ELLIPTIC SYSTEMS

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1 Introduction

Elliptic equations model the behaviour of *scalar* quantities u, such as temperature or gravitational potential, which are in an equilibrium situation subject to certain imposed boundary conditions. In his first four lectures, John Urbas discussed *linear*[1] elliptic equations. In his lectures on the minimal surface equation, Graham Williams discussed the minimal surface equation, a *quasi-linear*[2] elliptic equation in divergence form. Neil Trudinger and Tim Cranny will discuss *fully nonlinear*[3] elliptic equations.

Elliptic systems model *vector-valued* quantities in an equilibrium situation subject to certain imposed boundary conditions. Examples are a vector-field describing the molecular orientation of a liquid crystal, and the displacement of an elastic body under an external force.

Solutions of elliptic *equations* are typically as smooth as the data allows (e.g. are C^∞ if the given data is C^∞). Solutions of elliptic *systems* typically have singularities.

We use as reference [G] the book *Multiple Integrals in the Calculus of Variations* by M. Giaquinta.

2 A Model, Harmonic Map, Problem

Suppose $\Omega \subset I\!R^n$ is an elastic membrane, "stretched" via the function w over a part of the n-dimensional sphere $S^n \subset I\!R^{n+1}$, where w is specified on the boundary $\partial\Omega$. As a simple approximation to the physical situation, we can regard w as a minimiser of the *Dirichlet energy*

$$\frac{1}{2}\int_\Omega |Dw|^2, [4] \tag{1}$$

amongst all maps $w:\Omega \to I\!R^{n+1}$ such that

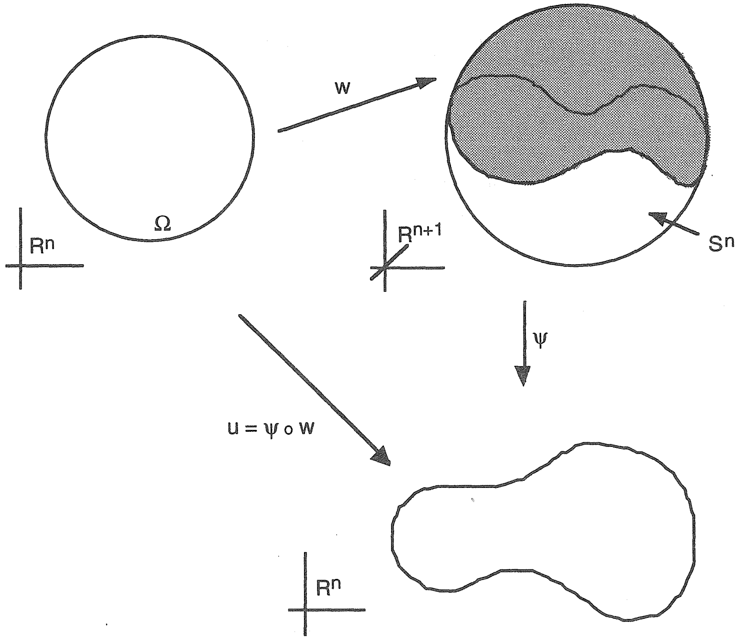
$$|w| = 1, \quad w|_{\partial\Omega} \text{ specified.}$$

[1]The unknown function u and its first and second derivatives occur linearly. The *coefficients* of u and its derivatives may be nonlinear, but usually smooth, functions of the domain variables $x_1, \dots, x_n$.

[2]Linear in the second derivatives of u, but not necessarily linear in u or its first derivatives.

[3]Not even linear in the second derivatives of u.

[4]Where $|Dw|^2 = \sum_{i,\alpha} |D_i w^\alpha|^2$. The $\frac{1}{2}$ is merely a convenient normalisation constant.



A simpler related problem, without the constraint $|w| = 1$, is obtained as follows. Let $\psi : S^n \to \mathbb{R}^n$ be stereographic projection from the north pole. If $w[\Omega]$ avoids a neighbourhood of the south pole then $u = \psi \circ w$ solves the problem:
Minimise

$$E(u) = \frac{1}{2}\int_\Omega a(u)|Du|^2,$$

amongst all maps $u : \Omega \to \mathbb{R}^n$ such that

$$u|_{\partial\Omega} \text{ specified.}$$

Here $a(u)$ is a smooth *positive* function (which is determined[5] by ψ).

We will consider this simpler problem

Euler Lagrange System We now derive the Euler Lagrange system for minimisers of E. Arguing formally, if u is a minimiser of E (subject to fixing the boundary values of u), then for all $\phi \in C^1_c(\Omega; \mathbb{R}^n)$ [6]

$$\begin{aligned} 0 &= \left.\frac{d}{dt}\right|_{t=0} \frac{1}{2}\int_\Omega a(u+t\phi)|D(u+t\phi)|^2 \\ &= \int_\Omega a(u)D_iu^\alpha D_i\phi^\alpha + \frac{1}{2}D_\alpha a(u)\phi^\alpha |Du|^2 \\ &= \int_\Omega a(u)DuD\phi + B(u)|Du|^2\phi. \end{aligned}$$

[5] $a(u) = |\nabla\psi|^{-2}$, where $\nabla\psi$ is the tangential gradient, defined in a natural manner.
[6] $C^1_c(\Omega; \mathbb{R}^n)$ consists of all compactly supported C^1 functions $\phi : \Omega \to \mathbb{R}^n$.

We sum over repeated indices in the second line, and in the last line we repress the indices.

If u satisfies the above integral equation for all $\phi \in C_c^1(\Omega; I\!R^n)$, we say that u is a *weak solution* of the system

$$D_i(a(u)D_iu^\alpha) = \frac{1}{2}D_\alpha a(u)|Du|^2 \tag{2}$$

for $\alpha = 1, \ldots, n$.[7] We abbreviate this to

$$D(a(u)Du) = B(u)|Du|^2. \tag{3}$$

If u is C^1 then being weak solution is the same as satisfying (3) in the usual sense.

Important features to note are the positivity of $a(u)$, which makes the system *elliptic*,[8] and the quadratic nature of $|Du|^2$ on the right.[9]

Solutions with Singularities In the theory of elliptic P.D.E's, you considered the class of $W^{1,2}(\Omega)$ functions largely for technical reasons.[10] It was "simple" to show the existence of weak solutions in this class, and then one considered the question of regularity of solutions. In the vector-valued setting, solutions need not be smooth, and it becomes even more natural to work in the $W^{1,2}$ setting.

Thus we define

$$W^{1,2}(\Omega; I\!R^N)$$

to be the class of functions $u : \Omega \to I\!R^N$ such that each component function belongs to $W^{1,2}(\Omega)$.

Note that the energy $E(u)$ is well defined for arbitrary functions $u \in W^{1,2}(\Omega; I\!R^n)$. In particular, the function $x/|x|$ has partial derivatives which "behave like" $1/|x|$, and so $x/|x| \in W^{1,2}(B_1(0); I\!R^n)$ if $n \geq 3$. But note that $x/|x|$ has a singularity at the origin.

Let $\Omega = B_1(0)$. The function

$$w(x) = (x/|x|, 0)$$

maps $B_1(0)$ "radially" onto the equator of $S^n \subset I\!R^{n+1}$. The function $x/|x|$, and hence w, is a $W^{1,2}$ function if $n \geq 3$. One can show that if $n \geq 7$ then w

[7]The fact that the number of "dependent" variables $u_1, \ldots, u_n$ and the number of "independent" variables $x_1, \ldots, x_n$ are the same is just a consequence of this particular problem. It is not the case in general.

[8]More generally, if instead of $a(u)D_iu^\alpha D_i\phi^\alpha$ we had $\sum_{\substack{i=1,\ldots,n \\ \alpha=1,\ldots,N}} A_{ij}^{\alpha\beta} D_iu^\alpha D_j\phi^\beta$, then we say the system is *elliptic* if $A_{ij}^{\alpha\beta}\xi_i^\alpha\xi_j^\beta \geq \lambda|\xi|^2$ for some constant $\lambda > 0$ and all $\xi \in I\!R^{n+N}$. In many physical problems it is important to have a weaker form of ellipticity, namely $A_{ij}^{\alpha\beta}\xi_i\eta^\alpha\xi_j\eta^\beta \geq \lambda|\xi|^2$ for some constant $\lambda > 0$ and all $\xi \in I\!R^n$, $\eta \in I\!R^N$.

[9]An exponent less than two is "easier" to handle; an exponent greater than two is more difficult. But two is the "natural" exponent for many problems, as is the case here.

[10]See also my lectures on measure theory.

has least energy amongst all functions mapping $B_1(0)$ onto the unit sphere and having the same boundary values as w. Similarly, if $n \geq 7$, $u = \psi \circ w$ minimises $E(u)$ in (2) amongst all maps having the same boundary values. In particular, u satisfies the system of equations (2), i.e. (3). If $3 \leq n < 7$ then u is no longer a minimiser, but it still satisfies the system (3). If $n = 2$ it turns out that solutions of (3), and in particular minimisers of $E(u)$, are smooth.

We have just noted that a solution of (3) may have a singularity. If $u(x)$ is a solution, then clearly so is $u(x-a)$ for any $a \in \mathbb{R}^n$. Since a sum of solutions is also solution, we obtain solutions with any finite number of singularities.

In general, a solution of (3) is said to be *stationary*, or an *equilibrium solution*, for the energy E. Thus minimisers are solutions of the Euler Lagrange system, but not necessarily conversely.[11] Since the energy is the Dirichlet Energy (for w, and also for u if we choose the appropriate metric), stationary functions for this particular problem are called *harmonic*.

3 A Simpler Model Problem

Our intention is to provide a reasonably complete analysis for solutions of systems of the form (3), but with *zero* right side. Thus we consider systems of the form

$$D(a(u)Du) = 0, \tag{4}$$

which may or may not be an Euler Lagrange system.

Systems of the type (4) were the first type of *nonlinear* elliptic system to be analysed. (In the next Section we briefly remark on *linear* elliptic systems.) If the right side is *nonzero*, as in (3), then the problem is considerably more complicated. In particular, minimisers will have "nicer" properties than merely stationary solutions. See [G] for more details.

We remark (4) may also have singular solutions. For example, $x/|x|$ is a weak solution of (4) if

$$A_{ij}^{\alpha\beta}(u) = \delta_{ij}\delta_{\alpha\beta} + \left(\delta_{\beta j} + \frac{4}{n-2}\frac{u^j u^\beta}{1+|u|^2}\right)\left(\delta_{\alpha i} + \frac{4}{n-2}\frac{u^i u^\alpha}{1+|u|^2}\right),$$

where $n = N \geq 3$. See [G, p. 57]. Note that the A are C^∞, in fact analytic. The system of equations in this case *is* an Euler Lagrange system for a certain energy functional. Moreover, for sufficiently large n, $x/|x|$ is the (unique) minimiser of this particular energy functional.

[11]The analogy is that a function E defined on $\mathbb{R}^k$ can have equilibrium points which are not minimisers.

4 Linear Elliptic Systems

For completeness, we briefly discuss linear elliptic systems. Suppose $\Omega \subset I\!R^n$ and

$$u : \Omega \to I\!R^N.$$

We say u satisfies a *linear elliptic system* in *integral form* if

$$\int_\Omega \sum_{\substack{i=1,\dots,n \\ \alpha=1,\dots,N}} A_{ij}^{\alpha\beta}(x) D_i u^\alpha D_j \phi^\beta = 0 \tag{5}$$

for all $\phi \in C_c^1(\Omega; I\!R^N)$. The $A_{ij}^{\alpha\beta}(x)$ are required to satisfy the *ellipticity condition*

$$A_{ij}^{\alpha\beta}(x)\xi_i^\alpha \xi_j^\beta \geq \lambda |\xi|^2$$

for some constant $\lambda > 0$ and all $\xi \in I\!R^{nN}$. Note that the coefficients $A_{ij}^{\alpha\beta}(x)$ depend only on x and not on u. The summation sign is usually dropped, and we even suppress all indices and write

$$\int_\Omega A(x) Du D\phi = 0. \tag{6}$$

The ellipticity condition is then written

$$A\xi\xi \geq \lambda |\xi|^2.$$

Assuming the $A_{ij}^{\alpha\beta}(x)$ are bounded, it is straightforward to show by an approximation argument that we may take $\phi \in W_0^{1,2}(\Omega; I\!R^N)$ in (5). Recall that $W_0^{1,2}(\Omega; I\!R^N)$ consists of those $W^{1,2}(\Omega; I\!R^N)$ functions which are zero on $\partial\Omega$ in a natural way.

Motivated by integration by parts, we usually write the system as

$$D_j\left(A_{ij}^{\alpha\beta}(x) D_i u^\alpha\right) = 0 \tag{7}$$

for $\beta = 1, \dots, N$. This abbreviates to

$$D(A(x)Du) = 0. \tag{8}$$

If $u \in W^{1,2}(\Omega; I\!R^N)$ satisfies (5) (i.e. (6)) we say u is a *weak solution* of the system (7) (i.e. (8)). If $A(x)$ and u are C^1, then it follows from integration by parts that a weak solution is a solution in the classical pointwise sense.

The theory of linear elliptic systems is similar to the theory of linear equations. In particular, one obtains an analogous Schauder theory (for $C^{k,\alpha}$ solutions) and Sobolev theory (for $W^{k,2}$ solutions).[12] The main difference is that if the functions $A_{ij}^{\alpha\beta}(x)$ are merely bounded, then there exist solutions with singularities. This is *not* the case for a single equation. See [G, p. 54]

[12]Although the details can be considerably more complicated, at least when one considers other than *second-order* elliptic systems.

5 Regularity Results, Summary

We now consider the question of *partial regularity* (i.e. smoothness) of solutions of (4).

More precisely, suppose $u \in W^{1,2}(\Omega; I\!R^N)$ and

$$D(A(u)Du) = 0, \tag{9}$$

where

1. $|A(z)| \leq M \ldots \forall z \in I\!R^N$,
2. $A\xi\xi \geq \lambda|\xi|^2 \ldots \forall \xi \in I\!R^{nN}$, where $\lambda > 0$,
3. $A \in C^0(I\!R^N)$ is uniformly continuous.

More precisely, we are using an abbreviated notation as in the previous section. By u satisfying the system (9) we mean that the corresponding integral equations (as in (5) or (6) but with $A_{ij}^{\alpha\beta}(u)$ instead of $A_{ij}^{\alpha\beta}(x)$), are satisfied for all test functions $\phi \in W_0^{1,2}(\Omega; I\!R^N)$

We will see that $u \in C_0^\alpha(\Omega_0)$ for some open $\Omega_0 \subset \Omega$, where $\Omega \setminus \Omega_0$ is a set of dimension $\leq n-2$ (in a sense to be explained later). If A is smoother than C^0, then u is correspondingly smoother in Ω_0. In particular, if A is C^∞ then $u \in C_0^\infty(\Omega_0)$.

More can be proved. It is only necessary that A be continuous, not uniformly continuous. Moreover, $\Omega \setminus \Omega_0$ is in fact a set of dimension p for some $p < n-2$, and is empty if $n = 2$.

The idea of the proof is that if the graph of a solution u is sufficiently "flat" in the L^2 sense near $x_0 \in \Omega$, then in fact u is smooth in a neighbourhood of x_0. We will see that the "flatness" condition holds at all except a "small" set of points.

The key technical point in the proof is to consider the quantity

$$U(x_0, R) = \fint_{B_R(x_0)} |u - (u)_{x_0,R}|^2,$$

for $B_R(x_0) \subset \Omega$. This measures the L^2 mean oscillation of u in $B_R(x_0)$. Here $\fint$ denotes the average, and is obtained by dividing by the volume $\omega_n R^n$ of $B_R(x_0)$. The quantity $(u)_{x_0,R}$ is the average of u in $B_R(x_0)$ and is given by

$$(u)_{x_0,R} = \fint_{B_R(x_0)} u.$$

We will see that if $U(x_0, R)$ is sufficiently small then in fact $U(x_0, r)$ approaches zero like a power of r. From this, one deduces the Hölder continuity of u in a neighbourhood of x_0. One also shows that except for a set of x_0 of dimension $n-2$, $U(x_0, R)$ *is indeed* small for some $R = R(x_0)$.

6 Some Important Preliminaries

We discuss a number of fundamental results that are used in the proof of partial regularity.

6.1 Integral Characterisation of Hölder Continuity

Theorem *If Ω has Lipschitz boundary, then*

$$u \in C^{0,\alpha}(\overline{\Omega})^{13} \iff \int_{B_R(x_0)} |u-(u)_{x_0,R}|^2 \leq cR^{n+2\alpha}$$

for all $B_R(x_0) \subset \Omega$, and some constant c.

Remark More precisely, if the integral condition holds, then the *precise representative* u^* of u, defined by

$$u^*(x_0) = \lim_{R\to 0} ⨍_{B_R(x_0)} u,$$

satisfies $u^* \in C^{0,\alpha}(\overline{\Omega})$. Since $u^* = u$ a.e., and changing u on a set of measure zero does not change the integral, this is the best one can expect.

PROOF: If u is Hölder continuous, the integral inequality is straightforward. For the other direction, one works from the definition of u^*, see [G; Ch. III,1]. ∎

6.2 Energy (or Caccioppoli) Inequality

Theorem *If u is a solution of (9) and $B_R(x_0) \subset \Omega$, then*

$$\int_{B_{R/2}(x_0)} |Du|^2 \leq \frac{c}{R^2}\int_{B_R(x_0)} |u|^2.$$

Philosophy The important point here is that we are bounding the L^2 norm of the *derivative* of u in some ball in terms of the L^2 norm of u in a larger ball. Such an estimate is not true for arbitrary functions u, but it *is* typical of solutions of elliptic equations or systems that we can often bound integrals of higher derivatives in terms of integrals of lower derivatives, usually over a slightly larger set.

[13]Suppose $0 < \alpha \leq 1$. Then

$$u \in C^{0,\alpha}(\overline{\Omega}) \iff |u(x)-u(y)| \leq M|x-y|^\alpha$$

for some $M > 0$ and all $x, y \in \Omega$. Note that if $\alpha > 1$ then the derivative of u would be everywhere zero, and so u is constant!

Conversely, bounding integrals of lower derivatives in terms of integrals of higher derivatives is something we can do for *arbitrary* functions, by means of Sobolev or Poincaré inequalities. In particular, note the *Poincaré inequality*

$$\int_{B_R(x_0)} |u - (u)_{x_0,R}|^2 \leq cR^2 \int_{B_R(x_0)} |Du|^2.$$

PROOF: Since the proof is one of the simplest examples of a test function argument, we sketch it here.

As is usual in P.D.E.'s, in the following, c denotes a constant which may change from line to line. But it will only depend on the dimension and constants such as M and λ which appear at the beginning of Section 5.

Let $\phi = \eta^2 u$, where η is smooth, $\eta \geq 0$, $\eta = 1$ on $B_{R/2}(x_0)$, $\eta = 0$ outside $B_R(x_0)$, and $|D\eta| \leq 3/R$. Substituting this in the integral form of (9),

$$\begin{aligned} 0 &= \int A\, Du\, D\phi \\ &= \int A\, Du\, (\eta^2 Du + 2\eta u\, D\eta) \end{aligned}$$

Hence

$$\int \eta^2 A\, Du\, Du = -\int 2A\,\eta\, D\eta\, u\, Du.$$

Hence

$$\begin{aligned} \lambda \int \eta^2 |Du|^2 &\leq c \int \eta |D\eta|\, |u|\, |Du| \\ &\leq \epsilon \int \eta^2 |Du|^2 + c(\epsilon) \int |D\eta|^2 |u|^2, \end{aligned}$$

by Young's inequality[14]. Taking $\epsilon = \lambda/2$,

$$\int_{B_{R/2}(x_0)} |Du|^2 \leq \frac{c}{R^2} \int_{B_R(x_0)} |u|^2,$$

as required. ∎

6.3 A Decay estimate for Solutions of Constant Coefficient Systems

Theorem *Suppose u satisfies (9) where the A are* **constant** *and* $\Omega = B_1(0)$ *for simplicity of notation. Then for* $0 < r \leq 1$,

$$U(0, r) \leq cr^2 U(0, 1)$$

for some constant c.

[14] See the last Section of my measure theory notes.

PROOF: We may assume $r \le 1/4$, since if $r > 1/4$ we can take $c \ge 4^{n+2}$.

Then

$$\begin{aligned} r^{-2}U(0,r) &= \omega_n^{-1} r^{-2-n} \int_{B_r(0)} |u-(u)_r|^2 \\ &\le c r^{-n} \int_{B_r(0)} |Du|^2 \quad \text{Poincaré's inequality} \\ &\le c \sup_{B_r(0)} |Du|^2 \\ &\le c \int_{B_{1/2}(0)} |Du|^2 \quad \text{a standard elliptic estimate} \\ &\le c \int_{B_1(0)} |u-(u)_1|^2 \quad \text{by Caccioppoli's inequality} \end{aligned}$$

The "standard elliptic estimate" above is that one can typically bound higher norms (here L^∞) of solutions and their derivatives in terms of lower norms (here L^2) over a larger domain. "Caccioppoli's inequality" is applied to the solution $u - (u)_1$.

This gives the result. ∎

7 Outline of Proof of Partial Regularity

Lemma *Suppose u is a solution of (9). Then there exist constants $\epsilon > 0$ and $\tau \in (0,1)$ such that*

$$U(x_0, r) < \epsilon$$

implies

$$U(x_0, \tau r) < \frac{1}{2} U(x_0, r).$$

PROOF: Suppose $\tau \in (0,1)$ and the conclusion of the lemma is false for each $\epsilon > 0$ (the intention is to obtain a contradiction if τ is sufficiently small).

Then there exist balls $B_{r_k}(x_k) \subset \Omega$ such that

$$U(x_k, r_k) = {\lambda_k}^2 \to 0 \qquad (10)$$

but

$$U(x_k, \tau r_k) \ge \frac{1}{2} {\lambda_k}^2. \qquad (11)$$

Rescale to the unit ball by setting

$$v_k(z) = \frac{u(x_k + r_k z) - a_k}{\lambda_k}$$

for $z \in B_1(0)$, where $a_k = (u)_{x_k, r_k}$.

Then, using the integral form of (9),

$$\int A(\lambda_k v_k + a_k) Dv_k \, D\phi = 0$$

for all $\phi \in W_0^{1,2}(B_1(0); \mathbb{R}^N)$.

Moreover, from (10) and (11),

$$\begin{aligned} (v_k)_1 &= 0 \\ \fint_{B_1} |v_k|^2 &= 1 \\ \fint_{B_\tau} |v_k - (v_k)_\tau|^2 &\geq 1/2. \end{aligned}$$

From Caccioppoli's inequality, $\int_{B_1} |Dv_k|^2$ is bounded independently of k. This allows one to pass to a subsequence of the v_k which converges *weakly* in $W^{1,2}$, *strongly* in L^2 and *pointwise a.e.*, to some function v. Moreover, $a_k \longrightarrow a$, say. From this it is not difficult to show that v will satisfy the "limit" equation

$$\int A(a) \, Dv \, D\phi = 0$$

for all $\phi \in W_0^{1,2}(B_1(0); \mathbb{R}^N)$.

From the decay estimate for constant coefficient equations,

$$\int_{B_\tau} |v - (v)_\tau|^2 < c\tau^2 \int_{B_1} |v - (v)_1|^2 = c\tau^2.$$

On the other hand,

$$\int_{B_\tau} |v - (v)_\tau|^2 \geq \frac{1}{2},$$

using continuity of the L^2 norm under L^2 convergence in both lines.

This is a contradiction for sufficiently small τ. ∎

The Lemma is now used as follows. The inequality $U(x_0, r) < \epsilon$ must hold in an open subset of Ω. Moreover, it can be iterated to show

$$U(x_0, \tau^j r) < \left(\frac{1}{2}\right)^j U(x_0, r).$$

This, together with the integral characterisation of Hölder continuity and elementary arguments, shows $u \in C^{0,\alpha}(\Omega_0)$ for some α. Higher smoothness follows by fairly standard iteration techniques (although $C^{0,\alpha}$ to $C^{1,\alpha}$ is not *quite* so standard).

The estimate on the dimension of $\Omega \setminus \Omega_0$ follows from noting

$$U(x_0, r) \leq cr^{2-n} \int_{B_r(x_0)} |Du|^2$$

by Poincaré's inequality, and the fact (using a Vitali covering argument) that the right side approaches zero except on a set E with $\mathcal{H}^{n-2}(E) = 0$.