Bernstein's Theorem

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Abstract: We present a proof of Bernstein's theorem for minimal surfaces which makes use of major techniques from geometric measure theory.

1. THE PROBLEM

Bernstein's theorem:

Suppose $u(x, y)$ is a C^2 function on $\mathbf{R}^2$ which solves the nonparametric minimal surface equation

$$\frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{1 + u_x^2 + u_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{1 + u_x^2 + u_y^2}}\right) = 0$$

in all of $\mathbf{R}^2$.
Then $u(x, y) = ax + by + c$ is affine, i.e. the graph of u is a plane.

Bernstein obtained this result in [BS] as a consequence of another theorem, "Bernstein's geometric theorem", stating that a function $f(x, y)$ whose graph has Gauss curvature $K \leq 0$ in the x, y-plane, and $K < 0$ at some point, cannot be bounded.

There was a gap in the proof of the geometric theorem and it was not until 1950 that a complete proof was given (Hopf [H], Mickle [M]).

In the meantime a proof of Bernstein's theorem was given by Radó [R], using complex analytic methods.

There are now many complex variables proofs - let us mention here those by Bers [BL] and Nitsche [N].

2. A CLASSICAL PROOF IN TWO DIMENSIONS

We sketch here the elegant proof by Nitsche [N] using complex variable methods.

We rewrite the minimal surface equation in the form

$$(1 + u_x^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_y^2)u_{yy} = 0.$$

u solving this equation is necessary and sufficient for the existence of a function $\phi(x, y)$ such that

$$\phi_{xx} = \frac{1 + u_x^2}{\sqrt{1 + u_x^2 + u_y^2}}$$

$$\phi_{xy} = \frac{u_x u_y}{\sqrt{1 + u_x^2 + u_y^2}}$$

$$\phi_{yy} = \frac{1 + u_y^2}{\sqrt{1 + u_x^2 + u_y^2}}$$

Such a function satisfies

$$\phi_{xx}\phi_{yy} - \phi_{xy}^2 = 1$$

which is the hypothesis of the following theorem by Jörgens:

Theorem: *Let $\phi(x, y)$ verify the equation*

$$\phi_{xx}\phi_{yy} - \phi_{xy}^2 = 1$$

in $\mathbf{R}^2$. Then ϕ is a polynomial of degree two.

Proof: We can assume that ϕ is convex (changing the sign of ϕ if necessary). Then the map $(x, y) \longmapsto (\xi, \eta)$ given by

$$\xi = x + \phi_x(x, y) \quad \eta = y + \phi_y(x, y)$$

is a diffeomorphism from $\mathbf{R}^2$ onto itself.

We set $\zeta = \xi + i\eta$ and

$$w(\zeta) = x - \phi_x(x,y) - i(y - \phi_y(x,y))$$

(here x and y have to be understood as functions of ξ and η). $w(\zeta)$ is an entire holomorphic function. Moreover

$$|w'(\zeta)|^2 = \frac{\phi_{xx} + \phi_{yy} - 2}{\phi_{xx} + \phi_{yy} + 2} < 1.$$

By Liouville's theorem w' is constant. Therefore the second derivatives

$$\phi_{xx} = \frac{|1-w'|^2}{1-|w'|^2} = c_1$$
$$\phi_{yy} = \frac{|1+w'|^2}{1-|w'|^2} = c_2$$

are constant and hence ϕ a polynomial of degree two.

Returning to the original problem, we see that u_x and u_y are constant so that u is linear.

3. ANOTHER PROOF IN TWO DIMENSIONS

The minimal surface equation given above is the Euler-Lagrange equation for the area functional

$$\mathcal{A}(u) = \int \sqrt{1 + u_x^2 + u_y^2}\, dx dy.$$

obtained by requiring the first variation of this functional to be zero.

Assume M to be a *minimizing* smooth surface in $\mathbf{R}^3$, i.e.

$$|M \cap K| \leq |S \cap K| \text{ for all compact } K \subset \mathbf{R}^3$$

and comparison surfaces S with $S \equiv M$ in $\mathbf{R}^3 \setminus K$. (Here $|\cdot|$ denotes the two-dimensional Hausdorff measure – see notes by M. Ross).

Then M is *stable*. As definition of stability we use the nonnegativity of the second variation of the area functional, i.e.

$$\delta^2 \mathcal{A} = \int_M \left(|\nabla^M \zeta|^2 - |A|^2 \zeta^2 \right) d\mathcal{H}^2 \geq 0$$

for every Lipschitz continuous function ζ with compact support in $\mathbf{R}^3$. Here ∇^M denotes the gradient on M and A the second fundamental form of M in $\mathbf{R}^3$ ([S1]; compare also notes by K. Ecker for first and second variation).

For $j \in \mathbf{N}$, set

$$\zeta_j(x) = \begin{cases} 1 & |\times| \leq j \\ 0 & |\times| \geq j^2 \\ 2 - \frac{\log |\times|}{\log j} & j < |\times| < j^2 \end{cases}$$

For a stable surface M we have for every j

$$\int_{M \cap B_j} |A|^2 d\mathcal{H}^2 \leq \int_M |\nabla^M \zeta_j|^2 d\mathcal{H}^2 \leq \int_M |\nabla^{\mathbf{R}^3} \zeta_j|^2 d\mathcal{H}^2.$$

Here B_j is the 3-dimensional ball about zero with radius j. Let $M_j = M \cap (B_{j^2} \setminus B_j)$. Then

$$\begin{aligned} \int_M |\nabla^{\mathbf{R}^3} \zeta_j|^2 d\mathcal{H}^2 &= \frac{1}{(\log j)^2} \int_{M_j} |x|^{-2} d\mathcal{H}^2 \\ &= \frac{1}{(\log j)^2} \int_0^\infty \mathcal{H}^2\{x \in M_j : |x|^{-2} > t\} dt \\ &\leq \frac{1}{(\log j)^2} \left(\int_{j^{-4}}^{j^{-2}} \mathcal{H}^2\{M \cap B_{t^{-\frac{1}{2}}}\} dt + j^{-4} \mathcal{H}^2(M_j) \right). \end{aligned}$$

Using the condition on M to be minimizing we have (compare to surface area of sphere).

$$\mathcal{H}^2(M \cap B_r) \leq cr^2.$$

We combine this and the above estimate to obtain

$$\int_M \left|\nabla^{R^3} \zeta_j\right|^2 d\mathcal{H}^2 \leq \frac{C}{\log j}.$$

Finally we let $j \to \infty$. This gives

$$|A|^2 = 0$$

and therefore M is a plane.

4. HIGHER DIMENSIONS

We remark that the classical proof used complex analysis methods. The second proof presented here used the fact that in two variables any bounded set has zero absolute capacity (i.e. $\inf\{\int |\nabla^{\mathbf{R}^3} f| dx : f \in C_0^1(R^2), f \geq 1$ on $E\} = 0$ for every bounded set $E \subset \mathbf{R}^2$ – this is positive for higher dimensions).

It was not until the sixties that Fleming [F] gave a new proof of the two-dimensional theorem, using a method independent of the number of dimensions and provided hope of proving the theorem in more than two variables.

In [F] Fleming is investigating the oriented Plateau problem set in the newly developed framework on integral currents by Federer and Fleming [FF]. The geometric measure theory techniques described there led to yet another solution of Bernstein's problem.

The main idea in the proof is to construct a sequence of surfaces by blowing down the original surface about a point. It is shown that this sequence converges to a minimizing cone. The question is then reduced to the existence of singular cones in $\mathbf{R}^n$. Since no such cones exist in $\mathbf{R}^3$, Fleming's argument gives the new proof in two dimensions.

De Giorgi [DG] improved the result showing that nonexistence of singular minimal k-cones in $\mathbf{R}^{k+1}$ would imply Bernstein's theorem for minimal graphs in $\mathbf{R}^{k+2}$.

In 1966, Almgren [A] proved that there exist no singular cones in $\mathbf{R}^4$. These two last results settled the theorem for three-dimensional surfaces in $\mathbf{R}^4$ and four-dimensional in $\mathbf{R}^5$.

In 1968 Simons [SJ] extended the result up to $\mathbf{R}^7$. The exciting discovery in this paper is the example of the cone

$$C = \{x \in \mathbf{R}^8 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2\}$$

which is stable (every compact variation of C increases the area).

Simons' cone is not only stable but even absolutely area minimizing as shown by Bombieri, De Giorgi and Giusti in [BdGG]. They also constructed complete minimal graphs over $\mathbf{R}^n$, for $n \geq 8$, different from hyperplanes.

Bernstein's theorem was now solved but new problems arise. Let us mention two questions:

- Are there any additional conditions on the function $u(x, y)$ which guarantee for the solution to be a plane even in higher dimensions?

- Even if the solutions are not planes, do they have other common characteristics? (such as behaviour at infinity.).

For further discussions we refer to [G1], [G2], [O], [S2].

THE PROOF

We discuss here the way in which geometric measure theory techniques are used for the proof.

Suppose M is a solution of Bernstein's theorem. We like to think of M as a smooth minimizing hypersurface in $\mathbf{R}^{n+1}$, i.e.

$$|M \cap K| \leq |S \cap K| \text{ for all compact } K \subset \mathbf{R}^{n+1}$$

and hypersurfaces S with $S \equiv M$ in $\mathbf{R}^{n+1} \setminus K$.

Assume $p = 0 \in M$ and define the rescaled surfaces

$$M_j = t_j^{-1} M.$$

We let $t_j \to \infty$ and show that there exists a subsequence $M_{j'}$ converging to a minimizing set M_∞.

In order to do this we bound the n-dimensional Hausdorff measures $|M_j \cap B_R|$, in balls B_R of radius R, to be able to use compactness. We have

$$\begin{aligned} |M_j \cap B_R| &= |t_j^{-1} M \cap B_R| = t_j^{-n} |M \cap B_{t_j R}| \\ &\leq t_j^{-n} c(n) t_j^n R^n = c(n) R^n. \end{aligned}$$

For the inequality we used the minimality of M and compared to the surface area of spheres.

We can now use the compactness theorem to conclude that $\lim_{j \to \infty} |M_j \cap B_R|$ exists for all $R > 0$ and that M_j converges to C.

Compactness theorem:

([S1], [G1]) *Let $\{M_j\}$ be a sequence of minimizing hypersurfaces in $\mathbf{R}^{n+1}$ such that $\partial M_j = \emptyset$ and $\mathcal{H}^n(M_j \cap B_R) \le C(R)$ for all $R > 0$. Then there exists a subsequence $t_{j'} \to \infty$ and a minimizing set M_∞ with $M_{t_{j'}} \to M_\infty$.*

We note that the set that we obtain does not need to be a smooth hypersurface any longer. Also it could depend on the choice of the subsequence $t_{j'}$ (but not on R). Actually the theorem is formulated for integral currents.

The next step is to study the structure of C; the goal is to show that C is a cone.

We prove that the quantity $\frac{|C \cap B_\sigma|}{\omega_n \sigma^n}$ is independent of σ (by ω_n we denote the volume of the n-dimensional ball). We have

$$
\begin{aligned}
\frac{|C \cap B_\sigma|}{\omega_n \sigma^n} &= \lim_{j \to \infty} \frac{|M_j \cap B_\sigma|}{\omega_n \sigma^n} \\
&= \lim_{j \to \infty} \frac{|M \cap B_{t_j \sigma}|}{\omega_n (t_j \sigma)^n} \\
&= \lim_{\rho \to \infty} \frac{|M \cap B_\rho|}{\omega_n \rho^n}
\end{aligned}
$$

We show that the last limit exists and hence $\frac{|C \cap B_\sigma|}{\omega_n \sigma^n}$ is independent of σ. In fact $\frac{|M \cap B_\rho|}{\omega_n \rho^n}$ is bounded (compare to the surface area of spheres, as before) and also monotonically increasing in ρ as obtained from the

Monotonicity theorem:

Let N be a minimizing hypersurface in $\mathbf{R}^{n+1}$. Then $\frac{|N \cap B_\rho|}{\omega_n \rho^n}$ is monotonically increasing in ρ.

Sketch of the proof: For almost every $\rho > 0$ $N \cap \partial B_\rho$ is a smooth (we assumed N to be a smooth hypersurface) submanifold.

We define $C_\rho = \{tx : t \in [0,1], x \in N \cap \partial B_\rho\} \cup (N \setminus B_\rho)$.

For almost every $\rho > 0$ C_ρ is an admissible comparison set, and since N minimal

$$
\begin{aligned}
|N \cap B_\rho| &\leq |C_\rho \cap B_\rho| \\
&= \frac{\rho}{n}\mathcal{H}^{n-1}(C_\rho \cap \partial B_\rho) \\
&= \frac{\rho}{n}\mathcal{H}^{n-1}(N \cap \partial B_\rho) \\
&\leq \frac{\rho}{n}\frac{d}{d\rho}|N \cap B_\rho|
\end{aligned}
$$

Here we used the fact that $C_\rho \cap B_\rho$ is a cone and the last inequality was obtained by the

Coarea formula: $\frac{d}{d\rho}|N \cap B_\rho| \geq |N \cap \partial B_\rho|$.

Therefore

$$
\frac{d}{d\rho}(\rho^{-n}|N \cap B_\rho|) \geq 0
$$

and this implies the monotonicity.

We actually know even more

$$
\frac{|N \cap B_\rho|}{\omega_n \rho^n} - \frac{|N \cap B_\sigma|}{\omega_n \sigma^n} = \int_{N \cap (B_\rho \backslash B_\sigma)} \frac{|x \cdot \nu|^2}{|x|^{n+2}} d\mathcal{H}^n
$$

(see [S1]), where x is the position vector and ν the normal to the surface.

Back to the proof we obtain

$$
0 = \frac{|C \cap B_\rho|}{\omega_n \rho^n} - \frac{|C \cap B_\sigma|}{\omega_n \sigma^n} = \int_{C \cap (B_\rho \backslash B_\sigma)} \frac{|x \cdot \nu|^2}{|x|^{n+2}} d\mathcal{H}^n
$$

so for $x \cdot \nu = 0$ almost everywhere. Since the position vector is a tangential vector, C is a cone.

For the last step of the proof we use the **regularity result** that there are no minimal cones with singularities for $n \leq 6$.

For $n \leq 6$ C is a smooth cone and therefore a plane.

The intuitive reasoning for that is that C as a cone is scale invariant. If we blow up C about a point, we obtain C itself. On the other hand blowing up gives us the tangent space at the point, which is a plane for a smooth surface.

For a plane we have

$$\frac{|C \cap B_\rho|}{\omega_n \rho^n} = 1 \text{ for all } \rho > 0.$$

The original hypersurface M was assumed to be regular. By the definition of density (M. Ross' notes)

$$
\begin{aligned}
1 &= \lim_{\rho \to 0} \frac{|M \cap B_\rho|}{\omega_n \rho^n} \\
&\leq \frac{|M \cap B_\rho|}{\omega_n \rho^n} \qquad \text{for any fixed } \rho \\
&\leq \lim_{\rho \to \infty} \frac{|M \cap B_\rho|}{\omega_n \rho^n} \\
&= \lim_{\rho \to \infty} \frac{|\rho^{-1} M \cap B_1|}{\omega_n} \\
&= \frac{|C \cap B_1|}{\omega_n} \\
&= 1
\end{aligned}
$$

(using again monotonicity).

Therefore $\frac{|M \cap B_\rho|}{\omega_n \rho^n} = 1$ for all $\rho > 0$. Arguing as before for the set C we conclude that M must be a plane as long as $n \leq 6$.

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