Representations of Derivatives of Functions in Sobolev Spaces in Terms of Finite Differences

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1. Introduction

In numerical analysis, one can approximate $f'(x)$, the derivative of a given function f at x, by a single difference $\frac{f(x)-f(x-h)}{h}$. However, such an approximation is not generally identical to $f'(x)$ if f is not a linear function. On the other hand, if we approximate $f'(x)$ by a sum of the two finite differences $[\frac{f(x+h)-f(x)}{2h}+\frac{f(x)-f(x-h)}{2h}]$, this approximation is identical to $f'(x)$ for each quadratic function f. This raises the questions: under what conditions can the derivative of a given function be expressed as a sum of finite differences and how many terms in such a sum of finite differences is sufficient?

In [2] p.187, it was shown that if f is a twice continuously differentiable function on R, there are constants a_1, a_2 and continuous functions f_1, f_2 such that

$$f'(x)=\sum_{j=1}^{2}[f_j(x)-f_j(x-a_j)], \text{ for all } x \text{ in R.}$$

In this paper, we apply some results of difference subspaces of $L^2(R)$ in [4] to show that for each positive integer m, if f is a function in the Sobolev space $H^{m+2}(R)$, the subspace of functions in $L^2(R)$ whose distributional derivatives up to order m are also in $L^2(R)$, and if f' is the distributional derivative of f, then there are constants a_1, a_2 and functions f_1, f_2 in $H^m(R)$ such that

$$f'=\sum_{j=1}^{2}[f_j-\delta_{a_j}*f_j] \text{ a.e.} \tag{1}$$

where $*$ is the usual convolution and δ_a is the Dirac measure at a in R, so that

$$(\delta_a * f)(x) = f(x-a) \quad \text{a.e. for } x \text{ in R.}$$

This result is closely related to the results of Meisters in [2], but by the present approach we can show that such a representation of a derivative by a sum of finite differences as in (1) is best possible in that it uses the minimum number of finite differences of functions.

In addition, we show that in order to represent the distributional derivative of a given function in $H^m(R)$ as a sum of finite differences of functions in $H^{m-1}(R)$, in general 3 terms is the best possible.

In the last section, we show that if Δ is the Laplace operator on the Sobolev space $H^m(R^n)$, then for each f in $H^m(R^n)$, Δf can be expressed as a sum of 5 second order differences of functions in $H^{m-2}(R^n)$ for $n \le 5$.

2. Difference Subspaces of $L^2(R^n)$

Throughout this paper, $\hat{f}$ will denote Fourier Transform of $f \in L^2(R^n)$ such that if f is in the subspace $L^1(R^n) \cap L^2(R^n)$, $\hat{f}(y)$ is given by $\int_{R^n} e^{-i<x,y>} f(x)dx$, where $< , >$ is the usual inner product in R^n. It is a standard result that for each positive integer m, the Sobolev space $H^m(R^n) = \{ f \in L^2(R^n) : \hat{f}(x)(1+|x|^2)^{\frac{m}{2}} \in L^2(R^n) \}$. ([1])

Definition

Let s be a given positive integer and V be a subspace of $L^2(R^n)$. Then the s-th order difference subspace of V, $D_s(V)$, is the space spanned by $\{ (\delta_0 - \delta_a)^s * f : f \in V, a \in R^n \}$. Hence, for a given function f in V, f is also in the subspace $D_s(V)$ if and only if there are vectors $a_1, a_2, \ldots, a_m$ in R^n and functions $f_1, f_2, \ldots, f_m$ in V such that

$$f = \sum_{j=1}^{m} (\delta_0 - \delta_{a_j})^s * f_j \quad \text{a.e.}$$

It can be shown that for $s = 1$, $D_1(V)$ is spanned by $\{ f - \delta_a * f : f \in V, a \in R^n \}$ while for the case $s = 2$, $D_2(V)$ is the subspace spanned by $\{ f - \frac{(\delta_a + \delta_{-a})}{2} * f : f \in V, a \in R^n \}$.

The following are the fundamental results of the difference subspace of $L^2(R^n)$:

Theorem 1 ([4])

Let s be a given positive integer. Then for each function f in $L^2(R^n)$, f is also in $D_s(L^2(R^n))$ if and only if $\int_{R^n} \frac{|\hat{f}(x)|^2}{|x|^{2s}} dx < \infty$.

Theorem 2 ([4])

Let f be a given function in $L^2(R^n)$ and $a_1, a_2, \ldots, a_m$ are given vectors in R^n. Then there are functions $f_1, f_2, \ldots, f_m$ in $L^2(R^n)$ such that $f = \sum_{j=1}^{m} (\delta_0 - \delta_{a_j})^s * f_j$ a.e. *if and only if*

$$\int_{R^n} \frac{|\hat{f}(x)|^2}{\sum_{j=1}^{m} \sin^{2s} \frac{<a_j, x>}{2}} dx < \infty.$$

3. Non-Liouville numbers and finite differences

By Theorem 2, a necessary condition for (1) to hold is to find some constants a_1, a_2 such that the integral in Theorem 2 is finite. In order to achieve such a goal, we need some results of number theory:

Let α be a given irrational number. Then α is a *non-Liouville number* if there are constants $C > 0$ and $k \geq 1$ such that

$$d_z(m\alpha) > C\,|m|^{-k} \text{ for all non-zero integers } m, \qquad (2)$$

where $d_z(x)$ is the distance of $x \in R$ to the nearest integer. It is known if α is an algebraic irrational number of degree j, then (2) holds with $k = j - 1$ (see [3] Theorem 2A). It can also be shown that the compliment of non-Liouville numbers has zero measure in R.

Proposition 3

Let α be a given non-Liouville number and C, k be the corresponding constants in inequality (2). Let $\{\delta_m\}$ be the decreasing sequence of positive numbers defined by

$$\delta_m = \frac{Cm^{-k}}{2(1+|\alpha|)}, \quad m \in \mathbf{N}.$$

Then for each positive integer m and non-zero number x,

$$|x - m| < \delta_m \text{ and } d_z(\alpha x) < \delta_m$$

cannot hold simultaneously.

Proof

We prove the result by contradiction. Assume the two inequalities hold simultaneously for some positive integer m and non-zero x. Then there is an integer p such that

$$|x - m| < \delta_m \text{ and } |\alpha x - p| < \delta_m.$$

By the above two inequalities, we have

$$|\alpha m - p| < (1 + |\alpha|)\delta_m = \frac{Cm^{-k}}{2}.$$

Since, $d_z(\alpha m) \le |\alpha m - p|$, we also have

$$d_z(\alpha m) < \frac{Cm^{-k}}{2}$$

which contradicts that α is a non-Liouville number. Hence, the result follows.

In the following proposition, we show that if α is a non-Liouville number, then the function $1 / (\sin^2 \pi x + \sin^2 \pi\alpha x)$ is bounded by some polynomials if x is sufficiently large.

Proposition 4

Let α be a non-Liouville number and C, k be the positive number such that inequality (2) holds. Let $\{\delta_m\}$ be the decreasing sequence of positive numbers as constructed in Proposition 3. Let m_α be the least positive integer such that

$$\text{if } n \geq m_\alpha, \text{ then } \delta_n < 1/2 \text{ and } n^{2k} \geq 4C^2.$$

Let f_α be the function on $[m_\alpha, \infty)$ defined by $f_\alpha(x) = 1/(\sin^2 \pi x + \sin^2 \pi\alpha x)$ for all $x > m_\alpha$. Then there is $\lambda > 0$ such that $f_\alpha(x) < \lambda x^{2k}$.

Proof

As α is irrational, f_α is continuous on $[m_\alpha, \infty)$. Thus, by the inequalities

$$4d_z(y) \leq |1 - e^{-2\pi i y}| = 2|\sin \pi y| \leq 2\pi d_z(y) \text{ for all real } y \in R \text{ ([2] p.178)}$$

and the continuity of f_α on $[m_\alpha, \infty)$, for each $n \geq m_\alpha$, there is $0 < \varepsilon_n < \delta_n$ such that for all $x \in [n, n+\varepsilon_n)$,

$$\begin{aligned} f_\alpha(x) &\leq f_\alpha(n) + \frac{1}{2^n} \\ &= \frac{1}{\sin^2 \pi\alpha n} + \frac{1}{2^n} \\ &\leq \frac{1}{4[d_z(\alpha n)]^2} + 1 \\ &\frac{n^{2k}}{4C^2} + 1 \quad \text{by (2)} \\ &\leq \frac{n^{2k}}{2C^2}, \quad \text{by definition of } m_\alpha. \end{aligned} \tag{3}$$

Suppose $x \in [n + \varepsilon_n, n + 1/2]$. If $|x - n| \geq \delta_n$, then $d_z(x) = |x - n| \geq \delta_n$. Meanwhile, if $|x - n| < \delta_n$, then $d_z(\alpha x) \geq \delta_n$ by Proposition 3. In either case, we have

$$
\begin{aligned}
f_\alpha(x) &= \frac{1}{\sin^2 \pi x + \sin^2 \pi\alpha x} \\
&\le \frac{1}{4[d_z(x)]^2 + 4[d_z(\alpha x)]^2} \\
&\le \frac{1}{4\delta_n^2} \\
&= \frac{n^{2k}}{C^2}(1+|\alpha|)^2, \text{ by definition of } \delta_n. \qquad (4)
\end{aligned}
$$

By inequalities (3) and (4), there is $\lambda_1 > 0$ so that for each $n \ge m_\alpha$, if $x \in [\, n, n + 1/2\,]$,

$$
\begin{aligned}
f_\alpha(x) &\le \lambda_1 n^{2k} \\
&\le \lambda_1 (2x)^{2k} \quad \text{because } n < x < 2x. \qquad (5)
\end{aligned}
$$

Similarly, there is $\lambda_2 > 0$ such that for each $n \ge m_\alpha$, if $x \in [\, n + 1/2, n + 1\,]$,

$$f_\alpha(x) \le \lambda_2 (2x)^{2k}. \qquad (6)$$

Hence, the result follows by inequalities (5) and (6).

In the following proposition, we apply the result of Proposition 4 to construct a sufficient condition for expressing a function in $D_1(L^2(R))$ as a sum of two finite differences.

Theorem 5

Let α be a given non-Liouville number such that inequality (2) holds with some positive integer k. Let m be a given integer ≥ 0. Let a_1, a_2 be given non-zero numbers such that $a_2/a_1 = \alpha$. Suppose f is a function in $D_1(L^2(R))$ which is also in the Sobolev space $H^{m+k}(R)$. Then there are function f_1, f_2 in $H^m(R)$ such that

$$f = \sum_{j=1}^{2} (f_j - \delta_{a_j} * f_j). \qquad (7)$$

Proof

We first prove the case $a_1 = 2\pi$. As $\sin x \approx x$ if $|x|$ is sufficiently small, by Theorem 1, there is $\varepsilon > 0$ such that $\int_{|x|<\varepsilon} \frac{|\hat{f}(x)|^2}{\sin^2 \pi x + \sin^2 \pi\alpha x} dx < \infty$. Let m_α and λ be the constants in Proposition 4. Let A be the compact set $[-m\alpha, -\varepsilon] \cup [\varepsilon, m\alpha]$. As $1/(\sin^2\pi x + \sin^2\pi\alpha x)$ is continuous on A, $1/(\sin^2\pi x + \sin^2\pi\alpha x)$ is bounded above on A. Hence, we have

$$\int_{|x|\le m_\alpha} \frac{|\hat{f}(x)|^2}{\sin^2 \pi x + \sin^2 \pi\alpha x} dx < \infty. \tag{8}$$

By the properties of f and Proposition 4,

$$\int_{|x|>m_\alpha} \frac{|\hat{f}(x)|^2}{\sin^2 \pi x + \sin^2 \pi\alpha x} dx \le \lambda \int_{|x|>m_\alpha} |\hat{f}(x)|^2 |x|^{2k} dx$$
$$< \infty. \tag{9}$$

Therefore, by inequalities (8) and (9), $\int_R \frac{|\hat{f}(x)|^2}{\sin^2 \pi x + \sin^2 \pi\alpha x} dx < \infty$. Similarly, we can also show that

$$\int_R \frac{|\hat{f}(x)|^2 (1+|x|^2)^m}{\sin^2 \pi x + \sin^2 \pi\alpha x} dx < \infty. \tag{10}$$

Now, we show that there are functions f_1, f_2 in the Sobolev space $H^m(R)$ such that (7) holds. For each $j = 1, 2$, let g_j be the function on R defined by

$$g_j(x) = \frac{\hat{f}(x)|1-e^{-ia_jx}|}{(1-e^{-ia_jx})\sum_{j=1}^{2}|1-e^{-ia_jx}|} \quad \text{a.e.}$$

By [4] Lemma 1, there is $c_1 > 0$ such that for all x in R,

$$4\sum_{j=1}^{2}\sin^2\left(\frac{a_j x}{2}\right)=\sum_{j=1}^{2}\left|1-e^{-ia_j x}\right|^2$$
$$\leq c_1\left(\sum_{j=1}^{2}\left|1-e^{-ia_j x}\right|\right)^2.$$

Thus, it can be shown that g_1, g_2 are functions in $L^2(R)$. By the Plancherel Theorem, there are functions f_1, f_2 in $L^2(R)$ such that $\hat{f}_j = g_j$ for $j = 1, 2$. Moreover, by the constructions of f_1 and f_2, for $j = 1, 2$,

$$\int_R \left|\hat{f}_j(x)\right|^2(1+|x|^2)^m dx = \int_R \frac{\left|\hat{f}(x)\right|^2(1+|x|^2)^m}{\left(\sum_{j=1}^{2}\left|1-e^{-ia_j x}\right|^2\right)}dx$$
$$\leq \frac{c_1}{4}\int_R \frac{\left|\hat{f}(x)\right|^2(1+|x|^2)^m}{\sin^2 \pi x + \sin^2 \pi\alpha x}dx$$
$$< \infty, \text{ by (10)}.$$

Therefore,, f_1, f_2 are functions in $H^m(R)$. By the definitions of f_1, f_2 and the Fourier Inversion Theorem, $f = \sum_{j=1}^{2}(f_j - \delta_{a_j} * f_j)$. Hence, the result follows for the case $a_1 = 2\pi$.

For the general case, choose F in $L^2(R)$ such that $\hat{F}(x) = \hat{f}(2\pi x/|a_1|)$ a.e. Then $\int_R \frac{\left|\hat{F}(x)\right|^2}{\sin^2 \pi x + \sin^2 \pi\alpha x}dx < \infty$. Use the substitution $y = 2\pi x/|a_1|$, then it can be shown the result also holds for this case.

As there is non-Liouville number α such that inequality (2) holds with $k = 1$. By Theorem 5, we then have the following result.

Theorem 6

Let m be a given integer ≥ 0. For each function f in the Sobolev space $H^{m+2}(R)$, if f' is the distributional derivative of f, then there are constants a_1, a_2 in R and functions f_1, f_2 in the Sobolev space $H^m(R)$ such that (1) holds. That is,

$$f' = \sum_{j=1}^{2}[f_j - \delta_{a_j} * f_j] \quad a.e.$$

Proof

By the properties of Fourier Transform of functions in $L^2(R)$, it can be shown that f' is a function in $D_1(L^2(R))$ which is also in the Sobolev space $H^{m+1}(R)$. Choose a non-Liouville number α such that inequality (2) holds with $k = 1$. By Theorem 5, there are constants a_1, a_2 in R and functions f_1, f_2 in $H^m(R)$ such that (1) holds.

The following counter example shows that Theorem 6 is best possible in the sense of minimum numbers of finite differences of functions.

Theorem 7

There is a function f in $H^m(R)$ for all positive integer m but there is no b in R and g in $L^2(R)$ such that $f' = g - \delta_b * g$.

Proof

Let f be the function in $L^2(R)$ such that $\hat{f}(x) = xe^{-x^2}$. Then f is in $H^m(R)$ for all positive integer m and f' is $D_1(L^2(R))$ and $|(f')^{\wedge}(x)| = x^2e^{-x^2}$. Let $0 < \delta < 1/2$ and $b \neq 0$ be given. Let $\{m_k\}$ be the sequence of positive numbers defined by

$$m_k = \min|x^2e^{-x^2}| \quad x \in [\, 2\pi k / b - \delta, 2\pi k / b + \delta \,].$$

$$\text{Then } \int_R \frac{x^2 e^{-x^2}}{\left|1-e^{-ibx}\right|^2} dx \geq \sum_{k=1}^{\infty} m_k \int_{2\pi k/b-\delta}^{2\pi k/b+\delta} \frac{dx}{\left|1-e^{-ibx}\right|^2}$$

$$= \sum_{k=1}^{\infty} m_k \int_{-\delta}^{\delta} \frac{dy}{\left|1-e^{-iby}\right|^2}$$

$$= \infty.$$

Hence, by Theorem 2, f' cannot be expressed as a single finite difference of function in $L^2(R)$.

4. Difference Subspaces of Sobolev Spaces

Theorem 6 shows that if m is a positive integer ≥ 2 and f is a function in the Sobolev space $H^m(R)$, then f', the distributional derivative of f, can be always expressed as a sum of two finite differences of functions in $H^{m-2}(R)$. Consequently, f' can be expressed as a sum of two finite differences of functions in $L^2(R)$ for $m \geq 2$. However, it is known that there is a function φ in $D_1(L^2(R))$ such that φ cannot be written as $\varphi = \sum_{j=1}^{2}(\varphi_j - \delta_{a_j} * \varphi_j)$, for any φ_1, φ_2 in $L^2(R)$and a_1 , a_2 in R (see Corollary 6.12 in [5]). If we choose f in the Sobolev space $H^1(R)$ such that $f' = \varphi$, then f' cannot be expressed as a sum of two finite differences of functions in $L^2(R)$. In order to represent the distributional derivative of a given function in $H^1(R)$ as a sum of finite differences of functions in $L^2(R)$, we need some results on the differences subspaces of Sobolev spaces.

Theorem 8

Let s and m be given positive integers. Then $D_s(H^m(R^n)) = D_s(L^2(R^n)) \cap H^m(R^n)$.

Proof

By definition, $D_s(H^m(R^n))$ is contained in $D_s(L^2(R^n)) \cap H^m(R^n)$. Therefore, it is sufficient to show $D_s(L^2(R^n)) \cap H^m(R^n)$ is contained in $D_s(H^m(R^n))$. Suppose f is a function in the subspace $D_s(L^2(R^n)) \cap H^m(R^n)$. By Theorem 1, $\int_{R^n} \frac{|\hat{f}(x)|^2}{|x|^{2s}} dx < \infty$. Let $k = \max\{2s+1, n\}$. Then adapting the proof of Theorem 3 in [4], it can be shown that there are vectors a_1, a_2,...a_k in R^n such that $\int_{R^n} \frac{|\hat{f}(x)|^2 (1+|x|^2)^m}{\sum_{j=1}^{k} \sin^{2s} \frac{<a_j, x>}{2}} dx < \infty$. For each $j \leq k$, let g_j be the function defined by $g_j(x) = \frac{\hat{f}(x)(|1-e^{-i<a_j,x>}|^s)}{(1-e^{-i<a_j,x>})^s \sum_{j=1}^{k} |1-e^{-i<a_j,x>}|^s}$ a.e., then g_j is a function in $L^2(R^n)$. By the Plancherel Theorem, for each $j \leq k$, there is f_j in $L^2(R^n)$ such that $\hat{f}_j = g_j$. Apply an argument similar to the proof of Theorem 6, it can be shown that for each $j \leq k$, f_j is in $H^m(R^n)$. By the Fourier Inversion Theorem, $f = \sum_{j=1}^{k} (\delta_0 - \delta_{a_j})^s * f_j$ a.e. Thus, f is in $D_s(H^m(R^n))$. So, $D_s(L^2(R^n)) \cap H^m(R^n)$ is contained in $D_s(H^m(R^n))$. Hence, the result follows.

By the properties of Fourier Transform and Theorem 8, we can have the followings.

Theorem 9

Let $s > 0$ and $m \geq 0$ be given integers. Then $D_s(H^m(R^n))$ is a Hilbert space in the norm $\|\cdot\|_{m,s}$ given by $\|f\|_{m,s} = (\int_{R^n} |\hat{f}(x)|^2 (1+|x|^2)^m (1+|x|^{-2})^s dx)^{1/2}$.

Theorem 10

Let m be a given positive integer. Suppose f is a function in the Sobolev space $H^m(R)$ and f' is the distributional derivative of f. Then there are constants a_1, a_2, a_3 in R and functions f_1, f_2, f_3 in $H^{m-1}(R)$ such that $f' = \sum_{j=1}^{3} (\delta_0 - \delta_{a_j}) * f_j$ a.e.

Theorem 11

Let $m \geq 2$ and $n \leq 5$ be given positive integers. Let Δ be the Laplace operator on $H^m(R^n)$. Suppose f is a function in the Sobolev space $H^m(R^n)$. Then there are vectors a_1, $a_2,\ldots,a_5$ in R^n and functions f_1, $f_2,\ldots$, f_5 in $H^{m-2}(R^n)$ such that

$$\Delta f = \sum_{j=1}^{5} f_j - \frac{(\delta_{-a_j} + \delta_{a_j})}{2} * f_j \quad \text{a.e.}$$

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