EXISTENCE OF A POTENTIAL BY KURZWEIL–HENSTOCK INTEGRATION.

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ABSTRACT. Applications of ideas from Kurzweil-Henstock integration to path independence of line integrals is discussed.

Introduction. This paper is an extract from [V3], consequently detailed statements and proofs are mostly omitted.

One of the main features of the Perron integral is that it integrates every derivative without any restriction. It seems that this property was not fully exploited because of the nonelementary character of Perron's definition. In 1957 Kurzweil [K], in connection with research in differential equations, gave an elementary definition equivalent to the Perron one, moreover the proof of the fundamental theorem became then extremely simple. For Kurzweil's own presentation of the theory see [K1]. Henstock later [He] independently rediscovered Kurzweil's approach and advanced it further [He 1–4]. In this talk we list some applications of K-H integration in analysis and prove the existence of a potential under assumptions substantially weaker than the classical ones

Notation, basic facts. A partition of a compact interval $[a,b]$ is a set of couples (ξ_k, I_k) such that the points $\xi_k \in [a,b]$, the intervals I_k are non-overlapping and

$$\bigcup_1^n I_k = [a,b]. \tag{1}$$

We shall call the point ξ_k the tag of I_k. A partition with the additional property that $\xi_k \in I_k$ is a P-partition. We shall be dealing only with P-partitions and shall omit the qualifying letter P. Often it will be convenient to have the intervals, $I_k = [u_k, v_k]$, ordered hence for a partition $\Pi \equiv \{\xi_k, [u_k, v_k]\}$ we have

$$\Pi \equiv a = u_1 \le \xi_1 \le v_1 = u_2 \le \xi_2 \le v_2 \le \cdots \le \xi_n \le v_n = b. \tag{2}$$

If $\delta : [a,b] \mapsto (0,\infty)$ then a partition Π for which

$$\xi_i - \delta(\xi_i) < u_i \le \xi_i \le v_i < \xi_i + \delta(\xi_i)$$

Key words and phrases. Perron integral, KH-integral, Fundamental Theorem, path independence, existence of a potential..

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for all i with $1 \leq i \leq n$ is called a δ-fine partition of $[a, b]$. The set of all δ-fine partition will be denoted by $\mathcal{P}(\delta)$. Hence, instead of saying that Π is δ-fine, we write $\Pi \in \mathcal{P}(\delta)$. For any positive function δ and any compact interval $[a, b]$ a δ-fine partition of $[a, b]$ always exists. The existence and the use of δ-fine partitions has been traced by Mawhin [Ma] to Cousin in the last century. We shall refer to the statement guaranteeing the existence of a δ-fine partition for any positive function δ as to Cousin's lemma. The contrast between the early discovery of Cousin's lemma and rather late arrival of Kurzweil-Henstock integral is a bit surprising. As Hadamard once said: "In mathematics simple ideas come late."

Real line applications. It has been shown, for instance in [V1] and [B], that δ-fine partitions have applications in analysis of $\mathbb{R}$.

Cousin's lemma has been shown to be equivalent to the least upper bound axiom [V1] and has been used in the proofs of the following theorems:

(1) the Bolzano–Cauchy convergence principle [V3],
(2) the Bolzano–Weierstrass theorem,[1]
(3) intermediate value theorem [V1],
(4) uniform continuity,[2]
(5) uniform approximation of continuous functions by continuous piecewise linear functions,
(6) Weierstrass' theorems on boudedness and extreme values of continuous functions [V1],
(7) Mean Value Theorems not only on $\mathbb{R}$ but also for vector valued functions [V3].
(8) Besides all this there is a host of theorems asserting something about the increament of a function from some information concerning the derivative. The simplest example is:
$f' > 0 \Rightarrow f$ increasing,
a more sophisticated example is:
f absolutely continuous, g increasing and $|f'| \leq g'$ a.e. implying

$$|f(b) - f(a)| \leq g(b) - g(a).$$

See [Bo], [V1], [V3].

The Kurzweil-Henstock definition. A number I is the integral of f from a to b if for everey positive ε there is a positive function δ such that for every partition $\Pi \in \mathcal{P}(\delta)$ (given by (2))

$$\left| \sum_{1}^{n} f(\xi_i)(v_i - u_i) - I \right| < \varepsilon.$$

The Fundamental theorem. In Kurzweil-Henstock theory the formula

$$\int_a^b F' = F(b) - F(a) \qquad (3)$$

[1] the proof is similar to the proof of 1) in [V3]
[2] similar to the proof by Borel's covering theorem

holds for a continuous F if the derivative F' exists except possibly a countable st M. There are other conditions under which the equation above holds but (3) suffices for our purposes.

The Fundamental Theorem can be used to improve classical versions of the following theorems

(1) Differentiation of series [V3],
(2) L'Hospital rule [V3],
(3) The Taylor Theorem and Taylor-like theorems [T], [T1], [V3].

Existence of a potential. The letters G and S will stand for an open set and an open star-shaped set in $\mathbb{R}^n$, $n \geq 2$, respectively. We denote by $d(M)$, $A(M)$ and $p(M)$ the diameter, the two dimensional area and the perimeter of M, in that order. A generic point in $\mathbb{R}^n$ will be denoted by $(x^1, x^2, \ldots, x^n)$, hence for a mapping $F : G \mapsto \mathbb{R}^n$ we have $F = (F^1, F^2, \ldots F^n)$ with $F^i : G \mapsto \mathbb{R}^1$. Partial derivatives will be denoted by subscripts, consequently $\partial/\partial x_i F^j = F^j_{,i}$. The word *path* will be used for a continuous map of bounded variation from an interval in $\mathbb{R}$ into $\mathbb{R}^n$. We shall allow a slight abuse of notation and use the same symbol for a closed path and its geometrical image (correspondingly oriented). By a line integral $\int_\varphi F(x)dx$ or $\int_\varphi \sum_1^n F^i(x)dx^i$ we understand the Kurzweil–Henstock limit of the Riemann sums

$$\sum_k \sum_{i=1}^n F^i(\varphi(\xi_k))(\varphi^i(v_k) - \varphi^i(u_k)).$$

It is a classical result that if F has continuous partial derivatives in S and

(4) $$F^i_{,j}(x) = F^j_{,i}(x)$$

for all $x \in S$ then the line integral is independent of the path and there exists a function U with $U_{,i}(x) = F^i(x)$ for all $x \in S$. Moreover U is obtained by choosing an arbitrary point x_0 in S and integrating F from x_0 to a variable point x along any path in S. Our aim in this section is to reduce the assumption of continuous derivatives to mere differentiabilty of F. Results of this nature can be also obtained by using the work of Jarník, Kurzweil, Mawhin and Pfeffer on Stokes' theorem. We shall need the following generalization of Cousin's lemma: If T is a triangle and T_k; $k = 1 \ldots r$ are nonoverlapping triangles with $\cup_1^r T_k = T$, points y_k belong to T_k and $\delta : T \mapsto (0, \infty)$ then we say that the set $\{(y_k, T_k); k = 1, 2, \ldots, r\}$ is a δ-fine partition of T if $d(T_k) < \delta(y_k)$. *For a triangle $T \in \mathbb{R}^n$ there always exists a δ-fine partition consisting of triangles* similar *to T*. An indirect proof can be given which follows the usual pattern of the one-dimensional bisection argument except that now T would be divided into four similar triangles formed by mid-points of sides of T. We shall say that assumption $\mathcal{D}$ is satisfied in G if F is continuous in G and there exists a countable set M such that F is differentiable and satisfies (4) in $G \setminus M$.

Lemma. *If $\mathcal{D}$ is satisfied in G and the triangle $T \subset G$ then*

(5) $$\int_T F(x)dx = 0.$$

Proof. The elements of M can be enumerated and for $w_m \in M$ and for arbitrary $\varepsilon > 0$ there exists a $\delta > 0$ such that

(6) $$|F(x) - F(w_m)| < \frac{\varepsilon}{2^m}$$

whenever $|x - w_m| < \delta$. For $y \in S \setminus M$ there is a $\delta > 0$ such that for $i = 1, 2, \ldots, n$

(7) $$\left|F^i(x) - F^i(y) - \sum_{j=1}^{n} F^i_{,j}(y)(x^j - y^j)\right| < \frac{\varepsilon}{n}|y - x|$$

whenever $|x - y| < \delta$. Let $T \subset G$ and $\{y_k, T_k\}$ a δ-fine partition of T with triangles similar to T. If $y_k \in M$ then $y_k = w_m$ for some m and we obtain from (6)

(8) $$\int_{T_k} F(x)dx < \frac{\varepsilon}{2^m}p(T_k) \le \frac{\varepsilon}{2^m}p(T).$$

If $y_k \notin M$ then because of (4) .

(9) $$\int_{T_k} \sum_{i=1}^{n} \left(F^i(y) + \sum_{j=1}^{n} F^i_{,j}(y)(x^j - y^j) \right) dx^i = 0.$$

This can be seen most easily by realizing that the integrand in (9) has a 'primitive' $\tilde{U}$, where $\tilde{U}(x) = \sum_{i=1}^{n} F^i(y)x^i + \frac{1}{2}\sum_{i,j=1}^{n} F^i_{,j}(y)(x^j - y^j)(x^i - y^i)$. For $y_k \notin M$ we get from (7) and(9) that

(10) $$\left|\int_{T_k} F(x)dx\right| < \varepsilon d(T_k)p(T_k).$$

The triangles T_k are similar to T, therefore there exists a constant C depending *only* on T (and independent of k) such that $d(T_k)p(T_k) < CA(T_k)$ for all k. Consequently (10) becomes

(11) $$\left|\int_{T_k} F(x)dx\right| < C\varepsilon A(T_k).$$

Obviously

$$\int_T F(x)dx = \sum_{k=1}^{r} \int_{T_k} F(x)dx,$$

which finally by (8) and (11) gives

$$\int_T F(x)dx < \varepsilon(CA(T) + p(T)). \quad \square$$

Theorem. Path independence. *If $\mathcal{D}$ is satisfied in S then there is a function $U : S \mapsto \mathbb{R}$ such that*

$$dU(x) = \sum_{i=1}^{n} F^i(x)dx^i \tag{12}$$

for all x in S. If $\varphi : [a,b] \mapsto S$ is a path lying in S then

$$\int_{\varphi} F(x)dx = U(\varphi(b)) - U(\varphi(a)). \tag{13}$$

Proof. It suffices to prove (12), equation (13) then follows by an argument similar to that one used in the proof of the Fundamental Theorem in K–H theory. To see this choose $\varepsilon > 0$. For every $\xi \in [a,b]$ there is a positive δ such that if $|z - \xi| < \delta$ then

$$\left|U(\varphi(z)) - U(\varphi(\xi)) - \sum_{1}^{n} F^i(\varphi(\xi))(\varphi^i(z) - \varphi^i(\xi))\right| < \varepsilon|\varphi(z) - \varphi(\xi))|.$$

Consequently, if $\xi - \delta < u \leq \xi \leq v < \xi + \delta$ then

$$|U(\varphi(v)) - U(\varphi(u)) - \sum_{1}^{n} F^i(\varphi(\xi))(\varphi^i(v) - \varphi^i(u))| < \varepsilon \operatorname*{Var}_{u}^{v} \varphi. \tag{14}$$

For Π a δ-fine partition of $[a,b]$ we obtain with the help of (14)

$$|U(\varphi(b)) - U(\varphi(a)) - \sum_{\Pi}\sum_{1}^{n} F^i(\varphi(\xi_k))(\varphi^i(v_k) - \varphi^i(u_k))| < \varepsilon \operatorname*{Var}_{a}^{b} \varphi.$$

This establishes the implications (12)$\Rightarrow$(13). We denote by $l(x)$ the path whose geometrical image joins the centre of the star-shaped region S with x and define

$$U(x) = \int_{l(x)} F(z)dz.$$

It follows from the Lemma that

$$U(x+h) - U(x) = \int_{\psi} F(z)dz,$$

where $\psi(t) = x + th$, with $0 \leq t \leq 1$. Routine continuity argument now gives (12).

GENERALIZATIONS. If φ_1 and φ_2 are two paths homotopic with fixed ends in G and assumption $\mathcal{D}$ is satisfied in G then

$$\int_{\varphi_1} F(z)dz = \int_{\varphi_2} F(z)dz.$$

This can be proved in two different ways. Firstly the integral $\int_\varphi F(x)dx$ is independent of the path locally, i.e. in some neighborhood of every point in G. Using this one can employ the usual homotopy argument (see e.g. [C]) to obtain the result in the large. Alternatively, one can use Cousin's lemma (with squares rather than triangles) for the homotopy square and proceed similarly as in the Lemma. This would necessitates the use of a differentiable (compare [V2]) homotopy, the general result must then be obtained by an approximation argument.

REMARK. The assumption $\mathcal{D}$ can be weakened without substantially changing the proofs. One needs to assume the differentiability only on two-dimensional planes only, the modified assumption reads as follows: We say that weak-$\mathcal{D}$ is satisfied in G if F is continuous in G and for every two-dimensional plane P there exists a countable set M_P such that for every $x \in G \cap (P \setminus M_P)$ there exists a symmetric n by n matrix $[a_{ij}]$ with the following property: Given $\varepsilon > 0$ and $x \in P \cap G$ there is a positive δ such that for every y and z in P with $|x-y| < \delta$ and $|x-z| < \delta$ we have

$$\left|\int_{yz}\sum_{i=1}^{n}\Big(F^i(u) - F^i(x) - \sum_{j=1}^{n} a_{ij}(u^i - x^i)\Big)du^i\right| < \varepsilon|y-z|\max(|x-y|,\ |x-z|).$$

The lemma and the theorem on path independence remain valid if the assumption $\mathcal{D}$ is replaced by weak-$\mathcal{D}$. This allows the set where F is not differentiable or equation (4) is not satisfied to be uncountable. It is an interesting problem to determine how big (say in measure theoretic terms) this set can be and still have the theorem on path independence valid.

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