 0$. If M_2 is only smooth, then in (ii) $f(U)$ does not necessarily lie on

M_2. In general, $f(U)$ has only infinite order of contact with M_2. Also, the condition $h \equiv 0$ must be understood in terms of formal power series.

Note that if the Levi form of M_1 degenerates, (i) may not be true, even if the Levi form of M_2 is nondegenerate.

Example 2. Let M_1 and M_2 be hypersurfaces in $\mathbb{C}^2$ given by the equations

$$M_1: \quad v = |z|^4,$$
$$M_2: \quad v = |z|^2.$$

Then the mapping

$$z \mapsto z^2, \quad w \mapsto w$$

obviously maps M_1 to M_2 and has vanishing Jacobian only for $z = 0$.

In the present paper we consider manifolds of codimension $k > 1$. The first difficulty that we encounter trying to generalize the result for hypersurfaces to the case of higher codimensions is that not every manifold with even nondegenerate Levi form for $k > 1$ can be given by an equation analogous to (3) (see [**B1**], [**B2**], [**B3**], [**L**]). Generally speaking, the second order term may depend on u and terms of type F_{j1}, F_{1l} may occur, i.e. we only have the representation (2) instead. At the moment we can not resolve this problem, and the possibility to write M_1 in the form (3) is our extra requirement.

THEOREM. *Let M_1 and M_2 be two manifolds of codimension $k \geq 1$ in $\mathbb{C}^N$. Suppose that M_2 is given in the form (2), M_1 is given in the form (3) and the Levi form of M_1 is nondegenerate.*

Let f be a holomorphic mapping defined in a neighborhood U of the origin such that $f(M_1) \subset M_2$, $f(0) = 0$ and $J_f(0) = 0$,

$$f: \quad z \mapsto g(z, w), \quad w \mapsto h(z, w).$$

Then

(i) $J_f \equiv 0$ in U, and moreover

(ii)$'$ there exist an integer $m \geq 1$ and a linear change of coordinates of the form

$$z \mapsto z, \quad w \mapsto Sw,$$

where S is a real $k \times k$-matrix, such that after applying it to M_2 one can write M_2 as the intersection $M_2' \cap M_2''$, where M_2' is given by the first m equations defining M_2, M_2'' by the last $k - m$ equations, and

$$f(U) \subset M_2' \cap \{w' = 0\},$$

with $w' = (w_1, \dots, w_m)$. *Here* $m = k - \text{rank}\, \frac{\partial h}{\partial w}(0)$.

The statement (ii)$'$ of the Theorem says that we can always rewrite the equations of M_2 as certain combinations of the original ones and split them into two groups such that the manifold of codimension $m \le k$ defined by the first group being intersected with its complex tangent space at the origin contains $f(U)$. For the case of hypersurfaces this coincides with the statement (ii) above.

Note that m may be strictly less than k, and therefore (ii) can not be generalized directly to the case of higher codimensions. An obvious obstruction for that is possible reducibility of manifolds in consideration. Indeed, take $M_1 = M_2 = M = M^1 \times M^2$, where $M^j \subset \mathbb{C}^{n_j+k_j}$, $CR\,\dim M^j = n_j$, $\text{codim}_{\mathbb{R}} M^j = k_j$, $n_1 + n_2 = n$, $k_1 + k_2 = k$, and define a mapping f as $f = f^1 \times f^2$, with $f^1 : M^1 \to M^1$, $f^1 = id$, $f^2 : M^2 \to M^2$, $f^2 \equiv 0$.

However, even for irreducible manifolds (those which can not be represented as a direct product in any holomorphic coordinates near the origin) there are examples of holomorphic self-mappings with $m < k$. The following example is due to Ezhov.

Example 3. Let $M_1 = M_2 = M \subset \mathbb{C}^5$ be given by

$$v_1 = |z_1|^2 - |z_2|^2, \qquad (5)$$
$$v_2 = |z_1|^2 - |z_3|^2,$$

and f be the following linear mapping

$$z_1 \mapsto z_1, \quad z_2 \mapsto z_1, \quad z_3 \mapsto z_3,$$
$$w_1 \mapsto 0, \quad w_2 \mapsto w_2.$$

Here $k = 2$ and $m = 1$. It also can be shown that M is irreducible. Indeed, as it is proved in [ES], for manifolds of the form

$$v = < z, z >$$

(often called quadrics) the irreducibility is equivalent to the irreducibility of the algebra $\mathfrak{A}$ consisting of pairs (D, d) of complex $n \times n$- and $k \times k$-matrices respectively such that $< Dz, z' > = d < z, z' >$ for all $z, z' \in \mathbb{C}^n$.

It is an easy computation to show that for the quadric (5) the corresponding algebra $\mathfrak{A}$ consists of pairs of the form

$$D = \begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{pmatrix}, \quad d = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}, \quad t \in \mathbb{C},$$

and being one-dimensional does not split.

The mapping f in the above example is linear. It turns out that this is a typical situation. Indeed, as we see from (ii)$'$, the only obstructions for generalizing (ii) to higher codimensions are linear mappings. Namely, it is easy to note that the linear mapping

$$\tilde{f}: \quad z \mapsto Az, \quad w \mapsto Dw$$

with $A = \frac{\partial g}{\partial z}(0)$, $D = \frac{\partial h}{\partial w}(0)$, is a mapping between the quadrics

$$\tilde{M}_1: \quad v = < z, z >_1,$$
$$\tilde{M}_2: \quad v = < z, z >_2,$$

where $< z, z >_j$ is the Levi form of M_j at 0, and it follows from (ii)$'$ that if the image of $\tilde{f}$ is in $\tilde{M}_2 \cap \{w = 0\}$ (i.e. if $m = k$, or equivalently $D = 0$), then the image of f is in $M_2 \cap \{w = 0\}$.

The proof of the Theorem is a generalization of the proof in [I] to higher codimensions and is also based on analysis of power series. It is very likely that the Theorem is true under much weaker assumptions for the manifold M_1. For example, it would be interesting to find a proof for essentially finite M_1 dropping all the other conditions for its power series.

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