General Relativity

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February 1995

Recent years have seen an upsurge of interest by mathematicians in problems arising from the Einstein equations, with many new and surprising results being established using techniques from pde and differential geometry. These three lectures provide an introduction to some of the problems of current interest, in particular those arising from spherical symmetry or motivated by known behaviour of spherically symmetric spacetimes.

1 The Einstein Equations

Minkowski space $\mathbf{R}^{3,1}$ is the simplest solution of the Eisntein equations. This is just $\mathbf{R}^4$ with the standard coordinates labeled $(x^\alpha) = (t, x)$, $\alpha = 0, \ldots, 3$, with the inner product $\eta = (\eta_{\alpha\beta}) = diag(-1, 1, 1, 1)$, so $\eta(x, x) = -t^2 + |x|^2$. The Lorentz group $O(3, 1)$ is the set of linear transformations preserving η,

$$\eta(Ax, Ax) = \eta(x, x), \qquad A \in O(3,1),\ x \in \mathbf{R}^{3,1}. \tag{1}$$

$O(3, 1)$ has 4 connected components, indexed by $sign(\det A)$ and $sign(A^0_0)$, with the identity component $SO(3, 1)$ consisting of the time and space orientation preserving linear transformations. Some examples of Lorentz transformations are:

1. spatial rotations
$$A = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}, \qquad R \in SO(3),$$
which preserve the future timelike vector $e_0 = (1, 0, 0, 0)^t$;

2. boosts in the (t, x^1) plane through hyperbolic angle ψ,
$$A = \begin{pmatrix} \cosh\psi & \sinh\psi & \\ \sinh\psi & \cosh\psi & \\ & & I_2 \end{pmatrix}.$$

$SO(3, 1)$ preserves the unit hyperboloid

$$H^+ = \{(t, x) : t = \sqrt{1 + |x^2|}\} \subset \{(x^\alpha), \eta(x, x) = -1\}, \tag{2}$$

just as the unit sphere S^n in Euclidean space is preserved by the rotation group $SO(n+1)$. Likewise, the curvature of the metric induced from η on H^+ is constant (-1) and the hyperbolic angle ψ measure geodesic distance, so the metric on H^+ in polar coordinates is

$$ds^2_{H^+} = d\psi^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2).$$

A spacetime is a 4-dimensional manifold with a metric $g = (g_{\alpha\beta})$ of Lorentzian signature. The Einstein equations for g are

$$G_{\alpha\beta} = 8\pi\kappa T_{\alpha\beta}, \tag{3}$$

where κ is Newton's gravitational constant and $G_{\alpha\beta}$ is the Einstein tensor, determined from the Ricci tensor and Ricci scalar by

$$G_{\alpha\beta} = Ric_{\alpha\beta} - \tfrac{1}{2}Rg_{\alpha\beta}. \tag{4}$$

Recall that

$$Ric_{\alpha\beta} = g^{\gamma\delta} R_{\alpha\gamma\delta\beta}, \qquad R = g^{\alpha\beta} Ric_{\alpha\beta}.$$

Finally, $T_{\alpha\beta}$ is the stress-energy tensor, which is determined by the physical matter fields. For example, the stress-energy tensor describing perfect fluid matter has the form

$$T_{\alpha\beta} = (\rho + p) u_\alpha u_\beta + p g_{\alpha\beta}, \tag{5}$$

where ρ is the local energy (mass) density, p is the pressure and $u = (u^\alpha)$ is the future timelike unit length vector describing the spacetime flow lines of the fluid particles. The laws of mass and momentum conservation for the matter described by $T_{\alpha\beta}$ are combined into the divergence identity

$$\nabla^\beta T_{\alpha\beta} = 0. \tag{6}$$

It is an exercise to show that (6) applied to the perfect fluid stress energy (5) reduces to the classical mass conservation and Euler's equation for fluid flow.

In order that (6) and (3) hold, the gravitational tensor $G_{\alpha\beta}$ must satisfy

$$\nabla^\alpha G_{\alpha\beta} = 0. \tag{7}$$

This follows from the second Bianchi identity

$$\nabla_\alpha R_{\beta\gamma\delta\epsilon} + \nabla_\beta R_{\gamma\alpha\delta\epsilon} + \nabla_\gamma R_{\alpha\beta\delta\epsilon} = 0,$$

by taking two traces,

$$\begin{aligned} 0 &= \nabla_\alpha R_{\beta\gamma}{}^{\gamma\beta} + \nabla_\beta R_{\gamma\alpha}{}^{\gamma\beta} + \nabla_\gamma R_{\alpha\beta}{}^{\gamma\beta} \\ &= \nabla_\alpha R - \nabla_\beta Ric_\alpha^\beta - \nabla_\gamma Ric_\alpha^\gamma \\ &= -2\nabla_\beta G_\alpha^\beta. \end{aligned}$$

The spacetime formulation of Maxwell's equations is particularly elegant and simpler than the classical form:

$$dF = 0, \qquad d^*F = j \tag{8}$$

where $F = (F_{\alpha\beta})$ is a 2-form, $F_{\alpha\beta} = -F_{\beta\alpha}$, and $j = (j_\alpha)$ is the current density determined by other charged matter. In index form these operators are

$$\nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = 0, \qquad \nabla^\beta F_{\alpha\beta} = j_\alpha. \tag{9}$$

By parameterising F with E_i, B_i,

$$E_i = F_{i0}, \qquad B_i = \tfrac{1}{2}\epsilon_{ijk}F_{jk}, \tag{10}$$

where the latin indices range $i, j, k = 1, \ldots, 3$ and ϵ_{ijk} is the totally skew tensor in $\mathbf{R}^3$, we may recover the classical form of Maxwell's equations,

$$\begin{aligned} \partial_t E &= \nabla \times B - 4\pi J, & \mathrm{div} E &= 4\pi\epsilon \\ \partial_t b &= -\nabla \times B, & \mathrm{div} B &= 0 \end{aligned} \tag{11}$$

where $j_0 = \epsilon$ is the charge density, and $J = (j_1, j_2, j_3)$ is the current density vector. The stress energy tensor for a Maxwell field is

$$T^{Maxwell}_{\alpha\beta} = F_{\alpha\gamma}F_{\beta}{}^{\gamma} - \tfrac{1}{4}F_{\gamma\delta}F^{\gamma\delta}\, g_{\alpha\beta}, \tag{12}$$

and the conservation law $\nabla^{\beta}T^{Maxwell}_{\alpha\beta} = 0$ follows from Maxwell's equations. Note that $T^{Maxwell}$ has the property that $g^{\alpha\beta}T_{\alpha\beta} = 0$, which is not true of the stress energy tensors for a perfect fluid or for a solution of the scalar wave equation. This tracefree property is related to the conformal invariance of Maxwell's equations.

The components of the stress energy admit physical interpretations, namely

$$\begin{aligned} T_{00} &= \text{local energy density}, \\ T_{0i}e_i &= \text{local momentum density}, \end{aligned}$$

where both densities are as measured by an observer with world vector e_0. Thus $-T_0{}^{\beta}e_{\beta}$ is the energy momentum vector measured by the observer e_0. On physical grounds we may impose the following *energy conditions* on T,

1. *Weak Energy Condition*, which states that the local energy density is non-negative,

$$T_{nn} = T(n, n) = T_{\alpha\beta}n^{\alpha}n^{\beta} \geq 0, \quad \text{for all future timelike vectors } n = (n^{\alpha}); \tag{13}$$

2. *Dominant Energy Condition*, that the observed energy momentum vector is always future timelike,

$$T(n, n) \geq |T(n, v)|, \tag{14}$$

for all unit length future timelike vectors n and orthogonal unit length spacelike vectors v. Another way of expressing this is to say that $-T^{\beta}_{\alpha}n^{\alpha}e_{\beta}$ is always future timelike.

For example, for a Maxwell field we have

$$T_{00} = \tfrac{1}{2}\left(|E|^2 + |B|^2\right), \qquad T_{0i} = (E \times B)_i$$

and it follows that the dominant energy condition is satisfied.

The uniqueness results for the wave equation $\Box u = 0$ described in the previous lecture may be recast using only the stress energy tensor. For simplicity we formulate this only in $\mathbf{R}^{3,1}$ but the argument carries over without major modification to fields on a spacetime manifold, after taking some care with the definition of the domain of dependence.

Theorem 1 *If $T_{\alpha\beta}$ satisfies the conservation law (6) and the dominant energy condition (14), and if $T_{\alpha\beta} = 0$ along an acausal spacelike hypersurface S, then $T_{\alpha\beta} = 0$ in $D^+(S)$.*

Proof : We integrate $D^\alpha T_{\alpha\beta} = 0$ over the region $\{(t,x), t < \tau\} \cap D^+(S)$, bounded by $S_\tau = S \cap \{t < \tau\}$, $P_\tau = \{t = \tau\} \cap D^+(S)$ and the null surface $H_\tau = (\partial D^+(S) \backslash S) \cap \{t < \tau\}$. Letting $X^\alpha = -T^\alpha_0$ so X is future timelike, since $\mathrm{div} X = D_\alpha X^\alpha = 0$ we have by Stokes' theorem

$$\int_{S_\tau} X^\alpha n_\alpha \, dv = \int_{P_\tau} X^0 \, dv + \int_{H_\tau} \iota_X \mu \qquad (15)$$

where n^α is the future unit normal to S_τ, dv is the induced Riemannian volume measure on the surfaces S_τ, P_τ, and $\iota_X \mu$ is a 3-form to be integrated over H_τ. Because the induced Riemannian measure vanishes on a null surfaces such as H_τ, we must directly compute the final integral. Writing

$$\mu = du \wedge dx^1 \wedge dx^2 \wedge dx^3$$

where $u = t - \phi(x)$ and $H_\tau \subset \mathrm{graph}(\phi)$, we have

$$\begin{aligned} \int_{H_\tau} \iota_X \mu &= \int_{H_\tau} du(X) \, dx^1 \wedge dx^2 \wedge dx^3 \\ &= \int_{H_\tau} (X^0 - \phi_i X^i) \, dx^1 \wedge dx^2 \wedge dx^3 \\ &= \int_{H_\tau} (T_{00} + D_i \phi \, T_{0i}) \, dx^1 \wedge dx^2 \wedge dx^3, \end{aligned}$$

and this integral is non-negative by the dominant energy condition ($|D\phi| = 1$ because H_τ is null). Likewise the integral over P_τ is also non-negative, and since $T_{\alpha\beta}$ vanishes on S_τ by hypothesis, it follows that the stress energy tensor must vanish on H_τ and P_τ, as required. ■

2 The Schwarzschild Spacetime

In general the Einstein equations are very difficult to analyse, and thus most of our knowledge about the behaviour of solutions comes from the study of exactly solvable reductions, typically assuming some type of symmetry group for the spacetime metric. The class of spherically symmetric spacetimes, whilst very specialised, can be analysed in some detail and yields considerable information which guides our understanding of more general metrics. In fact, the very special example of the Schwarzschild spacetime is the source of most of our intuitive (physical) understanding of the behaviour of strong gravitational fields, such as near a black hole.

In this lecture we will compute the Einstein equations for a general spherically symmetric spacetime, derive the Schwarzschild metric and describe its geometry.

Metrics admitting a symmetry action by $SO(3)$ with generic orbits S^2 can be put into the form

$$ds^2 = g_{ab}\,dx^a dx^b + r^2\,d\sigma^2 \tag{16}$$

where $d\sigma^2 = d\vartheta^2 + \sin^2\vartheta\,d\varphi^2$ is the standard metric on S^2 and where $g_{ab}\,dx^a dx^b$ is a metric on the 2-dimensional space $\mathcal{X}$, depending only on $x^a, a = 1,2$. The area of the S^2 orbits is described by the geometric function $r(x)$, which is assumed to be always non-negative. We shall take the existence of such coordinates as the definition of spherically symmetric.

To describe the Einstein equations for the metric (16) we introduce the very useful function

$$m(x) = \tfrac{1}{2}r(1 - g(\nabla r, \nabla r)) = \tfrac{1}{2}r(1 - g^{ab}r_a r_b) \tag{17}$$

where $r_a = D_a r$. Direct computation establishes the remarkable identities

$$\begin{aligned} G_{ab} &= -\,2r^{-1}\,r_{;cd}\,\epsilon^c{}_a\epsilon^d{}_b - 2mr^{-3}\,g_{ab} && (18)\\ D_a m &= \tfrac{1}{2}r^2 G_{bd} r_c \epsilon^{bc}\epsilon^d{}_a, && (19) \end{aligned}$$

where ϵ^{ab} is the skew 2-form of g_{ab}, normalised by $\epsilon_{12} = \sqrt{\det(g)}$, $r_{;ab}$ is the second covariant derivative, and G_{ab} represents the components of the spacetime gravitational tensor tangential to $\mathcal{X}$.

Introducing a Lorentz-orthonormal frame e_0, e_1 (where e_0 is future timelike, and e_1 is spacelike), we may rewrite (19) in the more concrete form

$$\begin{aligned} D_0 m &= \tfrac{1}{2}r^2(r_1 G_{01} - r_0 G_{11}) && (20)\\ D_1 m &= \tfrac{1}{2}r^2(r_1 G_{00} - r_0 G_{01}). && (21) \end{aligned}$$

One may ask whether these equations are equivalent to the full Einstein equations. If $T_{\alpha\beta}$ is a spherically symmetric 2-tensor (so $T_{22} = T_{33}$, $0 = T_{23} = T_{02} = T_{03} = T_{12} = T_{13}$), and if $T_{\alpha\beta}$ satisfies the conservation law (6), then the full Einstein equations

$$Ein_{\alpha\beta} := G_{\alpha\beta} - 8\pi\kappa T_{\alpha\beta} = 0 \tag{22}$$

will be satisfied if only the transverse components are satisfied, i.e. $Ein_{ab} = 0$, $a, b = 1, 2$. This follows because the second Bianchi identities impose differential constraints on the Einstein tensor. In fact it is not even necessary that all the transverse equations be satisfied everywhere:

Theorem 2 *Suppose $B_{\alpha\beta}$ is a spherically symmetric, symmetric 2-tensor which satisfies the spacetime conservation identity $\nabla^\beta B_{\alpha\beta} = 0$. Suppose further that $B(\ell, \ell) = 0$, $B(n, n) = 0$ where ℓ, n are radial future null vectors normalised by $g(\ell, n) = -2$, and that $B(\ell, n)(p) = 0$ for some point p such that $r(p) \neq 0$, $dr_p \neq 0$. Then $B(\ell, n) = 0$ at all points on the connected component of the level set of r passing through p.*

Proof : The terms in the conservation identity corresponding to $\alpha = \ell, n$ yield the directional derivative relations

$$\begin{aligned} 0 &= D_\ell(B_{\ell n}) + D_n(B_{\ell\ell}) + 4r^{-1}(r_\ell B_{22} - B(\ell, \nabla r)) - 2\chi_n B_{\ell\ell}, \\ 0 &= D_\ell(B_{nn}) + D_n(B_{\ell n}) + 4r^{-1}(r_n B_{22} - B(n, \nabla r)) + 2\chi_\ell B_{nn}, \end{aligned} \tag{23}$$

where χ_ℓ, χ_n are connection coefficients. Inserting the conditions $B_{\ell\ell} = B_{nn} = 0$ and noting that $\nabla r = -\frac{1}{2}(r_\ell n + r_n \ell)$ reduces this system to

$$\begin{aligned} D_\ell(r^2 B_{\ell n}) + 4r r_\ell B_{22} &= 0, \\ D_n(r^2 B_{\ell n}) + 4r r_n B_{22} &= 0. \end{aligned} \tag{24}$$

Eliminating B_{22} leads to

$$D_{*\nabla r}(r^2 B_{\ell n}) = 0, \tag{25}$$

where $*\nabla r = r_\ell n - r_n \ell$ is tangent to the level sets of r, and the result follows by integrating (25) along the level set of r. ■

Note that B_{22} will then vanish if $B_{\ell n}$ vanishes on a set intersecting all level curves of r. Other variations on this result are possible.

The Schwarzschild metric may easily be derived from (19) by inserting the vacuum condition $G_{ab} = 0$. Clearly $m(x) = const.$; to proceed further we introduce Schwarzschild coordinates as follows. We assume $dr \neq 0$ and for definiteness, we suppose ∇r is spacelike. Choose e_1 to be the unit vector proportional to ∇r and let e_0

be the orthogonal future timelike vector. We use r as one coordinate function, and construct the second coordinate t by requiring the level sets of t to be integral curves of e_1 or equivalently, to be orthogonal to e_0. In this Lorentz-orthogonal frame we have $D_0 r = r_0 = 0$, so that $m = \frac{1}{2}r(1 - (D_1 r)^2)$. In this way the metric g_{ab} in the 2-dimensional space $\mathcal{X}$ takes the form

$$ds^2_{\mathcal{X}} = -e^{-2\delta}(1 - 2m/r)\, dt^2 + (1 - 2m/r)^{-1}\, dr^2, \tag{26}$$

where $\delta = \delta(r, t)$ is some function to be determined. However, the Einstein equations in this coordinate system (without the assumption of vacuum stress energy) take the form

$$\begin{aligned} \frac{\partial m}{\partial r} &= \tfrac{1}{2}r^2 G_{00}, & (27)\\ \frac{\partial m}{\partial t} &= \tfrac{1}{2}r^2 e^{-\delta}(1 - 2m/r)G_{01}, & (28)\\ \frac{\partial \delta}{\partial r} &= \tfrac{1}{2}r(1 - 2m/r)^{-1}G_{11}. & (29) \end{aligned}$$

In particular if $G_{11} = 0$ then $\delta = \delta(t)$, and reparameterising with $t = t(\tau)$ allows us to normalise to $\delta = 0$, giving the metric

$$ds^2 = -(1 - 2m/r)\, dt^2 + (1 - 2m/r)^{-1}\, dr^2 + r^2\, d\sigma^2. \tag{30}$$

The assumption that ∇r is spacelike implies that $m = \frac{1}{2}r(1 - |\nabla r|^2) < \frac{1}{2}r$, or in other words, this metric form is valid only for $r > 2m$. However, the second case ∇r timelike is easily seen to lead to the same formal metric, but with $r < 2m$. This case will of course only arise if $m > 0$. It is not at all obvious that the two resulting vacuum metrics are related since the metric (30) is singular along $r = 2m$. To show that they are in fact part of a single vacuum spacetime, we need to make a more global choice of coordinates. This leads to the uniqueness theorem due originally to G. Birkhoff.

Theorem 3 *A vacuum spherically symmetric metric is locally isometric to part of the maximally extended Schwarzschild spacetime, which has metric*

$$ds^2 = -e^{2\mu}\, dudv + r^2\, d\sigma^2, \quad (u, v) \in \mathbf{R}^2, uv < 1, \tag{31}$$

where

$$\mu(u, v) = \tfrac{1}{2}\log(32m^3) - \tfrac{1}{2}\log r - r/4m \tag{32}$$

and r is defined implicitly from

$$uv = (1 - r/2m)e^{r/2m}. \tag{33}$$

Proof : Introduce double null coordinates locally, then the metric takes the form (31) for some function $\mu(u,v)$. The Einstein equations in the double null frame $\ell = e^{-\mu}\partial_v$, $n = e^{-\mu}\partial_u$ are

$$G_{\ell\ell} = -\frac{2}{r}e^{-2\mu}(r_{vv} - 2r_v\mu_v), \tag{34}$$

$$G_{nn} = -\frac{2}{r}e^{-2\mu}(r_{uu} - 2r_u\mu_u), \tag{35}$$

$$G_{\ell n} = \frac{2}{r}e^{-2\mu}r_{uv} + m/r^3 \tag{36}$$

$$G_{22} = -4e^{-2\mu}(\mu_{uv} + r^{-1}r_{uv}) \tag{37}$$

where the u,v subscripts denote partial derivatives, and e_2 is some angular unit vector. Setting $G_{\alpha\beta} = 0$ and noting that $m = \frac{1}{2}r(1+4e^{-2\mu}r_ur_v)$ is constant gives five equations for μ, r. The result follows after a little computation.

3 Mass and Spherical Symmetry

We may derive some information about a solution of the Einstein equations $Ein_{\alpha\beta} = 0$ without specifying the matter model, beyond assuming that the dominant energy condition (14) is satisfied. In this lecture we describe two such results, which show a close relationship between the sign of the mass function, and the presence of naked singularities and other regularity conditions.

It is easily seen that the dominant energy condition (14) requires

$$G_{\ell\ell} \geq 0, \qquad G_{\ell n} \geq 0, \qquad G_{nn} \geq 0, \tag{38}$$

whilst the weak energy condition requires only

$$G_{\ell\ell} \geq 0, \qquad G_{nn} \geq 0, \qquad G_{\ell n} \geq -\sqrt{G_{\ell\ell}G_{nn}}. \tag{39}$$

From (18) we find that for any null vector ℓ,

$$\nabla^2_{\ell\ell}r = -\tfrac{1}{2}rG_{\ell\ell}. \tag{40}$$

Now suppose $\tau \mapsto x(\tau)$ is a radial null geodesic with tangent vector $\ell = \dot{x}(\tau)$, and let

$$K(\ell) = 2\frac{d}{d\tau}(\log r(x(\tau))).$$

If the weak energy condition (39) holds and if

$$K(\ell)|_{\tau=0} \leq -2\epsilon < 0,$$

then there is $\tau^* < 2/\epsilon$ such that

$$\lim_{\tau \to \tau^*} r(x(\tau)) = 0.$$

To see this, we regard K and r as functions of τ by restricting to the geodesic $x(\tau)$ and compute

$$\frac{1}{2}\frac{dK}{d\tau} = \frac{1}{r}\frac{d^2 r}{d\tau^2} - \frac{1}{r^2}\left(\frac{dr}{d\tau}\right)^2.$$

But $d^2r/d\tau^2 = \nabla^2_{\ell\ell} r$ since ℓ is geodesic, and then the weak energy condition together with (40) shows that $d^2r/d\tau^2 \le 0$ and $dr/d\tau \le dr/d\tau(0)$. Hence

$$\frac{dK}{d\tau} \le -2\left(\frac{1}{r}\frac{dr}{d\tau}\right)^2 = -\tfrac{1}{2}K^2$$

which may be integrated over the interval $[0, \tau]$ to give

$$K^{-1}(\tau) \ge \tfrac{1}{2}\tau - 1/\epsilon.$$

Since $K^{-1}(\tau) < 0$ by continuity ($dr/d\tau$ remains negative), we have

$$\tau < 2/\epsilon,$$

and this bound must be satisfied by τ^*, the maximal time of existence of $x(\tau)$. Whilst $K^{-1}(\tau)$ remains negative, the geodesic $x(\tau)$ is extendible, so $\lim_{\tau \to \tau^*} K^{-1}(\tau) = 0$ and thus $r(\tau) \to 0$.

In summary, once the radius function r starts decreasing along a null geodesic, it very rapidly crashes to zero. If both inner and outer null geodesics have decreasing radius function (ie. both $D_\ell r$ and $D_n r$ are negative) then the sphere is a *trapped surface.* A theorem of R. Penrose states that the future domain of influence of a trapped surface must contain some form of geodesic incompleteness, which is usually interpreted as showing that some type of singularity is present.

Note that $m = \frac{1}{2}r(1 + \frac{1}{4}r^2 K(\ell)K(n))$ where n is a radial null vector satisfying $g(\ell, n) = -2$. The mean curvature vector of the $r = const.$ 2-sphere is $2H = -K(\ell)n - K(n)\ell$, so $K(\ell), K(n)$ are the null mean curvatures. The trapped surface condition says that $K(\ell)$ and $K(n)$ have the same sign and then $m > r/2$; since this condition distinguishes a region which will encounter some form of singularity from the asymptotic region where $2m/r \to 0$, the set where $r = 2m$ is called the *apparent horizon.* The exterior region where $r(x(\tau)) \to \infty$ for the outward null geodesic has inner boundary described by a null geodesic where $r(x(\tau)) \to c > 0$; this boundary set is called the *event horizon*, and it seems clear that the apparent horizon lies "inside" the event horizon.

Next we turn to the nature of m and in particular, the sign of m. If $\tau \to x(\tau)$ is now a timelike geodesic and $e_0 = \dot{x}(\tau)$ is the timelike unit tangent vector, then from (18) and the geodesic condition $\nabla_{e_0} e_0 = 0$ we derive

$$\ddot{r} = -\frac{m}{r^2} - \tfrac{1}{2} r G_{11}; \tag{41}$$

which we recognise as the relativistic analogue of the Newtonian gravitational attraction arising from a central mass m, with G_{11} representing a pressure term. This provides one reason to denote m the mass function, and leads to an expectation that on physical grounds, m should be non-negative for reasonable matter fields and geometries. We now indicate how this expectation may be justified.

Theorem 4 *Assume the dominant energy condition holds. If $m(p) < 0$ for some point p, and if $\rho \mapsto x(\rho)$ is any unit speed spacelike geodesic with $p = x(0)$ and ρ the arclength parameter, such that*

$$\frac{d(r \circ x)}{d\rho}(0) < 0, \tag{42}$$

then $m(\rho) \le m(p) < 0$ for all $\rho \ge 0$ and there is $\rho^* > 0$ such that $\lim_{\rho \to \rho^*} r(x(\rho)) = 0$.

Proof : Introduce a Lorentz orthonormal frame e_0, e_1 along $x(\rho)$ such that $e_1 = x_*(\partial_\rho)$. Regarding m, r as functions of ρ we have

$$m = \tfrac{1}{2} r \left(1 + (D_0 r)^2 - (D_1 r)^2\right) \tag{43}$$

$$\frac{\partial m}{\partial \rho} = D_1 m = \tfrac{1}{2} r^2 (D_1 r \, G_{00} - D_0 r \, G_{11}). \tag{44}$$

Since $m(0) < 0$, (43) shows $(D_1 r)^2 = (dr/d\rho)^2 \ge 1$, and thus by continuity, $dr/d\rho \le -1$ for $\rho \ge 0$. The existence of $\rho^* < r(0)$ follows, since $r(0) > 0$. Since $G_{00} \ge |G_{11}|$ by hypothesis, it follows from (44) that $\partial m / \partial \rho \le 0$ and thus $m(\rho) \le m(p) < 0$ for all $\rho > 0$. ■

In particular, $x(\rho^*)$ is a singular "point", since the curvature is $O(mr^{-3})$ which is unbounded as $\rho \longrightarrow \rho^*$. As an easy corollary we obtain the positive mass theorem for spherically symmetric spacetimes.

Corollary 5 *Assume the dominant energy condition holds. Suppose $\rho \mapsto x(\rho)$, $0 \le \rho < \infty$, defines a spacelike surface (curve) such that $m(0) = 0$, $r(0) = 0$ (ie. $\rho = 0$ is a regular symmetry axis), and $r(\rho) \to \infty$ as $\rho \to \infty$. Then $m(\rho) \ge 0$ for all ρ.*

Proof : If $m(\rho_0) < 0$ for any $\rho_0 > 0$ then the previous result applies and shows that $r(\rho^*) = 0$ and $m(\rho^*) \neq 0$ for some $\rho^* > \rho_0$. This contradicts either the axis regularity condition (at $\rho = 0$) or the asymptotic hypothesis $r \to \infty$. ∎

A more detailed argument shows that if $m(\rho_0) = 0$ then the resulting spacetime is flat in the domain of dependence of $x([0, \rho_0])$.

In the exterior region where $m < r/2$ we have $dr/d\rho > 0$ since $r \to \infty$, and thus by the dominant energy condition again we find $\partial m/\partial \rho \geq 0$. In summary, global regularity conditions (at $\rho = 0$ and at $\rho = \infty$) together with the dominant energy condition, force the mass function to be non-negative. Conversely, if the mass becomes negative (from a regular central axis), then every spacelike curve from the negative point and away from the axis will reach a singularity at finite geodesic radius — this says the singular set, where $r = 0$ and $m < 0$, is "timelike", and provides the model behaviour for a naked singularity.

Extending these ideas to non-spherical spacetimes is a challenging task, and there are many unresolved questions.

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