FUNDAMENTAL CONSTRUCTIONS IN GEOMETRIC MEASURE THEORY

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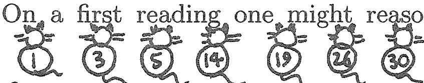
Contents.

A note on the notes.

In the main body of these note we give very little in the way of proofs. Statements which need proof are indicated by a . A  means the result is somewhere between "immediate" and "routine", and the proof can be regarded as an exercise. A  means the result is somewhere between "difficult" and "omigod". In an extended appendix we give references and proofs for all of the .

On a first reading one might reasonably attempt and/or read the proofs of 

. These are boxed in the notes.

References.

For the material covered in these notes, the best references are:

[EG] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton, 1992;

[Mo] F. Morgan, *Geometric Measure Theory: A Beginner's Guide*, Academic Press, Boston, 1988;

[Si] L. Simon, *Lectures on Geometric Measure Theory*, Proc. Centre Math. Anal., Aust. Nat. Univ., Canberra, 1984.

Two more good references are

[HS] R. Hardt and L. Simon, *Seminar on Geometric Measure Theory*, DMV **7**, Birkhäuser, Basel, 1986;

[Z] W. P. Ziemer, *Weakly Differentiable Functions*, GTM **120**, Springer-Verlag, Berlin, 1989.

The comprehensive but user-unfriendly reference is

[Fe] H. Federer, *Geometric Measure Theory*, Grundlehren **153**, Springer-Verlag, Berlin, 1969.

A complete list of the works cited is given at the end of the notes.

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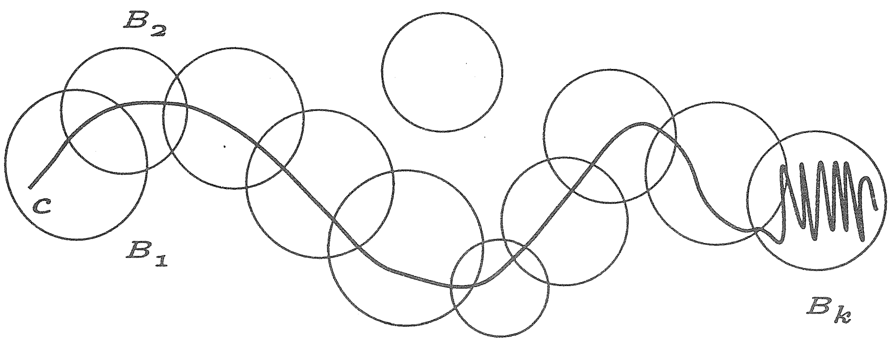
* *Destroyer of Empires, Scourge of Infidels.*

In a sentence, the idea behind geometric measure theory is to generalize the notion of "n-dimensional submanifold", allowing one to consider limits and subsequently to obtain existence (compactness) theorems. (More extensive motivation is given in the notes of Maria Athanassenas and Frank Morgan, contained in these Proceedings). Our intention here is to to give some of the underlying measure-theoretic constructions needed for this generalization procedure.

The fundamental notion is that of the *n-dimensional volume* of a (possibly nasty) subset of $\mathbb{R}^p$. (Recall from [TheHutch1,§1.2] that Lebesgue measure, $\mathcal{L}^p$, gives a notion of p-dimensional volume in $\mathbb{R}^p$, but Lebesgue gives no notion of lower dimensional volume in $\mathbb{R}^p$). This is the role played by *n-dimensional Hausdorff measure*, $\mathcal{H}^n$. To motivate the definition, consider a curve c in $\mathbb{R}^2$.



Covering c by balls (i.e. disks), we can hope that

$$\text{Length}\,(c) \overset{?}{\approx} \sum_{j=1}^{k} \text{diam}\,(B_j)\,.$$

There are two obvious problems with this approximation to Length (c):

(i) The sum may be too large because of wasted overlap or poorly placed balls. To compensate for this we need to take an *inf* over possible coverings. Of course this issue also arises in the definition of Lebesgue measure.

(ii) The sum may be two small because one big ball can cover a lot of lengthy wriggling of c (e.g. B_k in the picture above). To compensate for this we need to progressively consider coverings of c consisting of smaller and smaller sets. This issue does *not* arise in the definition of Lebesgue measure.

We note also:

(iii) For a technical reason it is helpful to consider coverings by arbitrary sets C_j rather than just balls B_j. (See Remark (b) after Theorem 1 below).

(iv) In approximating/defining n-dimensional volume, the quantity diam B_j is replaced by $\omega_n(\frac{\text{diam}\,C_j}{2})^n$, where $\omega_n = \mathcal{L}^n\,(B_1(0)) = \text{Vol}\,(\text{unit } n\text{-ball})$. (To see this quantity is reasonable, consider C_j a ball cutting off a piece of n-plane that passes through the centre of C_j).

(v) For non-compact sets we want to allow a covering to contain countably infinitely many sets.

Juggling all this motivation, we come up with ([EG,§2],[Mo,§2.3],[Si,§2]):

Definition ($\mathcal{H}^n_\delta$-approximating measure, $\mathcal{H}^n$-measure).

Suppose $n \geq 0$, $0 < \delta \leq \infty$ and $A \subseteq \mathbb{R}^p$. Then we define

$$\mathcal{H}^n_\delta(A) = \inf\left\{\sum_{j=1}^{\infty} \omega_n \left(\frac{\operatorname{diam} C_j}{2}\right)^n : A \subseteq \bigcup_{j=1}^{\infty} C_j,\ \operatorname{diam} C_j \leq \delta\right\}$$

$$\mathcal{H}^n(A) = \lim_{\delta \to 0+} \mathcal{H}^n_\delta(A)$$

Remarks.

(a) Since $\mathcal{H}^n_\delta(A)$ increases as δ decreases, $\mathcal{H}^n(A)$ is well-defined.

(b) We take $\omega_0 = 1$. This is justified by Theorem 1(vi) below.

(c) n need not be an integer in the above definitions, though it usually will be for us. When n is not an integer we take ω_n to be any positive constant. (For consistency, it is reasonable to take $\omega_n = \pi^{\frac{n}{2}}/\Gamma(\frac{n}{2}+1)$ where Γ is the gamma function - see [St,pp394-395]).

(d) It should be clear that Hausdorff measure can be similarly defined on any metric space. Much of our discussion below applies in this more general setting, but we shall not make further comment on this.

The following accumulation of facts shows that Hausdorff measure in general is well-behaved and in particular agrees with other notions of n-dimensional volume in familiar special cases.

Theorem 1 (Fundamental properties of Hausdorff measure).

(i) *$\mathcal{H}^n_\delta$ is a measure (i.e. an outer measure).*

(ii) *$\mathcal{H}^n$ is a Borel regular measure. $\mathcal{H}^n$ will not in general be Radon, but if $E \subset \mathbb{R}^p$ is $\mathcal{H}^n$-measurable with $\mathcal{H}^n(E) < \infty$ then the restriction $\mathcal{H}^n \lfloor E$ is Radon.*

(iii) *Suppose $m > n$. Then*

$$\begin{cases} \mathcal{H}^n(A) < \infty \implies \mathcal{H}^m(A) = 0, \\ \mathcal{H}^m(A) > 0 \implies \mathcal{H}^n(A) = \infty. \end{cases}$$

(iv) *$\mathcal{H}^n$ is invariant under isometries.*

(v) *Generalizing (iv), if $f:\mathbb{R}^p \to \mathbb{R}^q$ is Lipschitz and if $A \subseteq \mathbb{R}^p$ then*

$$\boxed{\mathcal{H}^n(f(A)) \leq (\operatorname{Lip} f)^n \, \mathcal{H}^n(A)}$$

(Recall that f is Lipschitz if there is a constant $K < \infty$ such that $|f(x) - f(y)| \leq K|x-y|$ for all $x, y \in \mathbb{R}^p$. Lip f *is the best such constant K).*

[6] (vi) *$\mathcal{H}^0$ is counting measure:*

$$\mathcal{H}^0(A) = \begin{cases} \textit{number of elements in } A & A \textit{ is finite,} \\ \infty & A \textit{ is infinite.} \end{cases}$$

[7] (vii) $\mathcal{H}^p = \mathcal{L}^p$ *on* $\mathbb{R}^p$.

[8] (viii) *If $M^n \subseteq \mathbb{R}^p$ is an embedded n-dimensional C^1-submanifold then*

$$\mathcal{H}^n(M) = \mathrm{Vol}(M) \qquad \textit{(e.g. by "}\sqrt{g}\textit{"-definition).}$$

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Remarks.

(a) The import of (iii) is that for $A \subseteq \mathbb{R}^p$ there is at most one exponent n such that $0 < \mathcal{H}^n(A) < \infty$. No such exponent need exist (see [25]), but we can always define the *Hausdorff Dimension* of A by

$$\dim A \equiv \sup\{n : \mathcal{H}^n(A) = \infty\} = \inf\{m : \mathcal{H}^m(A) = 0\}.$$

Also, by (vii), $A \subseteq \mathbb{R}^p \implies \dim A \leq p$. More generally, (viii) implies that (separable) immersed n-submanifolds of $\mathbb{R}^p$ are Hausdorff n-dimensional.

(b) A proof of (v) will clearly involve using f to transform coverings of A to $f(A)$. Notice than even if we begin with a covering of A by balls, the transformed covering need not consist of balls. It is for this reason that we allow coverings by arbitrary sets in the definition of $\mathcal{H}^n_\delta$.

(c) The proof of (vii) is quite involved - in particular one needs an application of the *Vitali Covering Theorem*, the statement and proof of which is given in [7].

(d) The proof of (viii) is not difficult, given (v), (vii) and the change of variables formula for Lebesgue integration: see ([TheHutch1,§3.2]) and [13]. (viii) is in fact a special case of an important result, the *Area Formula*, which we now describe.

Suppose $p \geq n$ and $f : \mathbb{R}^n \to \mathbb{R}^p$ is C^1. Given $a \in \mathbb{R}^n$ consider the derivative $Df(a)$, which we think of as both a linear map and as a $p \times n$ matrix. Since $Df(a)$ is not square (unless $p = n$) we cannot obtain a Jacobian factor by taking the determinant. However, we can write

[9] (♦) $$Df(a) = \rho \circ \sigma$$

where $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ is linear and $\rho : \mathbb{R}^n \to \mathbb{R}^p$ is a linear and orthogonal injection. Now we have

Definition. The *Jacobian* $Jf(a)$ of f at $a \in \mathbb{R}^n$ is defined by

$$Jf(a) \;=\; |\det \sigma|\;.$$

(10) *Remarks.*

(a) The decomposition ($\blacklozenge$) is not unique, but $Jf(a)$ is independent of the decomposition chosen.

(b) It is clear that when $p = n$ the above definition of $Jf(a)$ reduces to the usual one.

(11) (c) Note that

$$Jf(a) = 0 \iff \operatorname{rank}(Df(a)) < n\,.$$

(12) **Theorem 2 (Area Formula).** *Suppose $U \subseteq \mathbb{R}^n$ is open with $f : U \to \mathbb{R}^p$ a C^1 function. If $A \subseteq U$ is $\mathcal{L}^n$-measurable and f is injective on A then*

$$\boxed{\mathcal{H}^n\,(f(A)) \;=\; \int_A Jf\, d\mathcal{L}^n}$$

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(13) *Remarks.*

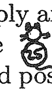
(a) Suppose $\phi : U \to \mathbb{R}^n$ is a coordinate map for an embedded C^1 submanifold $M^n \subset \mathbb{R}^p$ ([BartMan,§2],[Bo,p73]). Let $f = \phi^{-1}$ and set (as usual) $g_{ij} = \langle D_i f, D_j f\rangle$ and $g = \det(g_{ij})$. Then $Jf = \sqrt{g}$, and so Theorem 1(viii) is a special case of the Area Formula.

(b) Theorem 2 is more general than Theorem 1(viii) because there is no assumption that Df has rank n. In fact, from Remark (c) above,

$$\mathcal{H}^n\,(f(\{a : \operatorname{rank}(Df(a)) < n\})) \;=\; \mathcal{H}^n\,(f(\{a : Jf(a) = 0\})) \;= 0\,.$$

This is an important special case (combined with (c) below).

(c) There are a number of natural generalizations of the area formula. One possibility is to allow f to be non-injective, and we note that the claim in the previous remark continues to hold in this setting: see (12) (Part 5). For other generalizations, see [EG,§3.3], [Mo,§3.7], [Si,§§8,12], and (20) and (35) below.

Given $M \subseteq \mathbb{R}^p$ and $n \in \mathbb{Z}^+$, we have a notion of M being n-dimensional (Remark (a) after Theorem 1), but this is too weak to give a useful generalization of n-manifolds. For one thing, M being n-dimensional does not imply anything about the positivity or local finiteness (or σ-finiteness) of $\mathcal{H}^n(M)$ (see ). Moreover, even if $0 < \mathcal{H}^n(M) < \infty$, there is simply no reason why M should possess any weak manifoldish properties: see, for example, the Besicovitch Set below. Instead, we consider the much more restricted class of *countably n-rectifiable sets*, the fundamental objects of study in geometric measure theory ([Mo,§3],[Si,Ch3]). There are a number of equivalent definitions one can make: we start with the (apparently) weakest one.

Definition. A set $M \subseteq \mathbb{R}^p$ is *countably n-rectifiable* (or *rectifiable* for short) if we can write

$$\boxed{M = M_0 \cup \bigcup_{j=1}^{\infty} f_j(\Omega_j)}$$

where $\mathcal{H}^n(M_0) = 0$, each $\Omega_j \subseteq \mathbb{R}^p$, and each $f_j : \Omega_j \to \mathbb{R}^p$ is Lipschitz.

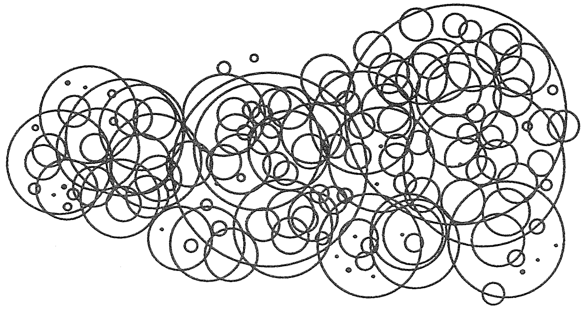
Clearly (separable) C^1 submanifolds are rectifiable. As well, the space of rectifiable sets is closed under the taking of subsets, countable unions and Lipschitz images.

Warning.

Our very first comment in §A might lead the reader to believe that the space of rectifiable sets has compactness properties, but this is not the case. The objects of study in geometric measure theory are defined in terms of rectifiable sets, but determining the precise spaces for which compactness theorems holds takes much more work. See [Mo,§5] and [Si,§32].

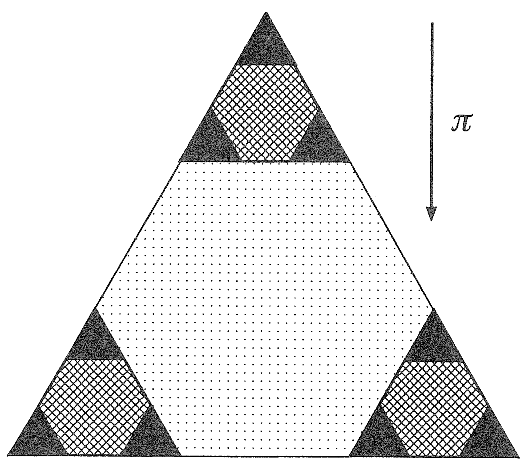
Example.

We give an example to show how bad rectifable sets can be; for another, similarly nasty, see [Mo,pp30-31]. Let $\{a_j\}_{j=1}^{\infty} \subset \mathbb{R}^2$ be an enumeration of the points with rational coordinates. Let S_j be the circle (not the disk) of radius $1/2^j$ about a_j. Now let $S = \cup_j S_j$ (see the picture on the following page). S is obviously 1-rectifiable but, topologically, S is pretty disgusting. Note that $\overline{S} = \mathbb{R}^2$ and that this is not changed by removing any set of $\mathcal{H}^n$-measure zero from S. S is not a set you would want to take home to meet your mother.



Non-non-example. We want give a non-example, a set which is not rectifiable. We first point out that the Cantor set C ([TheHutch1,§1.2.3]) is a non-non-example: since $\mathcal{H}^1(C) = \mathcal{L}^1(C) = 0$, the Cantor set is trivially and boringly 1-rectifiable. (For the calculation of the dimension of C, and of its variants, see ).

Non-example (due to Besicovitch). Let D_0 be the closed (filled-in) equilateral triangle of sidelength 1. Let $D_1 \subset D_0$ be the union of three triangles of sidelength $1/3$, as pictured. Proceeding iteratively, $D_j \subset D_{j-1}$ consists of 3^j triangles of sidelength $1/3^j$ placed in the obvious manner. We then define the *Besicovitch Set* to be $D = \cap_j D_j$.



We first prove $\mathcal{H}^1(D) = 1$, implying D is relevant to the discussion. For any δ, consider j with $\frac{1}{3^j} \leq \delta$. Using the obvious covering of D_j, we obtain

$$\begin{aligned} &\mathcal{H}^1{}_\delta(D) \leq \mathcal{H}^1{}_\delta(D_j) \leq 1 \\ \Longrightarrow\ &\mathcal{H}^1(D) \leq 1 \qquad\qquad (\text{letting } \delta \to 0) \end{aligned}$$

This is the easy direction - because $\mathcal{H}^1_\delta$ is an *inf*, upper bounds for Hausdorff measure are usually not difficult to obtain. In general, obtaining a good (sometimes any) lower bound can be difficult (see ). Here, however, we can use a trick. Let

$\pi : \mathbb{R}^2 \to \mathbb{R}$ be orthogonal projection onto the x_1-axis. Noting that $\operatorname{Lip} \pi = 1$, Theorem 1(v), (vii) give

$$\mathcal{H}^1(D) \geq \mathcal{H}^1(\pi(D)) = \mathcal{H}^1([0,1]) = \mathcal{L}^1([0,1]) = 1\,.$$

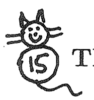
This gives us the desired conclusion that $\mathcal{H}^1(D) = 1$. (In this calculation, some work is hidden in the reference to Theorem 1(vii) and in the use of information about Lebesgue measure).

D certainly looks quite non-rectifiable. To clarify matters, we introduce a new notion ([Si,§13],[HS,p28]):

Definition.

$P \subset \mathbb{R}^p$ is *purely n-unrectifiable* if $\mathcal{H}^n(P \cap M) = 0$ for every countably n-rectifiable set M.

One can show directly that the Besicovitch set is purely 1-unrectifiable. Alternatively, this will follow from our density theorems below (see Corollary 10). In general we have



Theorem 3 (Decomposition Theorem).

Suppose $A \subseteq \mathbb{R}^p$ with $\mathcal{H}^n(A) < \infty$. Then

$$A = M \cup P$$

where M is countably n-rectifiable, P is purely n-unrectifiable and $M \cap P = \emptyset$. This decomposition is unique up to sets of zero $\mathcal{H}^n$-measure.

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We shall not consider them further, but there are some beautiful and important theorems on purely unrectifiable sets; we note in particular Federer's *Structure Theorem* ([Si,Th13.2],[Ma,§6],[Ros]). The best references for such material are probably [Fa] and [Ma].

Returning to the study of rectifiable sets, we want to show that such sets are not as bad as one might initially be led to believe. We do this by showing that Lipschitz functions are quite well behaved. To begin, we have

Theorem 4 (Lipschitz Extension Theorem). *Suppose $\Omega \subseteq \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^p$ is Lipschitz. Then there is a Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}^p$ such that $f = g$ on Ω.*

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The obvious consequence of Theorem 4 is

$$(M \subseteq \mathbb{R}^p \text{ is } n\text{-rectifiable})$$
$$\iff \left(M \subseteq M_0 \cup \bigcup_{j=1}^{\infty} f_j(\mathbb{R}^n) \text{ with } \begin{cases} \mathcal{H}^n(M_0) = 0 \\ f_j : \mathbb{R}^n \to \mathbb{R}^p \text{ Lipschitz} \end{cases} \right).$$

Next, we investigate the differentiability properties of Lipschitz functions; from Sobolev space theory we know that Lipschitz functions are weakly differentiable with L^∞ weak derivatives ([TheHutch1,§4.4],[EG,p131],[Z,Th2.1.4]); however, it is real-live classical derivatives we are interested in here. Of course a Lipschitz function need not be everywhere differentiable, as is illustrated by the function $f(x) = |x|$, but the next theorem indicates that this is almost the case.

Theorem 5 (Rademacher's Theorem).

Suppose $f : \mathbb{R}^n \to \mathbb{R}^p$ is Lipschitz. Then f is (classically) differentiable $\mathcal{L}^n$-almost everywhere.

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In fact, more than just being "almost differentiable", Lipschitz functions are "almost C^1":

Theorem 6 (Whitney's Extension Theorem).

Suppose $f : \mathbb{R}^n \to \mathbb{R}^p$ is Lipschitz and let $\epsilon > 0$. Then there is a C^1 function $g : \mathbb{R}^n \to \mathbb{R}^p$ and a closed set $C \subseteq \mathbb{R}^n$ such that:

(i) $\mathcal{L}^n(\mathbb{R}^n \sim C) \leq \epsilon$;

(ii) $f = g$ *and* $Df = Dg$ *on* C.

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Combining with the Area Formula (non-injective version: see Remarks (b) and (c) after Theorem 2), we have

Corollary 7.

$M \subseteq \mathbb{R}^p$ is countably n-rectifiable iff

$$M \subseteq N_0 \cup \bigcup_{j=1}^{\infty} N_j ,$$

*where $\mathcal{H}^n(N_0) = 0$ and each, for each $j \leq 1$, N_j is an n-dimensional regularly and properly embedded C^1 submanifold with boundary.**

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There is a general Littlewoodish Principle ([TheHutch1,§2.1.3]) at work here:

$$\left.\begin{array}{c} C^1 \text{ fact} \\ \bigoplus \\ \text{Whitney} \end{array}\right\} \implies \text{Lipschitz fact.}$$

*By the term "regularly embedded" we mean N_j is embedded in the nicest possible sense: for each a in the interior of N_j there is an open $U \subseteq \mathbb{R}^p$ containing a and a chart $\phi : U \to V$ for $\mathbb{R}^p$ such that $\phi(N_j \cap U) = \mathbb{R}^n \cap \phi(V)$; see [Bo,§3.5]. The charts around boundary points of N_j, if they exist, are similarly defined; of course, by Theorem 1(iii),(viii), the $\mathcal{H}^n$-measure of the boundary will be zero.

As another instance of this principle, we have

(20) **Corollary 8.** *The Area Formula (any version) holds for Lipschitz functions.*

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Geometric measure theory arguments are often based on the following generalization of this principle:

$$\left.\begin{array}{c}\text{Manifold fact}\\ \bigoplus \\ \text{Whitney}\end{array}\right\} \implies \text{Rectifiable fact.}$$

Illustrations of this principle are given in the following two sections.

C. Densities

Generalizing the notion of densities for Lebesgue measure ([EG,p45]), we have

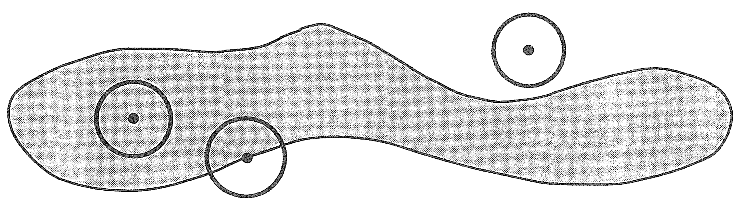
Definition.
Suppose $M \subseteq \mathbb{R}^p$ and $a \in \mathbb{R}^p$. Then the *n-dimensional density* of M at a is

$$\boxed{\Theta^n(M,a) = \lim_{r\to 0} \frac{\mathcal{H}^n(M \cap B_r(a))}{\omega_n r^n}}$$

where $B_r(a)$ is the closed ball of radius r about a.

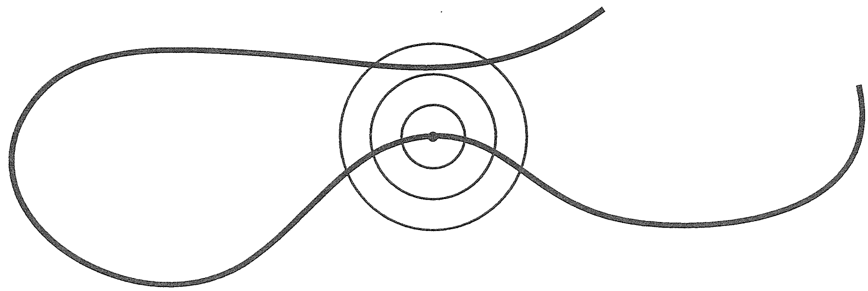
From our Lebesgue knowledge ([EG,p45], or [TheHutch1, Th3.4] applied to $f = \chi_M$) we know that if $M \subseteq \mathbb{R}^n$ is $\mathcal{L}^n$-measurable then

$$\begin{cases} \Theta^n(M,a) = 1 & \text{for } \mathcal{H}^n\text{-a.e. } a \in M, \\ \Theta^n(M,a) = 0 & \text{for } \mathcal{H}^n\text{-a.e. } a \in \sim M. \end{cases}$$



We remark that the measurability of M is only needed for the second of these density results; see (26).

(21) Using the Area Formula, one sees that these results also hold (more easily) if M is a properly and regularly embedded submanifold with boundary ([Bo,p77]. Note Example (b) below).



In general these equations need not hold; in fact, the density of M need not even be well-defined at any $a \in M$ (see Example (a) below). We are forced to consider the *upper density* and *lower density* of $M \subseteq \mathbb{R}^p$, these quantities given respectively by

$$\Theta^{n*}(M,a) = \overline{\lim_{r\to 0}} \frac{\mathcal{H}^n(M \cap B_r(a))}{\omega_n r^n}$$

$$\Theta^n_*(M,a) = \underline{\lim_{r\to 0}} \frac{\mathcal{H}^n(M \cap B_r(a))}{\omega_n r^n}$$

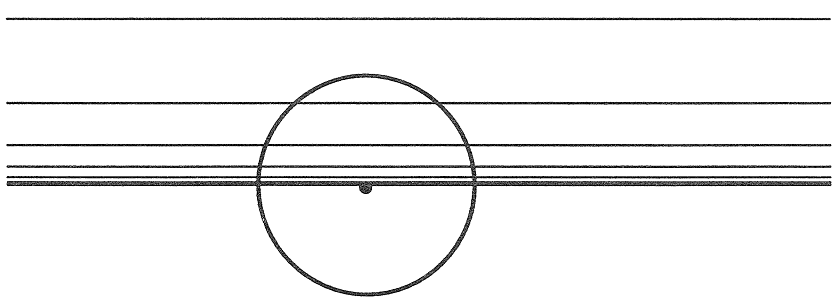
Examples.

(a) It is not hard to show that for the Besicovitch Set D

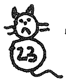
$$\begin{cases} \Theta^{1*}(D,a) \geq 1/2 \\ \Theta^1_*(D,a) \leq 1/2 \end{cases} \quad \text{for all } a \in D.$$

With a bit of work, the estimate on the lower density can be improved to $\frac{3}{10}$ for $\mathcal{H}^1$-almost all $a \in D$. Consequently the 1-density of D is undefined at almost all points in D.

(b) Let $A = \cup_{j=1}^{\infty}(\mathbb{R} \times \{1/j\}) \subset \mathbb{R}^2$. Clearly $\Theta^1(A,a) = \infty$ for every $a \in \mathbb{R} \times \{0\}$.



This second example shows that for nice density results we need some finiteness condition on $\mathcal{H}^n(A)$, even for rectifiable sets. (Local finiteness is enough, but we won't spell that out each time).



Theorem 9.

Suppose $M \subset \mathbb{R}^p$ *with* $\mathcal{H}^n(M) < \infty$. *Then*

(i) $\frac{1}{2^n} \leq \Theta^{n*}(M,a) \leq 1$ *for* $\mathcal{H}^n$*-a.e.* $a \in M$,

(ii) *If* M *is* $\mathcal{H}^n$*-measurable then* $\Theta^n(M,a) = 0$ *for* $\mathcal{H}^n$*-a.e.* $a \in \sim M$.

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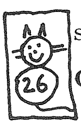
Remarks.

(a) The proofs of these results involve the Vitali Covering Theorem (see  and ).

(b) The previously mentioned density results for Lebesgue measure follow easily from Theorem 9 (ii).

(c) [Fe,§3.3.19] shows that the lower bound $\frac{1}{2^n}$ is optimal. The Besicovitch set also shows that the bound $\frac{1}{2^n}$ does not hold for lower densities. In fact no non-trivial lower bound exists: one can construct a Besicovitch-type set $\widehat{D}$ for which $\Theta_1^*(\widehat{D},a) = 0$ for all $a \in \widehat{D}$.

As a simple corollary to Theorem 9, we conclude that, densitywise, rectifiable sets behave like manifolds:



Corollary 10.

Suppose M *is countably* n*-rectifiable and* $\mathcal{H}^n$*-measurable with* $\mathcal{H}^n(M) < \infty$. *Then*

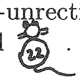
$$\begin{cases} \Theta^n(M,a) = 1 & \text{for } \mathcal{H}^n\text{-a.e. } a \in M \\ \Theta^n(M,a) = 0 & \text{for } \mathcal{H}^n\text{-a.e. } a \in \sim M \end{cases}$$

♣◇♡♠

Remarks.

(a) It is worth noting that, as is the case for Lebesgue density, the measurability of M is only needed for the second of these density results. See the proof of  .

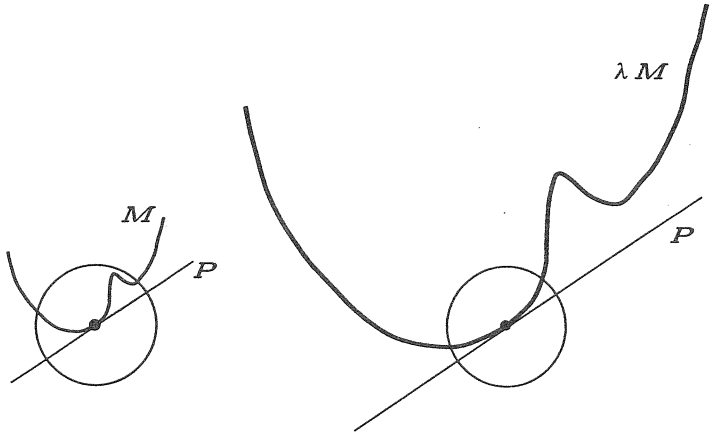
(b) We also point out that only Theorem 9(ii) is used in the proof of Corollary 10. Thus, in the context of rectifiable sets, we need not actually deal with upper and lower densties.

(c) The pure 1-unrectifiability of the Besicovitch Set follows readily from Corollary 10 and  .

D. Tangent Planes

Continuing our analysis of rectifable sets as approximate manifolds, we consider the existence of tangent planes. Of course we cannot in general hope for the existence of classical tangent planes: the rectifiable set given in §B will not have a classical tangent at any point, even if we ignore sets of $\mathcal{H}^1$-measure zero. We must instead consider a weak, measure-theoretic notion of tangency ([Si,§11], [HS,§2.3]).

Given $M \subseteq \mathbb{R}^p$, $a \in \mathbb{R}^p$ and an n-plane $P \subseteq \mathbb{R}^p$ passing through a, we shall define the notion of P being an *approximate tangent plane* to M at a. For notational simplicity, we'll consider $a = 0$ and P passing through 0. Given a compactly supported *test-function* $\phi \in C_0(\mathbb{R}^p)$, integrating ϕ over M (with respect to $\mathcal{H}^n$) should approximately be the same as integrating ϕ over P; if we scale M (i.e. replace M by λM with λ large) then the approximation should become better and better.



Definition.
P is an *approximate tangent plane to* M *at* 0 if

$$(\blacklozenge) \qquad \boxed{\lim_{\lambda \to \infty} \int_{\lambda M} \phi \, d\mathcal{H}^n = \int_P \phi \, d\mathcal{H}^n \qquad \text{for all } \phi \in C_0(\mathbb{R}^p)}$$

Remarks.

(a) We can write ($\blacklozenge$) as the weak convergence of measures ([HS,p22],[EG,§1.9]):

$$\mathcal{H}^n \lfloor (\lambda M) \rightharpoonup \mathcal{H}^n \lfloor P.$$

(b) Scaling ϕ instead of A, we can write ($\blacklozenge$) as

$$\lambda^n \int_M \phi(\lambda x) \, d\mathcal{H}^n(x) \longrightarrow \int_P \phi \, d\mathcal{H}^n.$$

(c) For tangents at a general $a \in \mathbb{R}^p$, the appropriate scaled set is $a + \lambda(M - a)$.

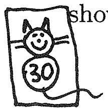
(d) The point a need not be in M for M to have a tangent plane at a. Note, however, 30 below.

28 (e) If the tangent plane to M at a exists then it is unique.

29 (f) It is not hard to see that classical tangent planes to regularly embedded submanifolds are also approximate tangent planes. It is also easy to see the converse is false, since approximate tangent planes ignore sets of $\mathcal{H}^n$-measure zero.

(g) More generally, one has the very important notion of a *tangent cone* to a given set. See [Mo,§9] and [Si,§35].

There are close ties between tangent planes and densities. Choosing the test function ϕ to approximate the characteristic function of a ball, it is not hard to show:



$$M \text{ has an approximate tangent plane at } a \implies \Theta^n(M,a) = 1\,.$$

More importantly,

Lemma 12.
Suppose $M, A \subseteq \mathbb{R}^p$, $a \in \mathbb{R}^p$, and suppose $\Theta^n(M \sim A, a) = \Theta^n(A \sim M, a) = 0$. Then M has an approximate tangent plane at a iff P does, and the tangent planes agree.

♣♢♡♠

This lemma, together with Theorem 9(ii), is quite powerful. In particular, it is easy to prove

Theorem 13.
Supppose $M \subseteq \mathbb{R}^p$ is countably n-rectifiable and $\mathcal{H}^n$-measurable with $\mathcal{H}^n(M) < \infty$. Then M has an approximate tangent plane at $\mathcal{H}^n$-a.e. $a \in M$. In fact, in the notation of Corollary 7, the tangent plane to M is the same as the tangent plane to N_j at almost all points of $M \cap N_j$.

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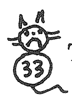
The converse is true, though less routine:



Theorem 14.
Supppose $M \subseteq \mathbb{R}^n$ is $\mathcal{H}^n$-measurable with $\mathcal{H}^n(M) < \infty$, and suppose M has an approximate tangent plane at $\mathcal{H}^n$-a.e. $a \in M$. Then M is countably n-rectifiable.

♣♢♡♠

Final Remarks.

(a) These results make concrete the Decomposition Theorem: if $A \subseteq \mathbb{R}^p$ is $\mathcal{H}^n$-measurable with $\mathcal{H}^n(A) < \infty$ then the rectifiable part of A is exactly the set of points in A at which A has an approximate tangent plane.

(b) A very important consequence of Theorem 13 is that if $M \subseteq \mathbb{R}^p$ is countably n-rectifiable and $f : M \to \mathbb{R}^p$ is Lipschitz then, at $\mathcal{H}^n$ a.e. $a \in M$, we can define the *tangential derivative* $D^M f$ of f at a. As a consequence we can define a suitable Jacobian factor $J^M f$, and we have the appropriate *Area Formula*:

$$\mathcal{H}^n(f(A)) = \int_A J^M f \, d\mathcal{H}^n \qquad (A \subset M,\ f \text{ injective})$$

The fact that rectifiable sets obey such an area formula, together with other results of a similar nature, is what allows us to successfully treat such sets analytically.

(1) [EG,p61].

The issue, of course, is to show that $\mathcal{H}^n_\delta$ is *countably subadditive* ([TheHutch1, §1.3]); the proof is identical to that for Lebesgue measure. Suppose $A \subseteq \cup_k A_k$ and, for each k, let $\{C_{jk}\}$ be a covering of A_k. Given $\epsilon > 0$, we can choose the C_{jk} so that $\operatorname{diam} C_{jk} \leq \delta$ and

$$\sum_{j=1}^{\infty} \omega_n \left(\frac{\operatorname{diam} C_{jk}}{2} \right)^n \leq \mathcal{H}^n_\delta(A_k) + \frac{\epsilon}{2^k}.$$

Then, since $A \subseteq \cup C_{jk}$,

$$\mathcal{H}^n_\delta(A) \leq \sum_{k=1}^{\infty} \mathcal{H}^n_\delta(A_k) + \epsilon.$$

Letting $\epsilon \to 0$ gives the desired result.

(2) [EG,pp61-63], [Si,p7].

Letting $\delta \to 0$, it easily follows from (1) that $\mathcal{H}^n$ is a measure. To prove $\mathcal{H}^n$ is Borel, we apply the *Carathéodory Criterion* ([EG,Th1.1.5],[Si,Th1.2],[TheHutch1, p6]): the point is to prove

$$\operatorname{dist}(A, B) > \delta \implies \mathcal{H}^n_\delta(A \cup B) = \mathcal{H}^n_\delta(A) + \mathcal{H}^n_\delta(B),$$

(which is easy), and then let $\delta \to 0$.

To show that $\mathcal{H}^n$ is regular ([TheHutch1,§1.4.1]), first note that in the definition of $\mathcal{H}^n_\delta$ we need only consider coverings of a given A by *closed* sets (because, for any set C, $\operatorname{diam}(\overline{C}) = \operatorname{diam} C$). This implies that for any $k \in \mathbb{N}$, we can find a Borel set $D_k \supseteq A$ for which

$$\mathcal{H}^n_{1/k}(D_k) \leq \mathcal{H}^n_{1/k}(A) + 1/k.$$

(D_k can be a countable union of closed sets from a suitable covering of A). Setting $D = \cap D_k$, we see $D \supseteq A$ is Borel with $\mathcal{H}^n(D) = \mathcal{H}^n(A)$.

For $n > p$, $\mathcal{H}^n$ will not be Radon on $\mathbb{R}^p$ ([TheHutch1,§1.4.2]), as Theorem 1(vii) implies that $\mathcal{H}^n$ will not be finite on compact sets. The fact that $\mathcal{H}^n \lfloor A$ will be Radon for suitable A follows from the definitions: see [EG,§1.1].

(3) [EG,p65].

The critical (and not difficult) fact is

$$\mathcal{H}^m_\delta(A) \le \frac{\omega_m}{\omega_n}\left(\frac{\delta}{2}\right)^{m-n} \mathcal{H}^n_\delta(A)\,.$$

The desired results now follow by letting $\delta \to 0$.

(4) [EG,p63].

The result is trivial, since isometries don't change diameters.

(5) [EG,p75].

If $A \subseteq \cup C_j$ then $f(A) \subseteq \cup f(C_j)$. Also $\operatorname{diam}(f(C_j)) \le (\operatorname{Lip} f)(\operatorname{diam} C_j)$. Thus,

$$\mathcal{H}^n_{(\operatorname{Lip} f)\delta}((f(A)) \le \mathcal{H}^n_\delta(A)\,.$$

Letting $\delta \to 0$ gives the result.

(6) [EG,p63].

Given any $a \in \mathbb{R}^p$, it is easy to show that $\mathcal{H}^0(\{a\}) = 1$. The desired result follows from countable additivity.

(7) [EG,§2.2], [Mo,Cor2.8], [Si,pp7-10].

This one takes work. We shall actually prove that, for any $\delta \in (0, \infty)$, $\mathcal{H}^n_\delta = \mathcal{L}^n$ on $\mathbb{R}^n$. The proof will be in four parts, followed by the proof of two subsidiary results:

Part 1

$$\mathcal{L}^n(A) \le \mathcal{H}^n_\delta(A) \qquad\qquad A \subseteq \mathbb{R}^n.$$

Part 2

$$\mathcal{L}^n(A) = 0 \implies \mathcal{H}^n_\delta(A) = 0 \qquad\qquad A \subseteq \mathbb{R}^n.$$

Part 3

$$\mathcal{H}^n_\delta(B) \le \mathcal{L}^n(B) \qquad\qquad B \text{ a closed ball of diameter } \le \delta.$$

Part 4

$$\mathcal{H}^n_\delta(A) \le \mathcal{L}^n(A) \qquad\qquad A \subseteq \mathbb{R}^n.$$

★ *PROOF OF PART 1.*

We need a big fact: *of all sets with a given diameter $d < \infty$, the ball of radius $d/2$ has the largest volume.* That is,

$$\mathcal{L}^n(C) \le \left(\frac{\operatorname{diam} C}{2}\right)^n \omega_n \qquad\qquad C \subset \mathbb{R}^n .$$

We prove this result, called the *Isodiametric Inequality*, after Part 4 below. Given the Isodiametric Inequality, Part 1 is easy. Covering A by sets C_j of diameter at most δ,

$$\mathcal{L}^n(A) \le \sum_{j=1}^{\infty} \mathcal{L}^n(C_j) \le \sum_{j=1}^{\infty} \left(\frac{\operatorname{diam} C_j}{2}\right)^n \omega_n .$$

Taking the *inf* over all possible coverings gives the result.

★ *PROOF OF PART 2.*

Consider, for Lebesgue purposes, a covering of A by (unequal-sided) cubes. By cutting up the cubes, we can assume that

- $\operatorname{diam}(I) < \delta$ for each cube I.
- Each cube I is close to having sides of equal length, say that the longest side of I is no longer than twice the shortest side of I. Consequently

$$(\operatorname{diam} I)^n \le (2\sqrt{n})^n \mathcal{L}^n(I) .$$

Taking the *inf* over such coverings, it easily follows that

$$\mathcal{H}^n_\delta(A) \le \omega_n (\sqrt{n})^n \mathcal{L}^n(A) ,$$

from which Part 2 follows immediately.

★ *PROOF OF PART 3.*

By definition of ω_n, and since Lebesgue measure scales,

$$\mathcal{L}^n(B) = \left(\frac{\operatorname{diam} B}{2}\right)^n \omega_n .$$

Covering B by itself, Part 3 follows immediately.

★ *PROOF OF PART 4 (the idea).*

We want to decompose A into a set E of zero Lebesgue measure and a disjoint union of small balls B_j, after which we can apply Parts 2 and 3. To do this we need a *covering theorem.*

Definition.

A *fine covering* $\mathcal{K}$ of $A \subseteq \mathbb{R}^n$ is a collection of sets such that, for each $a \in A$,

$$\inf\{\operatorname{diam} B : a \in B \in \mathcal{K}\} = 0 .$$

Vitali Covering Theorem.

Suppose $A \subseteq \mathbb{R}^n$ is covered by a collection $\mathcal{K}$ of closed balls of uniformly bounded and non-zero diameter. Then there is a pairwise disjoint countable subcollection $\mathcal{B} \subset \mathcal{K}$ having the following property:

(♦) *For any* $C \in \mathcal{K}$ *there is a* $B \in \mathcal{B}$ *with* $C \cap B \neq \emptyset$ *and* $C \subseteq 5B$.
(Here, $5B$ *is the ball concentric with* B *of 5 times the radius).*
Consequently

(a)

$$A \subset \bigcup_{B \in \mathcal{B}} 5B \qquad \textit{(Unsubtle Vitali Lemma);}$$

(b) *If* $\mathcal{K}$ *is a fine covering of* A *then for any finite subcollection* $\mathcal{J} \subset \mathcal{B}$,

$$A \subset \left(\bigcup \mathcal{J}\right) \cup \left(\bigcup_{\mathcal{B} \sim \mathcal{J}} 5B\right) \qquad \textit{(Subtle Vitali Lemma);}$$

(c) *If* $\mathcal{K}$ *is a fine covering of* A *then*

$$\mathcal{L}^n\left(A \sim \cup \mathcal{B}\right) = 0 \qquad \textit{(Vitali Theorem).}$$

The proof of Vitali is given below, after that for the Isodiametric Inequality.
★ *PROOF OF PART 4 (the work).*

Given $A \subseteq \mathbb{R}^n$, let $\epsilon > 0$ and choose an open $W \supseteq A$ such that $\mathcal{L}^N(W) < \mathcal{L}^n(A) + \epsilon$. (We can do this because $\mathcal{L}^n$ is Radon: see [EG,pp6-8],[Si,§1.3],[TheHutch1,§1.5]).

Let

$$\mathcal{K} = \{B : B \subset W \text{ is a closed ball of non-zero diameter } \leq \delta\}\,.$$

$\mathcal{K}$ is certainly a fine covering of A. Thus, by the Vitali Theorem, there is a countable pairwise disjoint subcollection $\{B_j\}$ with

$$\mathcal{L}^n\left(A \sim \bigcup_{j=1}^{\infty} B_j\right) = 0\,.$$

Thus, using Theorem 1(i),

$$\begin{aligned}
\mathcal{H}^n_\delta(A) &\leq \mathcal{H}^n_\delta\left(A \sim \bigcup_{j=1}^{\infty} B_j\right) + \sum_{j=1}^{\infty} \mathcal{H}^n_\delta(B_j) \\
&\leq \sum_{j=1}^{\infty} \mathcal{L}^n(B_j) && \text{(Parts 1 and 2)} \\
&= \mathcal{L}^n(\cup_j B_j) && \text{(}\{B_j\}\text{ pairwise disjoint)} \\
&\leq \mathcal{L}^n(W) \\
&\leq \mathcal{L}^n(A) + \epsilon && \text{(Choice of } W\text{).}
\end{aligned}$$

Letting $\epsilon \to 0$, we get the desired result.

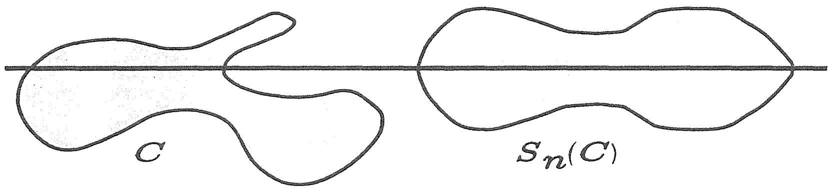
★ *PROOF OF THE ISODIAMETRIC INEQUALITY* ([EG,§2.2], [Mo, Cor2.8]).

Increasing the diameter of C by ϵ and letting $\epsilon \to 0$ at the end of the argument, we can assume that C is open. We now consider the *Steiner Symmetrization* $S_n(C)$ of C with respect the nth coordinate x_n: first, given $a \in \mathbb{R}^{n-1}$, define

$$l(a) \equiv \mathcal{H}^1(\{t : (a,t) \in C\}) .$$

Now define

$$S_n(C) \equiv \left\{(a,t) : |t| < \frac{1}{2} l(a)\right\} .$$



Note that:

- $S_n(C)$ is symmetric with respect to x_n.
  (Obvious).
- $S_n(C)$ is open
  (This comes from the fact that the function $a \mapsto l(a)$ is continuous, which itself follows easily from the fact that A is open).
- $\mathcal{L}^n(S_n(C)) = \mathcal{L}^n(C)$.
  (By Fubini ([EG,§1.4],[TheHutch1,§3.1]). Note that C and $S_n(C)$ are open and thus measurable).
- $\operatorname{diam} S_n(C) \le \operatorname{diam} C$.
  (If $(a_1,t_1),(a_2,t_2) \in S_n(C)$ then $|t_1 - t_2| < \frac{1}{2}(l(a_1) + l(a_2))$. But it is clear that, for any $\epsilon > 0$, there exist s_1 and s_2 such that $(a_1,s_1),(a_2,s_2) \in C$ and $|s_1 - s_2| > \frac{1}{2}(l(a_1) + l(a_2) - \epsilon)$).

We can perform this symmetrization with respect to any coordinate x_j, and it follows easily from the definition that

- S_j and S_k commute.

Thus the set

$$S(C) \equiv S_1(S_2(\ldots S_n(C)))$$

is symmetric with respect to *all* coordinates, and thus is symmetric with respect to the origin. Since $\operatorname{diam} S(C) \le \operatorname{diam} C$, this implies that

$$S(C) \subseteq B_{\frac{\operatorname{diam} C}{2}}(0) .$$

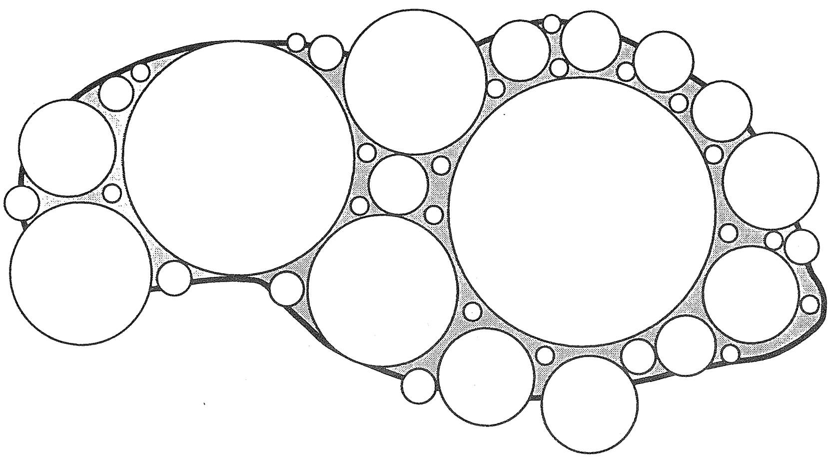
Since we also have $\mathcal{L}^n(C) = \mathcal{L}^n(S(C))$, this establishes the Isodiametric Inequality.

★ *PROOF OF THE VITALI THEOREM* ([Mo,pp27-29], [Si,pp11-12]).

Consider those pairwise disjoint subcollections $\mathcal{G}$ of $\mathcal{K}$ with the following property:

($\lozenge$) If $C \in \mathcal{K}$ with $C \cap (\cup \mathcal{G}) \neq \emptyset$ then $C \subseteq 5B$ for some $B \in \mathcal{G}$ with $C \cap B \neq \emptyset$.

The uniform bound on the diameters promises the existence of at least one such subcollection: if $B \in \mathcal{K}$ satisfies $\operatorname{diam} B \geq \frac{1}{2} \sup\{\operatorname{diam} C : C \in \mathcal{K}\}$ then $\mathcal{G} = \{B\}$ satisfies ($\lozenge$). As well, any union of subcollections satisfying ($\lozenge$) also satisfies ($\lozenge$). We can therefore apply the Hausdorff Maximal Principle to obtain a maximal subcollection $\mathcal{B}$ satisfying ($\lozenge$).



By maximality, it is clear that $\mathcal{B}$ satisfies ($\blacklozenge$) in the statement of the Vitali Theorem. Also, since the diameters of the balls are non-zero, $\mathcal{B}$ must be countable. It is also immediate that $\mathcal{B}$ satisfies (a).

To see (b), note that $\cup \mathcal{J}$ is closed. So if $a \in A \sim \mathcal{J}$ then, by the fineness of $\mathcal{K}$, $a \in C$ for some $C \in \mathcal{K}$ with $C \cap (\cup \mathcal{J}) = \emptyset$. So, by ($\blacklozenge$), $a \in C \subseteq 5B$ for some $B \in \mathcal{B} \sim \mathcal{J}$. (b) follows.

To prove (c), first intersect A with an arbitrary ball $B_R(0)$ and let $\mathcal{B}^* \subseteq \mathcal{B}$ be the collection of balls $B \in \mathcal{B}$ for which $5B$ intersects $A \cap B_R(0)$. Notice that $\cup \mathcal{B}^*$ is bounded and thus, In particular,

$$(\blacktriangle) \qquad \sum_{B \in \mathcal{B}^*} \mathcal{L}^n(B) = \mathcal{L}^n(\cup \mathcal{B}^*) < \infty .$$

It is enough to show that

$$(\blacktriangleleft) \qquad \mathcal{L}^n\left((A \cap B_R(0)) \sim \cup \mathcal{B}^*)\right) = 0 .$$

Now, by (b), for any finite $\mathcal{J} \subset \mathcal{B}$

$$\mathcal{L}^n(A - \cup \mathcal{J}) \leq \mathcal{L}^n\left(\bigcup_{\mathcal{B}^* \sim \mathcal{J}} 5B\right) \leq \sum_{\mathcal{B}^* \sim \mathcal{J}} \mathcal{L}^n(5B) = 5^n \sum_{\mathcal{B}^* \sim \mathcal{J}} \mathcal{L}^n(B) .$$

But, by (▲),

$$\sum_{\mathcal{B}^* \sim \mathcal{J}} \mathcal{L}^n(B) \longrightarrow 0 \quad \text{as } \mathcal{J} \nearrow \mathcal{B},$$

and (◀) follows.

(8) [EG,pp102-103].

This is a special case of the Area Formula, noting Theorem 1(vii). See (13).

(9) [EG,p87], [Si,p46].

We have $Df(a) : \mathbb{R}^n \to \mathbb{R}^p$. The image V of $\mathbb{R}^n$ under $Df(a)$ is at most n-dimensional, and so there is a linear isometry $\iota : \mathbb{R}^p \to \mathbb{R}^p$ with $\iota(V) \subseteq \mathbb{R}^n$. Then

$$\begin{aligned} Df(a) &= \iota^{-1} \circ (\iota \circ Df(a)) \\ &= \rho \circ \sigma, \end{aligned}$$

where $\sigma = \iota \circ (Df(a)) : \mathbb{R}^n \to \mathbb{R}^n$ and $\rho = \iota^{-1} : \mathbb{R}^n \to \mathbb{R}^p$. This is a decomposition of $Df(a)$ of the desired type.

(10) [EG,p88].

One can show directly that $Jf(a)$ is independent of the decomposition but, for work below, it is worthwhile to first give an equivalent definition of the Jacobian.

Given a linear map $\lambda : \mathbb{R}^n \to \mathbb{R}^p$, let $\lambda^* : \mathbb{R}^p \to \mathbb{R}^n$ be the *adjoint map* (or *transpose*). Then $\lambda^* \circ \lambda : \mathbb{R}^n \to \mathbb{R}^n$, and thus we can define

$$[\lambda] \equiv \sqrt{\det(\lambda^* \circ \lambda)}\,.$$

We shall show that

$$(▲) \qquad Jf(a) = [Df(a)]\,.$$

Of course the independence of $Jf(a)$ is immediate from (▲).

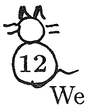
To prove (▲), we write

$$(Df(a))^* \circ Df(a) = \sigma^* \circ \rho^* \circ \rho \circ \sigma\,,$$

where σ and ρ are as in (9). Now $\rho^* \circ \rho$ is easily seen to be the identity on $\mathbb{R}^n$, and clearly $\det \sigma^* = \det \sigma$; (▲) follows immediately.

(11) [Mo,§3.6].

This follows immediately from the decomposition definition of $Jf(a)$.



[EG,§3.3], [Mo,pp25-27].

We give here a complete proof of the Area Formula, including, for completeness, a proof of the change of variables formula for Lebesgue integration (Part 3).

★ *PART 1: $f:\mathbb{R}^n\to\mathbb{R}^n$ LINEAR.*

In the case that $f:\mathbb{R}^n\to\mathbb{R}^n$ is linear we want to show

$$(\blacktriangle)\qquad \mathcal{L}^n(f(A)) = |\det f|\mathcal{L}^n(A) \qquad A\subseteq\mathbb{R}^n.$$

The key to proving this is write $f = p\circ d\circ q$ where $p,q:\mathbb{R}^n\to\mathbb{R}^n$ are linear isometries and $d:\mathbb{R}^n\to\mathbb{R}^n$ satisfies $d(x_1,\dots,x_n)=(d_1x_1,\dots,d_nx_n)$. It is clear that $\mathcal{L}^n=\mathcal{H}^n$ is invariant under p and q; it is also clear that, under d, $\mathcal{L}^n$ scales by $|d_1\cdot d_2\cdots d_n| = |\det f|$. This establishes ($\blacktriangle$).

★ *PART 2: MEASURABILITY LEMMA*

Parts 3 and 4 below involve decomposing $f(A)$ into small pieces and then summing estimates made on these pieces. In order to sum the estimates we need to know that the pieces of $f(A)$ are measurable. The fact we need is:

$$\left.\begin{array}{r} f:A\to\mathbb{R}^p \text{ is Lipschitz}\\ A\subseteq\mathbb{R}^n \text{ is } \mathcal{L}^n\text{-measurable}\end{array}\right\}\implies f(A)\text{ is }\mathcal{H}^n\text{-measurable.}$$

(We have worded this to be general enough to satisfy all our measurability needs. Notice that, by the Mean Value Theorem, a C^1 function is at least locally Lipschitz, and thus the Lemma is certainly applicable to the functions we are considering here).

To prove the Lemma, we use the fact $\mathcal{L}^n$ is Radon. This allows us to write $A = A_0\cup\bigcup K_j$ where $\mathcal{H}^n(A_0)=0$ and each K_j is compact (see [TheHutch1,p5],[EG,p8]). Now

- ▶ $\mathcal{H}^n(f(A_0))=0$ (by Theorem 1(v)), and so $f(A_0)$ is $\mathcal{H}^n$-measurable;
- ▶ Each $f(K_j)$ is compact ⇒ closed ⇒ $\mathcal{H}^n$-measurable.

Thus $f(A)=f(A_0)\cup\bigcup f(K_j)$ is $\mathcal{H}^n$-measurable, as desired.

★ *PART 3: CHANGE OF VARIABLES FORMULA.*

If $U\subseteq\mathbb{R}^n$ is open and $f:U\to\mathbb{R}^n$ is a C^1 diffeomorphism between U and $f(U)$, then we want to show

$$(\blacktriangledown)\qquad \mathcal{L}^n(f(A)) = \int_A |\det Df|\,d\mathcal{L}^n \qquad A\subseteq U\ \mathcal{L}^n\text{-measurable.}$$

To see this, fix $\epsilon>0$. For $a\in U$ let $\lambda(x)=f(a)+Df(a)\cdot(x-a)$ be the affine approximation to f at a. Then, for $\delta=\delta(a,\epsilon)$ small enough, we have

$$\begin{cases} \mathrm{Lip}(f\circ\lambda^{-1})\le 1+\epsilon & \text{on } \lambda(B_\delta(a)),\\ \mathrm{Lip}(\lambda\circ f^{-1})\le 1+\epsilon & \text{on } f(B_\delta(a)),\\ (1-\epsilon)|\det Df(a)|\le|\det Df|\le(1+\epsilon)|\det Df(a)| & \text{on } B_\delta(a).\end{cases}$$

(These results follow easily from the continuity of Df and the fact that $D(f \circ \lambda^{-1})(a) = I$).

Applying these inequalities, together with Theorem 1(v) and (▲) above, one readily obtains

$$\frac{1-\epsilon}{(1+\epsilon)^n}\mathcal{L}^n(f(E)) \leq \int_E |\det Df| \leq (1+\epsilon)^{n+1}\mathcal{L}^n(f(E)) \qquad E \subseteq B_\delta(a) \text{ measurable.}$$

Chopping A into small pieces and applying the Measurability Lemma, we obtain the same inequalities for any $\mathcal{L}^n$-measurable $A \subseteq U$. (A suitable chopping of A exists, e.g., by the local compactness of U). Finally, letting $\epsilon \to 0$, we obtain (▼).

★ *PART 4: $f : \mathbb{R}^n \to \mathbb{R}^p$ NONSINGULAR.*

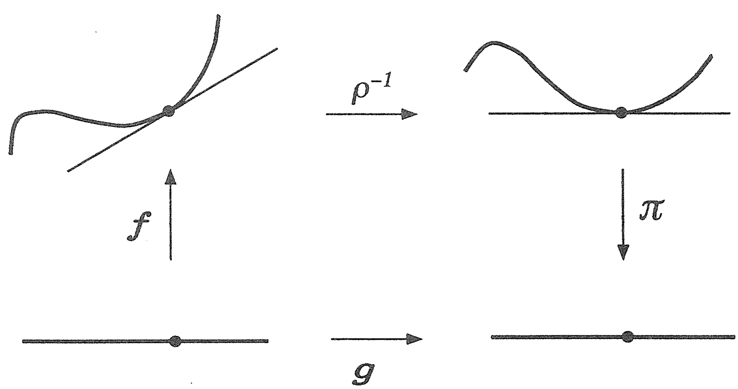
Here we prove the Area Formula,

$$(\blacklozenge) \qquad \mathcal{H}^n(f(A)) = \int_A Jf \, d\mathcal{L}^n \qquad A \subseteq U \subseteq \mathbb{R}^n \text{ measurable,}$$

under the assumption that $f : U \to \mathbb{R}^p$ is injective on A and rank $Df = n$ everywhere.

Given $a \in U$, write $Df(a) = \rho \circ \sigma$ as in the definition of $Jf(a)$. Let $\pi : \mathbb{R}^p \to \mathbb{R}^n$ be orthogonal projection onto $\mathbb{R}^n$ and define $g : \mathbb{R}^n \to \mathbb{R}^n$ by

$$g = \pi \circ \rho^{-1} \circ f\,.$$



Then

- ▶ $Jf(a) = |\det \sigma| = |\det Dg(a)|$.
- ▶ $Jf(x) \geq |\det D(g)(x)|$ for any $x \in U$.

(To see the inequality, write $Df(x) = \rho_x \circ \sigma_x$ and notice that $|\det(\pi \circ \rho^{-1} \circ \rho_x)| \leq (\text{Lip}(\pi \circ \rho^{-1} \circ \rho_x))^n \leq 1$).

Given $\epsilon > 0$, it follows that for $\delta = \delta(a, \epsilon)$ small enough

- ▶ $Jf(x) \leq (1+\epsilon)|\det D(g)(x)|$ for any $x \in B_\delta(a)$.

Now, the fact that f is nonsingular means, choosing δ smaller if need be, we can apply the Inverse Function Theorem to g on $B_\delta(a)$ ([Bo:§II.6]); it follows that $\rho^{-1} \circ f(B_\delta(a))$ can be written as a graph over $g(B_\delta(a))$. Abusing notation slightly, we refer to the graph function as π^{-1}, and we note that $D\pi^{-1}(g(a)) = 0$. Therefore, making δ smaller again, we can assume

▶ $\operatorname{Lip} \pi^{-1} \leq 1 + \epsilon$ on $g(B_\delta(a))$.

Combining the change of variables formula for g, the above estimates and Theorem 1(v), we find that

$$(1-\epsilon)^n \mathcal{H}^n(f(E)) \leq \int_E Jf \leq (1+\epsilon)\mathcal{H}^n(f(E)) \qquad E \subseteq B_\delta(a)\ \mathcal{L}^n\text{- measurable.}$$

Chopping up and applying the Measurability Lemma, we find the same estimate holds for measurable $A \subseteq U$, noting the injectivity assumption for f on A. Then, letting $\epsilon \to 0$,we obtain (♦).

★ *PART 5: GENERAL f.*

We want to show (♦) holds for general injective f. We do this by setting

$$Z = \{a : Jf(a) = 0\}$$

and showing that

$$(\lozenge) \qquad \mathcal{H}^n(f(Z)) = 0\,.$$

Of course, working locally, we can assume that f is Lipschitz and that $\mathcal{L}^n(Z) < \infty$. However, *our proof of* ($\lozenge$) *will not rely upon any injectivity assumption on* f.

Fix $\epsilon > 0$ and define $g : U \to \mathbb{R}^p \times \mathbb{R}^n$ by $g(x) = (f(x), \epsilon x)$. Then

$$Jg(a) \leq \epsilon\left((\operatorname{Lip} f)^2 + (\epsilon)^2)\right)^{(n-1)/2} \qquad a \in Z.$$

(To see this, notice that there is some $v \in \mathbb{R}^n$ for which $Df(a) \cdot v = 0$, and thus $|Dg(a) \cdot v| = \epsilon$; in any other direction, $Dg(a)$ stretches by at most $\sqrt{(\operatorname{Lip} f)^2 + \epsilon^2}$). g is injective and nonsingular. So, applying (♦) to g and using Theorem 1(v), we find that

$$\mathcal{H}^n(f(Z)) \leq \epsilon((\operatorname{Lip} f)^2 + (\epsilon)^2))^{(n-1)/2}\mathcal{L}^n(Z)\,.$$

Letting $\epsilon \to 0$, we obtain ($\lozenge$), completing the proof.

13 [EG,p102].

Noting the expression for $Jf(a)$ in 10 above, the result follows from the calculation

$$((Df)^* \circ Df)_{ij} = \sum_{k=1}^{p} (Df)^*_{ik}(Df)_{kj} = \sum_{k=1}^{p} (D_i f_k)(D_j f_k) = g_{ij}\,.$$

(14)

The only part which takes some thought is the claim that the Lipschitz image of a rectifiable set is rectifiable. This follows from:

- The composition of Lipschitz functions is Lipschitz;
- The Lipschitz image of a set of $\mathcal{H}^n$-measure zero has measure zero (by Theorem 1(v)).

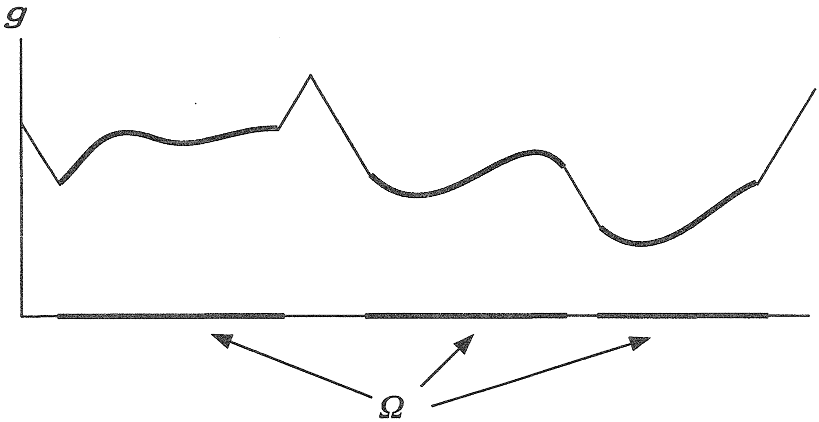
(15) [Si,p70].

Let $K = \sup\{\mathcal{H}^n(A \cap R) : R \text{ is rectifiable}\}$. For each j, let M_j be a rectifiable set with $\mathcal{H}^n(A \cap M_j) \geq K - 1/j$. Then set $M = A \cap (\cup_j M_j)$ and $P = A \sim M$. It is clear that this gives the desired decomposition and that it is unique up to sets of $\mathcal{H}^n$ measure zero.

(16) [EG,p80], [Si,Th5.1].

If $f : \Omega \to \mathbb{R}$ (i.e. $p = 1$) then we can define

$$g(x) = \inf_{a \in \Omega} \{f(a) + (\text{Lip}\, f)|x - a|\} .$$



It is easy to check that g is a Lipschitz extension of f with $\text{Lip}\, g = \text{Lip}\, f$. For general $f : \Omega \to \mathbb{R}^p$, we can apply this argument to each component of f, giving a Lipschitz extension g with $\text{Lip}\, g \leq \sqrt{n}\, \text{Lip}\, f$.

It can in fact be shown that a Lipschitz extension g exists with $\text{Lip}\, g = \text{Lip}\, f$, but the proof is difficult; see [Fe,§2.10.43].

(17) [EG,§3.1], [Mo,§3.2], [Si,Th5.2].

First note that Raemacher's Theorem is standard if $n = p = 1$: if $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz then f is absolutely continuous. Furthermore, in this case f' is $\mathcal{L}^n$-measurable and

$$(\blacktriangle) \qquad \int f'\,d\mathcal{L}^n = f(b) - f(a) \qquad a, b, \in \mathbb{R}\,.$$

(See [Roy,§5.4]. Note the application of the Vitali Theorem (§5.1)).

Now dealing with coordinate functions, it is enough to consider $f:\mathbb{R}^n \to \mathbb{R}$ (i.e. $p = 1$). By a standard argument, one can show that the partial derivatives $D_j f$ are measurable on the sets where they are defined ([EG,p82]). Then, applying Fubini's Theorem and ($\blacktriangle$), we find that:

- The partial derivatives $D_j f$ exist $\mathcal{L}^n$-almost everywhere;
- *Integration by parts* holds for Lipschitz functions; if $k:\mathbb{R}^n \to \mathbb{R}$ is Lipschitz with compact support then

$$(\blacktriangledown) \qquad \int k D_j f\,d\mathcal{L}^n = -\int (D_j k) f\,d\mathcal{L}^n \qquad j = 1, \ldots, n\,.$$

(Note that ($\blacktriangledown$) states that f has weak derivatives in the sense of [TheHutch1,§4.4]. Note also that Rademacher's theorem states more than just the partial derivatives of f exist almost everywhere: we have to show the existence of affine approximations to f).

More generally, for $v \in \mathbb{S}^{n-1}$, we can consider the *directional derivative*

$$D_v f(a) \equiv \lim_{t\to 0} \frac{f(a+tv) - f(a)}{t}\,.$$

By the rotational invariance of $\mathcal{L}^n = \mathcal{H}^n$, it follows that D_v exists $\mathcal{L}^n$-almost everywhere, is $\mathcal{L}^n$-measurable, and

$$(\blacktriangleleft) \qquad \int k D_v f\,d\mathcal{L}^n = -\int (D_v k) f\,d\mathcal{L}^n$$

for any $k:\mathbb{R}^n \to \mathbb{R}$ Lipschitz with compact support.

The proof is now completed in two steps:

(1) For any fixed $v = (v_1, \ldots, v_n) \in \mathbb{S}^{n-1}$, we show

$$(\lozenge) \qquad D_v f(a) = \sum_{j=1}^{n} D_j f(a) v_j \qquad \mathcal{L}^n\text{-a.e. } a \in \mathbb{R}^n.$$

(2) Let $D \subseteq \mathbb{S}^{n-1}$ be countable and dense. Then, for $\mathcal{L}^n$-almost all $a \in \mathbb{R}^n$, ($\blacktriangleleft$) holds for *every* $v \in D$. We show that f is differentiable at any such a.

To see (1), notice first that ($\lozenge$) holds for any C^1 function k, being a special case of the chain rule. Applying this, together with ($\blacktriangledown$) and ($\blacktriangleleft$), we find that

$$\int k D_v f\,d\mathcal{L}^n = \int k \sum_{j=1}^{n} D_j f(a) v_j\,d\mathcal{L}^n \qquad k \in C_0^1(\mathbb{R}^n).$$

($\lozenge$) holding almost everywhere now follows by a standard argument, using the arbitrariness of k.

To prove (2), Let a be such that ($\lozenge$) holds for all $v \in D$. Fixing $\epsilon > 0$, we show that, for $h = (h_1, \ldots, h_h) \in \mathbb{R}^n$ suitably small,

$$(\blacklozenge) \qquad \left| f(a+h) - f(a) - \sum_{j=1}^{n} D_j f(a) h_j \right| \leq M|h|\epsilon ,$$

where M is a constant independent of h.

Since $\mathbb{S}^{n-1}$ is compact, we can choose a *finite* $F \subset D$ for which

$$\mathbb{S}^{n-1} \subseteq \bigcup_{w \in F} B_\epsilon(w) .$$

So, for any $h \neq 0$, there will be some $v = v(h) = (v_1, \ldots, v_n) \in F$ for which

$$|h - |h|v| < \epsilon|h| .$$

Now, by ($\lozenge$),

$$\left| f(a+h) - f(a) - \sum_{j=1}^{n} D_j f(a) h_j \right| \leq |f(a+h) - f(a + |h|v)|$$

$$+ |f(a+|h|v) - f(a) - |h| D_v f(a)| + \sum_{j=1}^{n} |D_j f(a)(|h|v_j - h_j)|$$

It is easy to see that first term is bounded by $(\mathrm{Lip}\, f)|h|\epsilon$, and that the third term is bounded by $(\sum |D_j f(a)|)|h|\epsilon$. Finally, using the finiteness of F, the second term is bounded by $|h|\epsilon$ for small h. Together, these estimates give ($\blacklozenge$), completing the proof.

(18) [EG,pp245-252], [W], [Z,§§3.3.5-6].
(Whitney's Theorem, in its full glory, is much more general than our Theorem 6: see [W], [Z] or [Fe,§3.1.14]).

The proof given here, as is that in [EG], is based upon that in [Fe]. The proof is divided into six parts: in Part 1 we define the set C, f being particularly nice on C; in Part 2 we define a well-behaved locally finite covering of $A = \mathbb{R}^n \sim C$, and in Part 3 we construct a correspondingly well-behaved partition of unity subordinate to this covering; in Part 4 we define g; finally, in Part 5 we show g is differentiable with $Df = Dg$ on C, and in Part 6 we show Dg is continuous.

★ *PART 1: DEFINITION OF C.*

We assume throughout that $f : \mathbb{R}^n \to \mathbb{R}$ (i.e. $p = 1$). Now, by Rademacher's Theorem and Lusin's Theorem ([TheHutch1,§2.1.4],[EG,p15]), there is a closed $G \subseteq \mathbb{R}^n$ with $\mathcal{L}^{(}\mathbb{R}^n - G) < \epsilon/2$, and such that f is differentiable on G with $Df_{|G}$ continuous. Next, for $k \in \mathbb{N}$, define $h_k : \mathbb{R}^n \to \mathbb{R}$ by

$$h_k(c) \equiv \sup_{0<|x-c|<\frac{1}{k}} \frac{|f(x) - f(c) - Df(c)\cdot(x-c)|}{|h|}\,.$$

$h_k \to 0$ pointwise on G. So, by Egoroff's Theorem ([TheHutch1, §2.1.5],[EG,p16]) there is a closed $C \subseteq G$ with $\mathcal{L}^n(G \sim C) < \epsilon$ and such that

$$(\blacktriangle) \qquad h_k \longrightarrow 0 \text{ uniformly on bounded subsets of } C.$$

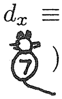
★ *PART 2: COVERING OF* $\mathbb{R}^n \sim C$.

Let

$$A = \mathbb{R}^n \sim C\,,$$

and for $x \in A$ define

$$d_x \equiv \text{dist}(x, A)\,.$$

By the Unsubtle Vitali Lemma (), There is a countable $D \subset A$ such that

$$(\triangle) \qquad A = \bigcup_{b\in D} B_{\frac{d_b}{4}}(b)$$

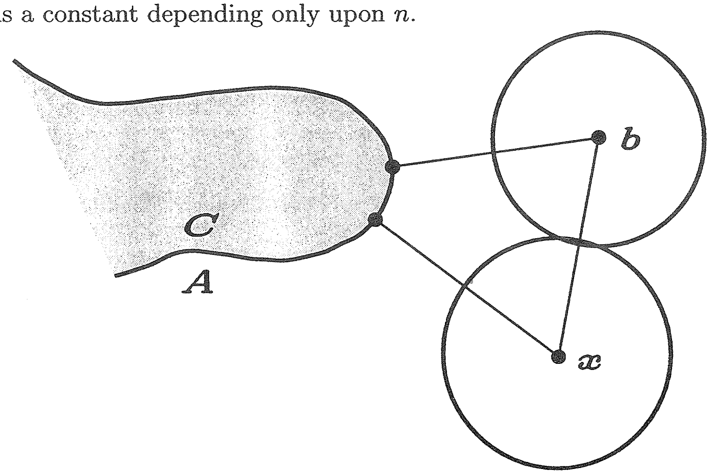
and such that the collection $\{B_{\frac{d_b}{20}}(b)\}_{b\in D}$ is pairwise disjoint. (Notice that $0 < d_x \le (\epsilon/\omega_n)^{\frac{1}{n}}$ for all $x \in A$, by the hugeness of C). We now want to prove that this covering of A is locally finite. Specifically, defining

$$D_x \equiv D \cap \left\{b : B_{\frac{d_x}{2}}(x) \cap B_{\frac{d_b}{2}}(b) \ne \emptyset\right\} \qquad x \in A,$$

we shall show that

$$(\lozenge) \qquad \text{Card}(D_x) \le N\,,$$

where N is a constant depending only upon n.



To prove (◊), notice that for any $x, b \in A$,

$$(\blacktriangledown) \qquad |x-b| \le \frac{d_x}{2} + \frac{d_b}{2} \implies \begin{cases} \dfrac{d_b}{3} \le d_x \le 3d_b\,, \\ |x-b| \le 2d_x\,. \end{cases}$$

(To see the first inequality, apply the hypothesis together with the general fact that $|d_x - d_b| \le |x-b|$; the second inequality follows easily from the first).

It follows that

$$B_{\frac{d_b}{20}}(b) \subset B_{3d_x}(x) \qquad b \in D_x\,.$$

Thus, considering $\mathcal{L}^n$ on the disjoint collection $\{B_{\frac{d_b}{20}}(b)\}_{b\in D}$, we find that

$$\sum_{b\in D_x} \left(\frac{d_b}{20}\right)^n \le (3d_x)^n\,,$$

and (◊) follows, with $N = (180)^n$.

★ *PART 3: PARTITION OF UNITY.*

$\{v_b\}_{b\in D}$ is to be a C^1 *partition of unity* subordinate to $\{B_{\frac{d_b}{2}}(b)\}_{b\in D}$; thus, each $v_b \ge 0$ is to be C^1, supported inside $B_{\frac{d_b}{2}}(b)$, and such that

$$(\triangledown) \qquad \sum_D v_b \equiv 1 \qquad \text{on } A.$$

The existence of partitions of unity is standard ([Bo,§V.4],[Z,p53]), but we want to ensure that $\{v_b\}$ has the extra property that

$$(\triangleleft) \qquad \sup|Dv_b| \le \frac{M}{d_b}\,,$$

where M is a constant depending only upon n. To do this, begin with a C^1 function $\phi\colon \mathbb{R}^n \to \mathbb{R} \ge 0$ satisfying

$$\begin{cases} \phi = 1 & \text{on } B_1(0), \\ \phi = 0 & \text{on } \mathbb{R}^n \sim B_2(0), \\ |D\phi| \le 2 & \text{on } \mathbb{R}^n. \end{cases}$$

We can then define

$$\begin{cases} \phi_b(x) \equiv \phi\left(\dfrac{4(x-b)}{d_b}\right)\,, \\ \Phi(x) \equiv \displaystyle\sum_{b\in D} \phi_b(x)\,, \end{cases}$$

and, finally,

$$v_b(z) \equiv \begin{cases} \dfrac{\phi_b(x)}{\Phi(x)} & x \in A, \\ 0 & x \in C. \end{cases}$$

Clearly

$$(\square) \qquad \begin{cases} |x-b| \le \dfrac{d_b}{4} \implies v_b \equiv 1\,, \\ |x-b| \ge \dfrac{d_b}{2} \implies v_b \equiv 0\,. \end{cases}$$

So, by ($\triangle$) and ($\Diamond$),

$$0 \le v_b \le 1 \le \Phi < \infty \qquad \text{on } A,$$

and $\{v_b\}$ is a well-defined partition of unity. It is also clear that

$$\sup |D\phi_b| \le \frac{8}{d_b}\,,$$

and ($\lhd$) follows easily, with $M = 16(1+N)$.
★ *PART 4: DEFINITION OF g.*
For each $b \in D$ choose $b^* \in C$ with

$$d_b = |b - b^*|\,,$$

and let $\lambda_b : \mathbb{R}^n \to \mathbb{R}$ be the affine approximation to f at b^*:

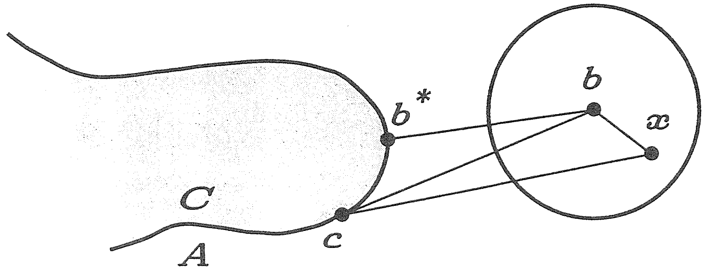
$$\lambda_b(x) = f(b^*) + Df(b^*) \cdot (x - b^*)\,.$$

Now define

$$g(x) \equiv \begin{cases} f(x) & x \in C, \\ \displaystyle\sum_{b \in D} \lambda_b(x) v_b(x) & x \in A. \end{cases}$$

Trivially, $g = f$ on C and Dg is continuous on A. Now we show the hard bits.
★ *PART 5: DIFFERENTIABILITY OF g.*
We want to show that g is differentiable at $c \in C$ with $Dg(c) = Df(c)$. Since f is differentiable on C, we need only worry about $c \in \partial C$, and we need only worry about x approaching c with $x \in A$. The key point is that $g(x)$ is defined in terms of approximations to f about points b^* close to c. In fact, from ($\square$) and ($\blacktriangledown$),

$$(\blacklozenge) \qquad \begin{aligned} v_b(x) \neq 0 &\implies |x-b| \le 2|x-c| \\ &\implies |x-b^*| \le 5|x-c|\,. \end{aligned}$$



(As the definition of b^* gives $|b - b^*| \leq |b - c|$, this follows from the triangle inequality).

Also for, for $x \in A$,

$$\begin{aligned} &|g(x) - g(c) - Df(c) \cdot (x - c)| \\ &= \left| \left(\sum_{b \in D} \lambda_b(x) v_b(x) \right) - f(c) - Df(c) \cdot (x - c) \right| \\ &\leq \left(\sum_{b \in D} v_b(x) \, |\lambda_b(x) - f(x)| \right) + |f(x) - f(c) - Df(c) \cdot (x - c)| \, . \end{aligned}$$

Therefore, if $k \in \mathbb{N}$ and $|x - c| < \frac{1}{5k}$, then ($\Diamond$) gives

$$\begin{aligned} &|g(x) - g(c) - Df(c) \cdot (x - c)| \\ &\leq N(\sup_b h_k(b^*))|x - b^*| + |f(x) - f(c) - Df(c) \cdot (x - c)| \, , \end{aligned}$$

where h_k is from Part 1 and the *sup* is over those b with $v_b(x) \neq 0$. By ($\blacklozenge$), any such b satisfies $|b - c| \leq 3|x - c|$, and is thus in a fixed compact set for x close to c. The fact that $Dg(c)$ exists and equals $Df(c)$ now follows from ($\blacktriangle$) and ($\blacklozenge$).

★ *PART 6: CONTINUITY OF Dg.*

We want to show Dg is continuous. Since $Df_{|C}$ is continuous, we need only worry about continuity at $c \in \partial C$ and x close to c with $x \in A$. Now

($\maltese$) $$Dg(x) = \sum_{b \in D} v_b(x) Df(b) + \sum_{b \in D} \lambda_b(x) Dv_b(x) \qquad x \in A \, .$$

As $x \to c$, the first sum appproaches $Df(c)$, by ($\blacklozenge$) and the continuity of Df at c. On the other hand, using $\sum Dv_b = D(\sum v_b) \equiv 0$, together with ($\lhd$), we have

$$\begin{aligned} \left| \sum_{b \in D} \lambda_b(x) Dv_b(x) \right| &= \left| \sum_{b \in D} (\lambda_b(x) - f(x)) Dv_b(x) \right| \\ &\leq M \sum_b \frac{|\lambda_b(x) - f(x)|}{d_b} \, . \end{aligned}$$

The sum is over those b for which $v_b(x) \neq 0$, and we note that we then also have $|x - b^*| \leq \frac{3d_b}{2}$. So, if $|x - c| < \frac{1}{5k}$ then (♦) and (◊) imply

$$\left|\sum_{b \in D} \lambda_b(x) D v_b(x)\right| \leq \frac{3MN}{2} \sup_b h_k(b^*)\,.$$

Aplying (▲) and returning to (✠), we find that Dg is continuous at c, as desired.

★ *PART 7.*

And in the Seventh Part, We Rested.

(19) [Mo,p30], [Si,p59].

The "⇐" direction is trivial. For the "⇒" direction, we may as well assume, by Theorem 4, that $M \subseteq f(\mathbb{R}^n)$ with $f : \mathbb{R}^n \to \mathbb{R}^p$ Lipschitz. Applying Whitney's Theorem with $\epsilon = \frac{1}{k}$, we obtain C^1 functions $g_k : \mathbb{R}^n \to \mathbb{R}^p$ such that

$$(♦) \qquad M \subseteq N_0 \cup \bigcup_{k=1}^{\infty} g_k(\mathbb{R}^n)\,.$$

Next, set

$$Z_k = \{a : Jg_k(a) = 0\}\,.$$

By the Area Formula,

$$\mathcal{H}^n\left(g_k(Z_k)\right) = 0\,.$$

(We are using here the non-injective version of the area formula. See Remark (c) after Theorem 2 and the proof of (12)). On the other hand, by Remark (b) after Theorem 2, $g_k(\mathbb{R}^n \sim Z_k)$ is an immersed C^1 submanifold and can thus be written as a countable union of regularly and properly embedded submanifolds with boundary. Combining with (♦), this gives the desired covering of M.

(20) [EG,§3.3], [Mo,pp25-27], [Si,§12].

We shall prove Lipschitz versions of the Area Formula as stated in Theorem 2 and the subsequent Remark (b). (In [EG] and [Mo], the Area Formula is proved directly for Lipschitz functions, rather than by first establishing a C^1 version).

Suppose $A \subseteq \mathbb{R}^n$ is $\mathcal{L}^n$-measurable and $f : A \to \mathbb{R}^p$ is Lipschitz. Considering the two distinct cases, we want to show

$$(♦) \qquad \begin{cases} f \text{ injective on } A \Longrightarrow \mathcal{H}^n(f(A)) = \displaystyle\int_A Jf\, d\mathcal{L}^n\,, \\ Jf \equiv 0 \text{ on } A \Longrightarrow \mathcal{H}^n(f(A)) = 0\,. \end{cases}$$

First note that by the Lipschitz Extension Theorem, we may assume that f is defined and Lipschitz on all of $\mathbb{R}^n$; then, by Rademacher's Theorem, $Jf(a)$ is well-defined, and independent of the extension chosen, on almost all of A.

By Whitney's Theorem there is a C^1 function $g : \mathbb{R}^n \to \mathbb{R}^p$ such that $g = f$ and $Dg = Df$ on a closed $C \subseteq \mathbb{R}^n$ with $\mathcal{L}^n(\mathbb{R}^n \sim C) \leq \epsilon$. Furthermore, by the arbitrariness of ϵ, we may as well assume that

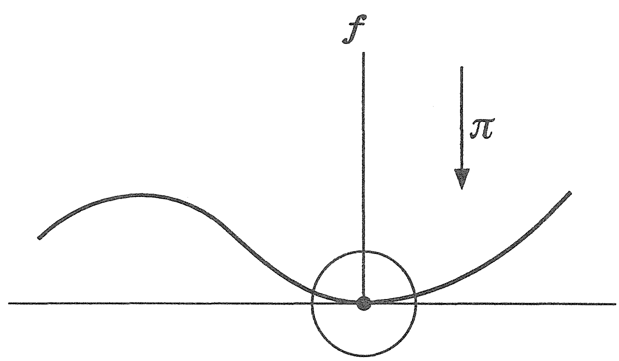
$$A \subseteq C\,.$$

(One proves $(\blacklozenge)$ with $A_\epsilon = A \cap C$ in place of A. There is then no problem in taking the limit as $\epsilon \to 0$). But now we see the result is trivial: clearly $Jg = Jf$ on A, and $(\blacklozenge)$ is immediate from applying the Area Formula to g.

21

The picture is clear: it is just a question of defining the type of embedding carefully enough to rule out naughtiness.

If $a \in \mathbb{R}^p \sim M$ then the properness of the embedding of M ensures that there is an open $U \subseteq \mathbb{R}^p \sim M$ containing a, and thus $\Theta^n(M, a) = 0$ trivially. Points on the boundary of M can be ignored since, by Theorem 1(iii),(viii), these points form a set of $\mathcal{H}^n$-measure zero. Finally, if a is a point in the interior of M, we want to show that $\Theta^n(M, a) = 1$. The easiest approach is to write a small piece of M about a as a graph $\{(x, f(x))\}$ with $a = (0, f(0))$ and $Df(0) = 0$: the regularity of the embedding allows us to ignore the rest of M.



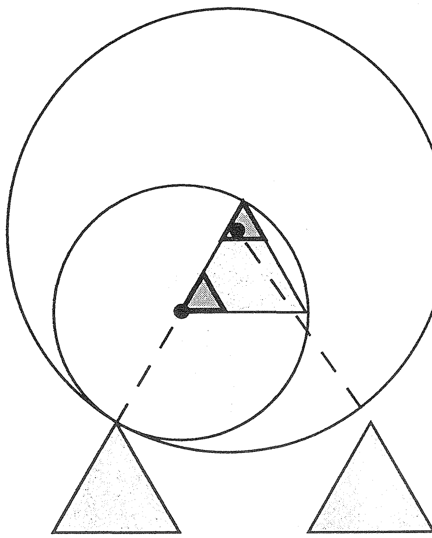
For any $\epsilon > 0$ we can find δ such that $\operatorname{Lip} f \leq \epsilon$ on $B^n_\delta(0)$. Theorem 1(v) now gives

$$\begin{cases} \mathcal{H}^n\left(M \cap B_{r\sqrt{1+\epsilon}}\right) \geq \omega_n r^n \\ \mathcal{H}^n\left(M \cap B_r\right) \leq \omega_n (1+\epsilon)^n r^n \end{cases} \qquad r < \delta.$$

(For the first inequality consider the orthogonal projection $\pi : \mathbb{R}^p \to \mathbb{R}^n$, and for the second inequality consider the lift $x \mapsto (x, f(x))$. Letting $\epsilon \to 0$, we obtain the desired density result.

22

We refer to the notation and diagram used in the definition of the Besicovitch Set. Fix $a \in D$ for consideration. For any k, the ball $B_{\frac{1}{3^k}}(a)$ contains all of one particular triangle making up D_k and does not interesect any of the other triangles at this scale (except, maybe, at one point).



Therefore, for all n, we have

$$\mathcal{H}^1\left(D \cap B_{\frac{1}{3^k}}(a)\right) = \frac{1}{3^k}.$$

Thus consideration of the sequence of radii $r_k = \frac{1}{3^n}$ proves both density estimates.

We now want to show that for $\mathcal{H}^1$-almost all $a \in D$ the lower density estimate can be improved to $\frac{3}{10}$. To see this, suppose that a is in the *upper triangle* of both D_k and D_{k+1}. Then the radius $\frac{1}{3^k}$ can be improved to give

$$\mathcal{H}^1\left(D \cap B_{\frac{5}{3^{k+1}}}(a)\right) = \frac{1}{3^k}.$$

This estimate shows that if, for infinitely many k, a lies in the *same-type triangle* of D_k and D_{k+1} then $\Theta^1_*(D,a) \leq \frac{3}{10}$. We just have to show that almost all a satisfy this same-triangle condition.

Let $E \subseteq D$ be the set of a such that, for all but finitely many k, a lies in different-type triangles of D_k and D_{k+1}. We want to show that $\mathcal{H}^1(E) = 0$. To see this first write $E = \cup E_m$ where $E_m \subseteq D$ is the set of a which are in different-type triangles of D_k and D_{k+1} for all $k \geq m$; so, we just have to fix m and show $\mathcal{H}^1(E_m) = 0$.

Each triangle of D_m contains three triangles of D_{m+1}, only two of which can contain any points from E_m. Thus

$$\mathcal{H}^1(E_m) \leq \tfrac{2}{3}\mathcal{H}^1(D)\,.$$

Next, considering any triangle from D_{m+1}, we note that it contains three triangles from D_{m+2}, only two of which can interesect E_m. So, combining with the previous inequality, we find that

$$\mathcal{H}^1(E_m) \leq \left(\tfrac{2}{3}\right)^2 \mathcal{H}^1(D)\,.$$

Proceeding inductively, we have, for any k,

$$\mathcal{H}^1(E_m) \leq \left(\tfrac{2}{3}\right)^k \mathcal{H}^1(D)\,.$$

Letting $k \to 0$ gives $\mathcal{H}^1(E_m) = 0$, as desired.

23 [EG,§2.3], [Si,§3].

There are three claims in the statement of Theorem 9, the first of which is not too difficult and we prove directly in Part 1. In Part 2 we prove a lemma, from which the other claims follow quite simply (Parts 3 and 4).

★ *PART 1.*

We consider $M \subset \mathbb{R}^p$ and we assume that $\mathcal{H}^n_\delta(M) < \infty$ for all $\delta > 0$ (which is all we need here). Then we show $\Theta^{n*}(M, a) \geq \frac{1}{2^n}$ for $\mathcal{H}^n$-a.e. $a \in M$. To do this, fix $t < \frac{1}{2^n}$ and set

$$A = M \cap \left\{ a : \frac{\mathcal{H}^n(M \cap B_r(a))}{\omega_n r^n} < t \text{ for } 0 < r < \delta \right\}.$$

We shall show that $\mathcal{H}^n(A) = 0$: since A is measurable, we can take limits (first in δ then in t), giving the desired result.

Fix $\epsilon > 0$ and let $\{C_j\}$ be a covering of A by sets of diameter at most δ and for which

$$\sum_j \omega_n \left(\frac{\operatorname{diam} C_j}{2}\right)^n \leq \mathcal{H}^n_\delta(A) + \epsilon\,.$$

We can assume that each C_j contains some $a_j \in A$, and then we note

$$C_j \subseteq B_{\operatorname{diam} C_j}(a_j)\,.$$

Therefore,

$$\begin{aligned}
\mathcal{H}^n_\delta(A) &\leq \sum_j \mathcal{H}^n_\delta\left(A \cap B_{\operatorname{diam} C_j}(a_j)\right) \\
&\leq t \sum_j \omega_n \left(\operatorname{diam} C_j\right)^n \\
&\leq t 2^n \left(\mathcal{H}^n_\delta(A) + \epsilon\right).
\end{aligned}$$

Letting $\epsilon \to 0$, we obtain a contradiction unless $\mathcal{H}^n_\delta(A) = 0$ for every δ, implying $\mathcal{H}^n(A) = 0$.

★ *PART 2.*

Supposing $t > 0$ we show that

$$(\blacklozenge) \qquad A \subseteq \{a : \Theta^{n*}(M,a) > t\} \implies t\mathcal{H}^n(A) \leq \mathcal{H}^n(M)\,.$$

Fixing $\delta > 0$, we apply the Subtle Vitali Lemma, giving a countable pairwise disjoint collection $\mathcal{B}$ of balls B satisfying

$$\begin{cases} \operatorname{diam}(5B) < \delta\,, \\ \mathcal{H}^n(A \cap B) > t\omega_n \left(\dfrac{\operatorname{diam} B}{2}\right)^n\,, \\ A \subset \bigcup \mathcal{J} \cup \bigcup\limits_{B \in \mathcal{B} \sim \mathcal{J}} (5B) \qquad \mathcal{J} \subseteq \mathcal{B} \text{ finite.} \end{cases}$$

Then

$$\begin{aligned} \mathcal{H}^n_\delta(A) &\leq \sum_{B\in\mathcal{J}} \omega_n \left(\frac{\operatorname{diam} B}{2}\right)^n + \sum_{B\in\mathcal{B}\sim\mathcal{J}} \omega_n \left(\frac{\operatorname{diam} 5B}{2}\right)^n \\ &\leq \frac{1}{t} \sum_{B\in\mathcal{J}} \mathcal{H}^n(M \cap B) + \frac{5^n}{t} \sum_{B\in\mathcal{B}\sim\mathcal{J}} \mathcal{H}^n(M \cap B) \\ &\leq \frac{1}{t}\mathcal{H}^n(M) + \frac{5^n}{t} \sum_{B\in\mathcal{B}\sim\mathcal{J}} \mathcal{H}^n(M \cap B)\,, \end{aligned}$$

where we have used the fact that $\mathcal{H}^n \lfloor M$ is a Borel measure to estimate the first sum (the measure is Borel even if M is not measurable: see [EG,p2]). For the same reason, the second sum is bounded by $\mathcal{H}^n(M) < \infty$, and thus converges to 0 as $\mathcal{J} \to \mathcal{B}$. Finally, letting $\delta \to 0$, we obtain ($\blacklozenge$).

★ *PART 3.*

Suppose $\mathcal{H}^n(M) < \infty$. Fixing $t > 1$, set

$$A = M \cap \{a : \Theta^{n*}(M,a) > t\}\,.$$

We want to show $\mathcal{H}^n(A) = 0$, implying the second claim of Theorem 9(i). To do this, fix $\epsilon > 0$ and choose an open $V \supseteq A$ with

$$\mathcal{H}^n(V \cap M) \leq \mathcal{H}^n(A) + \epsilon\,.$$

(Letting $B \supseteq M$ be a Borel set with $\mathcal{H}^n(B) = \mathcal{H}^n(M) < \infty$, V exists because the measure $\mathcal{H}^n \lfloor B$ is Radon: see [EG,p5]).

Since V is open,

$$\Theta^{*n}(V \cap M, a) > t \qquad \text{for } a \in A,$$

Thus, by (♦),

$$t\mathcal{H}^n(A) \leq \mathcal{H}^n(V \cap M) \leq \mathcal{H}^n(a) + \epsilon\,.$$

Letting $\epsilon \to 0$, we obtain a contradiction unless $\mathcal{H}^n(A) = 0$.

★ *PART 4.*

Suppose M is $\mathcal{H}^n$-measurable with $\mathcal{H}^n(M) < \infty$. Fixing $t > 0$, set

$$A = \sim M \cap \{a : \Theta^{n*}(M, a) > t\}\ .$$

We want to show $\mathcal{H}^n(A) = 0$, implying Theorem 9(ii).

Fix $\epsilon > 0$. By the hypotheses on M, the measure $\mathcal{H}^n \lfloor M$ is Radon ([EG,p5]), and thus we can find a closed $C \subseteq M$ with

$$\mathcal{H}^n(M \sim C) \leq \epsilon\,.$$

Notice that

$$\Theta^{n*}(M, a) > t \qquad \text{for } a \in A.$$

So, by (▲),

$$t\mathcal{H}^n(A) \leq \mathcal{H}^n(M \sim C) \leq \epsilon\,.$$

Letting $\epsilon \to 0$, we obtain $\mathcal{H}^n(A) = 0$, as desired.

(24)

Theorem 1(vii) and Theorem 9(ii) immediately imply the second Lebesgue density result:

$$\Theta^n(M, a) \;=\; 0 \qquad \text{for } \mathcal{L}^n\text{-a.e. } a \in \sim M,\ M \subseteq \mathbb{R}^n \text{ measurable.}$$

(Note that M automatically has locally finite measure, since $\mathcal{L}^n$ is Radon). Applying this result to $\mathbb{R}^p \sim M$ and noting,

$$\mathcal{L}^n\left(M \cap B_r(a)\right) + \mathcal{L}^n\left((\mathbb{R}^p \sim M) \cap B_r(a)\right) \;=\; \mathcal{L}^n(B_r(a)) \;=\; \omega_n r^n\,,$$

we get the other density result as well:

$$\Theta^n(M, a) \;=\; 1 \qquad \text{for } \mathcal{L}^n\text{-a.e. } a \in M,\ M \subseteq \mathbb{R}^n \text{ measurable.}$$

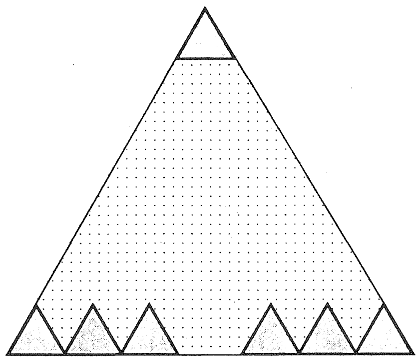
In the final remarks of we show how to eliminate the measurability hypothesis for this second result.

(25) [Fa, §1.5], [Ma,§4.7], [TheHutch2, Th5.3].

We give two variations of the Besicovitch Set: the first, $\widehat{D}$, will have the property that $\Theta^1_*(\widehat{D}, a) = 0$ for $\mathcal{H}^1$- almost all $a \in \widehat{D}$ (removing the set of measure zero, we can get the lower density to be zero at *all* points of the set); the second, $\widetilde{D}$, will be 1-dimensional but will fail to have finite $\mathcal{H}^1$- measure in any neighbourhood of any point in $\widetilde{D}$ (addressing our remark at the beginning of §B). We also give an example of a 1-dimensional set E for which $\mathcal{H}^1(E) = 0$: this can also be done using variations of the Besicovitch Set, but starting with the Cantor Set is a little easier, and so that will be our strategy.

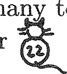
★ *PART 1.*

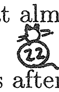
Let $\{N_k\}$ be a sequence of odd integers ≥ 3 and $\widehat{D}_0$ the equilateral triangle of sidelength 1. $\widehat{D}_k$ is defined in terms of $\widehat{D}_{k-1}$, as for the Besicovitch Set, but each triangle of $\widehat{D}_{k-1}$ is replaced by N_k triangles scaled by $\frac{1}{N_k}$, placed as indicated. (So, for the Besicovitch Set, $N_k \equiv 3$). Set $\widehat{D} = \cap_k \widehat{D}_k$.



The projection argument gives $\mathcal{H}^1(\widehat{D}) = 1$, just as for the Besicovitch Set. We now impose the condition

$$\lim_{k \to \infty} N_k = \infty .$$

Thus, for large k, points in the "top triangle" of $\widehat{D}_k$ are very isolated (at that scale). In fact, it is easy to show that any $a \in \widehat{D}$ which is in infinitely many top triangles must satisfy $\Theta^1_*(\widehat{D}, a) = 0$. (The argument is the same as that for  , keeping in mind that each triangle of $\widehat{D}_k$ contains the same amount of $\widehat{D}$).

We now impose a further condition on $\{N_k\}$ to ensure that almost all points in $\widehat{D}$ are in infinitely many top triangles. In fact, arguing as in , it is clear that the measure of the set of points which are in *no* top triangles after the m'th stage is the infinite product

$$\prod_{k=m+1}^{\infty} \left(\frac{N_k - 1}{N_k} \right) .$$

Thus, if we $N_k \to \infty$ slowly enough so that $\prod^{\infty}(1 - \frac{1}{N_k}) = 0$, then $\widehat{D}$ will have the desired property.

★ *PART 2.*

Let $\{t_k\}$ be a sequence of real numbers with $0 \leq t_k < \frac{1}{2}$. We define $\widetilde{D}$ exactly as for the Besicovitch Set except, at the k'th stage, the triangles are scaled by the factor $\frac{1+t_k}{3}$. (So $t_k \equiv 0$ for the Besicovitch Set). The standard projection argument shows that $\mathcal{H}^1(\widetilde{D}) \geq 1$, and thus $\widetilde{D}$ is at least 1-dimensional. Next, under the assumption that $t_k \to 0$ fast enough, we show that, for $s > 1$, $\mathcal{H}^s(\widetilde{D}) = 0$, implying that $\mathcal{H}^n(\widetilde{D})$ is precisely 1-dimensional. To see this, fix $\delta > 0$, take m large and consider the natural covering of $\widetilde{D}_m \supseteq \widetilde{D}$. This gives

$$\mathcal{H}^1_\delta(\widetilde{D}) \leq 3^m \frac{\omega_s}{2^s} \prod_{k=1}^{m} \left(\frac{1+t_k}{3}\right)^s = \frac{\omega_s}{2^s} 3^{m(1-s)} \left(\prod_{k=1}^{m}(1+t_k)\right)^s .$$

Thus $\widetilde{D}$ will be 1-dimensional if

$$\prod_{k=1}^{m}(1+t_k) = o\left((3^{\frac{s-1}{s}})^m\right) \qquad \text{as } m \to \infty, \text{ for each } s > 1.$$

(This will be true, e.g., if $\prod^m (1+t_k) = O(\log m)$).

We now show that if

$$\prod_{k=1}^{\infty}(1+t_k) = \infty$$

then $\mathcal{H}^1(\widetilde{D} \cap T) = \infty$ for any triangle T of any $\widetilde{D}_m$. Going to the $m+n$'th stage, $\widetilde{D} \cap T$ is contained in 3^n triangles of diameter $\prod^{m+n} \frac{1+t_k}{3}$. Applying the projection argument to *each* of these triangles, we find that

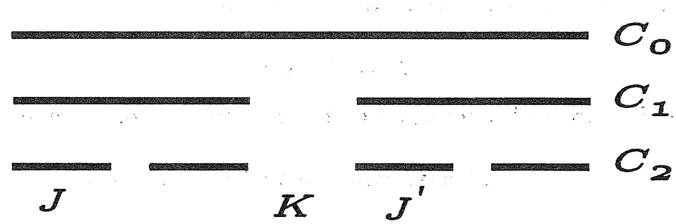
$$\mathcal{H}^1(\widetilde{D} \cap T) \geq 3^n \frac{1}{3^{m+n}} \prod_{k=1}^{m+n}(1+t_k) .$$

Letting $n \to \infty$ gives the desired result.

★ *PART 3.*

In principle, E is the easiest set to construct: let E_j be any t_j-dimensional set with $t_j \nearrow 1$, and define $E = \cup_j E_j$. Clearly E is 1-dimensional with $\mathcal{H}^1(E) = 0$, and so the only problem is to show the E_j exist. This, however, is not trivial (our projection argument only works for integer dimensions), and we have to perform some honest labour.

Fixing $s > 2$, we define a Cantorlike set $C^* = \cap C_k^*$: we set $C_0^* = [0,1]$, and C_{k+1}^* is obtained by replacing each interval of C_k^* by two intervals scaled by $\frac{1}{s}$. (so $s = 3$ gives the standard Cantor Set: see [TheHutch1,§1.2.3]).



Setting

$$t = \frac{\log 2}{\log s},$$

we show that

$$(\blacklozenge) \qquad \mathcal{H}^t(C^*) = \frac{\omega_t}{2^t}$$

implying C^* is t-dimensional.

Taking the obvious covering of C_k^* and considering $\delta < \frac{1}{s^k}$, we have

$$\begin{aligned} \mathcal{H}_\delta^t(C^*) \le \mathcal{H}_\delta^t(C_k^*) &\le \omega_t 2^k \left(\frac{1}{2s^k}\right)^t \\ &= \frac{\omega_t}{2^t} \qquad\qquad \text{(since } s^t = 2\text{)}. \end{aligned}$$

Letting $\delta \to 0$, we obtain one half of ($\blacklozenge$).

To show the reverse inequality, we have to show that the obvious coverings are the best. Suppose $\{I_j\}$ is a covering of C^*. Referring to the previous calculation, it is enough to show

$$(\blacktriangle) \qquad \sum_j (\operatorname{diam} I_j)^t \ge 1.$$

Now we can actually assume that $\{I_j\}$ is quite special: we can assume

- ▶ Each I_j is an interval.
  (Taking the "convex hull" of I_j does not increase $\operatorname{diam} I_j$).
- ▶ Each I_j is open with endpoints in $\sim C^*$.
  (Increase the size of each I_j slightly, and note that C^* has no interior).
- ▶ $\{I_j\}$ is finite.
  (C^* is compact).

Let d be the minimum distance from C^* to the boundary points of the I_j, and let k be large enough so that $\frac{1}{s^k} \le d$. Then every interval of C_k^* is contained in some I_j. If each I_j is contained at most one such interval, then ($\blacktriangle$) would be clear. To reduce to this case, consider an I_j which contains more than one interval from C_k^*; then I_j also contains an interval $K \subset \sim C_k^*$. Choosing K to be as large as possible, the construction of C_k^* shows that we can assume

$$I_j = J \cup K \cup J'$$

where J and J' are each intervals containing an interval of C_k^*, and for which

$$(s-2)\max(\operatorname{diam} J, \operatorname{diam} J') \le \operatorname{diam} K.$$

Thus

$$(\operatorname{diam} I_j)^t \geq \left(\frac{s}{2}\operatorname{diam} J + \frac{s}{2}\operatorname{diam} J'\right)^t = 2\left(\frac{1}{2}\operatorname{diam} J + \frac{1}{2}\operatorname{diam} J'\right)^t$$
$$\geq (\operatorname{diam} J)^t + (\operatorname{diam} J')^t \,,$$

where the last line follows from the concavity of the function $x \mapsto x^t$. Thus it is more efficient to replace I_j by J and J', and repeated application of the argument eventually gives (▲).

(26) [Mo,Prop3.12], [Si,§11].

We only need to prove the first statement:

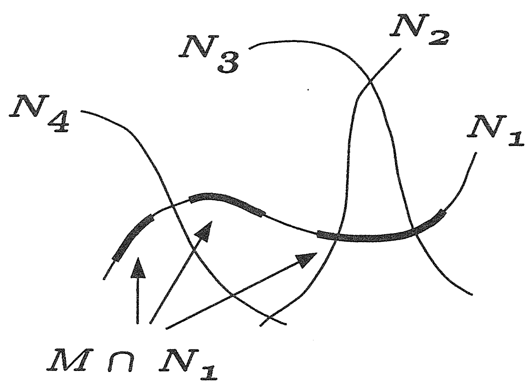
(♦) $$\Theta^n(M,a) = 1 \qquad \text{for } \mathcal{H}^n\text{-a.e. } a \in M.$$

By Corollary 7, we can write

$$M \subseteq N_0 \cup \bigcup_{j=1}^{\infty} N_j \,,$$

where $\mathcal{H}^n(N_0) = 0$ and, for $j \geq 1$, N_j is a regularly embedded C^1 submanifold with boundary. It is enough to focus on points in $M \cap N_1$, and we will show

(◊) $$\Theta^n(M,a) = 1 \qquad \text{for } \mathcal{H}^n\text{-a.e. } a \in M \cap N_1.$$



Write

$$\widehat{N} = N_0 \cup \left(\bigcup_{j=2}^{\infty} N_j\right) \sim N_1 \,.$$

Then we have

(◄) $$\Theta^n(N_1, a) = 1,$$

(▼) $$\Theta^n(N_1 \sim M, a) = 0, \qquad \text{for } \mathcal{H}^n\text{-a.e. } a \in M \cap N_1.$$

(▲) $$\Theta^n(\widehat{N}, a) = 0.$$

(◄) follows from (21) - see also its proof; (▼) and (▲) follow from Theorem 9(ii).
(◄) and (▼) imply

$$\Theta^n(M \cap N_1, a) = 1 \qquad \text{for } \mathcal{H}^n\text{-a.e. } a \in M \cap N_1.$$

(▲) implies

$$\Theta^n\left(M \cap \widehat{N}, a\right) = 0 \qquad \text{for } \mathcal{H}^n\text{-a.e. } a \in M \cap N_1.$$

Combining these last two results gives (◊), and thus (♦).

Finally, we want to show that (♦) continues to holds even if M is not measurable. By the Borel regularity of $\mathcal{H}^n$, we can find a Borel set $R \supseteq M$ satisfying $\mathcal{H}^n(R) = \mathcal{H}^n(M)$. Notice that if M is rectifiable then we can take R to be rectifiable as well, and in fact this must be the case. Notice also that

(△) $$\mathcal{H}^n(R \cap E) = \mathcal{H}^n(M \cap E) \qquad \text{for any } \textit{measurable } E.$$

Now, by (♦), $\Theta^n(R, a) = 1$ for $\mathcal{H}^n$-a.e. $a \in R$. Applying (△) with $E = B_r(a)$, the same result must hold for M.

(27)

Let M be the rectifiable part of the Besicovitch set D. By Corollary 10

$$\left\{\begin{aligned} \Theta^1(M, a) &= 1 \\ \Theta^1(D \sim M, a) &= 0 \end{aligned}\right. \qquad \text{for } \mathcal{H}^1\text{-a.e. } a \in M.$$

It follows that $\Theta^1(D, a) = 1$ for $\mathcal{H}^1$-a.e. $a \in M$, and thus $\mathcal{H}^1(M) = 0$ by (22).

(28) [Si,§11.3].

Consider two different n-planes P and Q through a. All we need to do is come up with a $\phi \in C_0(\mathbb{R}^p)$ such that

$$\int_P \phi \, d\mathcal{H}^n \neq \int_Q \phi \, d\mathcal{H}^n.$$

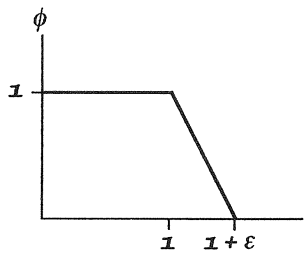
This is easy.

(29)

We can assume that $a = 0$ and write M near a as a graph $\{(x, f(x)\}$, as in (21). Expanding by λ, the graph becomes $\{(y, \lambda f(y/\lambda))\}$ (with $x \leftrightarrow y/\lambda$). Now, as $\lambda \to \infty$, we have $\lambda f(y/\lambda) \to 0$ uniformly on compact subsets of $\mathbb{R}^n$ (using $Df(0) = 0$). The result now follows easily.

(30) [HS,p22].

We can assume $a = 0$ and that $\mathbb{R}^n$ is tangent to M at 0. Now set $r = |x|$ and let $\phi = \phi(r)$ be an approximation to the characteristic function of $B_1(0)$, as pictured.



Then

$$\mathcal{H}^n\left(M \cap B_{\frac{1}{\lambda}}(0)\right) \le \int_M \phi(\lambda r)\, d\mathcal{H}^n(r) \le \mathcal{H}^n\left(M \cap B_{\frac{1+\epsilon}{\lambda}}(0)\right).$$

Dividing by ω_n/λ^n and letting $\lambda \to \infty$, Remark (b) after the definition of tangent plane gives

$$\Theta^{n*}(M, 0) \le \frac{1}{\omega_n} \int_{\mathbb{R}^n} \phi(r)\, d\mathcal{H}^n \le (1+\epsilon)^n \Theta^n_*(M, 0).$$

The result now follows by letting $\epsilon \to 0$.

(31)

Take $a = 0$. Let $E \subseteq \mathbb{R}^p$ and $\phi \in C_0(\mathbb{R}^p)$, and suppose ϕ is supported in $B_R(0)$. Then

$$\begin{aligned} \lambda^n \int_E \phi(\lambda x)\, d\mathcal{H}^n(x) &\le (\sup \phi) \frac{\mathcal{H}^n(E \cap B_{R/\lambda}(0))}{1/\lambda^n} \\ &= R^n (\sup \phi) \frac{\mathcal{H}^n(E \cap B_{R/\lambda}(0))}{(R/\lambda)^n}. \end{aligned}$$

Thus, if $\Theta^n(E, 0) = 0$ then $\lambda^n \int_E \phi(\lambda x) d\mathcal{H}^n(x) \to 0$ as $\lambda \to \infty$. Applying this with $E = M \sim A$ and $E = A \sim M$, the result follows.

(32) [Mo,Prop3.12], [Si,§11.6].

This follows readily from Corollary 7, Theorem 9(ii) and Lemma 12. The argument is the same as that given in the the proof of Corollary 10.

(33) [HS, Th 2.5], [Si, TH 11.8], [Ros, Th 11.7].

The proof here follows [Ros] and is based upon [Fe2,§3.3.6]. There are a number of parts: note the use of Lemma 12 and Theorem 9(ii) in Part 1, in order to fine-tune the hypotheses on A, making it easier to see A as rectifiable.

★ *PART 1: GRASSMAN MANIFOLD PRELIMINARIES.*

Given an n space $V \subseteq \mathbb{R}^p$, let $\pi_V : \mathbb{R}^p \to \mathbb{R}^p$ be orthogonal projection onto V. We can then define a metric on the space $G(n,p)$, the space of n-subspaces of $\mathbb{R}^p$, by

$$\mathrm{d}(V,W) \equiv \|\pi_V - \pi_W\| .$$

Thus we are identifying $G(n,p)$, the so-called *Grassman Manifold*, with a space $\{\pi_V\}$ of linear maps on $\mathbb{R}^p$ - see [Bo,p63]. Note that $\{\pi_V\} = G(n,p)$ is compact: this is quite easy to see by writing

$$\pi_V = \rho_V \circ \pi \circ (\rho_V)^{-1}$$

where $\pi = \pi_{\mathbb{R}^n}$ is projection onto $\mathbb{R}^n$ and ρ_V is a linear isometry of $\mathbb{R}^p$.

The point of all this is that we can assume

► All tangent planes $V + a$ to A satisfy $\mathrm{d}(V,\mathbb{R}^n) < \frac{1}{3}$.

This follows easily from (14), Theorem 9(ii) and Lemma 12, together with the compactness of $G(n,p)$ and its obvious symmetries.

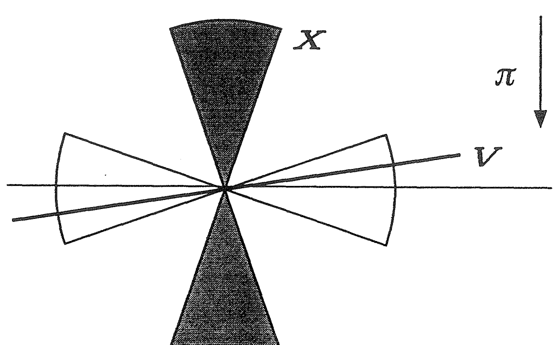
★ *PART 2: CONE DENSITIES.*

We shall show, in a moment, that

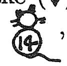
$$(\blacklozenge) \qquad \Theta^n\left(A \cap (X+a), a\right) = 0 \qquad a \in A$$

where X is the cone defined by

$$X = \mathbb{R}^p \cap \left\{ x : |\pi(x)| < \frac{1}{3}|x| \right\} .$$



This is the only use we make of the existence of tangent planes, and thus we actually show that any measurable set A of finite measure satisfying (♦) is rectifiable.

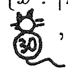
To make (♦) more concrete, let $M = M(n)$ be a constant to be chosen later. Then, by 

, we can assume the existence of a fixed $\delta > 0$ such that

$$(\blacktriangle) \qquad \frac{\mathcal{H}^n(A \cap (X + a) \cap B_r(a))}{r^n \omega_n} \leq \frac{1}{M} \qquad \text{for } a \in A \text{ and } 0 < r < \delta\,.$$

To prove (♦), we may as well assume that $a = 0$, and let V be the tangent plane to A there. It follows easily from the definition that

$$\mathrm{d}(V, \mathbb{R}^n) < \frac{1}{3} \Longrightarrow V \subset \left\{ x : |\pi(x)| \geq \frac{2}{3}|x| \right\}.$$

Now choose $\phi \in C_0(\mathbb{R}^n)$ satisfying $\phi = 1$ on $(X \cap (B_2(0) \sim B_1(0))$ with $\operatorname{spt}\phi \subset \{x : |\pi(x)| < \frac{2}{3}|x|\}$. Then, from the definition of tangent plane and arguing as in 

, we find that

$$\lim_{r \to 0} \frac{\mathcal{H}^n(A \cap X \cap (B_r \sim B_{\frac{r}{2}}))}{\omega_n r^n} = 0\,.$$

Recalling the definition of density, and summing up the estimates on the annuli $B_t \sim B_{\frac{t}{2}}$ with $t = r/2^j$, we easily obtain (♦).

★ *PART3: UNRECTIFIABLE FORMULATION.*

Since we want to use (▲) to show A is rectifiable, we may as well subtract off the rectifiable part of A, and thus we can assume

► A is purely unrectifiable.

So, now our task is to show that $\mathcal{H}^n(A) = 0$. As a simplification for later, recall that $\mathcal{H}^n \lfloor A$ is Radon (Theorem 1(ii)); thus, by 

, Theorem 9(ii) and Lemma 12, we may as well assume

► A is closed.

Now, our plan is to show the existence of a constant $N = N(n)$ such that

$$(\blacktriangledown) \qquad \Theta^{n*}(A, a) \leq \frac{N}{M} \qquad a \in A.$$

Then, choosing $M = 2^{n+1}N$, the fact that $\mathcal{H}^n(a) = 0$ will follow immediately from Theorem 9(i).

★ *PART 4: GEOMETRY OF CONES AND CYLINDERS.*

Fix $a \in A$, $r < \frac{\delta}{4}$, and set

$$A_r = A \cap B_r(a)\,.$$

(▼) will follow if we can show

$$(\blacktriangleleft) \qquad \mathcal{H}^n(A_r) \leq \frac{N\omega_n r^n}{M}\,.$$

Define a sharper cone $\widehat{X}$ by

$$\widehat{X} = \mathbb{R}^p \cap \left\{ x : |\pi(x)| < \frac{1}{9}|x| \right\},$$

and set

$$R = A_r \cap \left\{ a : A_r \cap (\widehat{X} + a) = \emptyset \right\}.$$

Notice that $\pi_{|R}$ is injective and that $\operatorname{Lip}(\pi_{|R})^{-1} \le 9$. Consequently, R is rectifiable and thus, by our unrectifiability assumption on A,

$$(\lozenge) \qquad \mathcal{H}^n(R) = 0.$$

Next, for $x \in A_r \sim R$, choose $x^* \in A_r \cap (\widehat{X} + x)$ such that

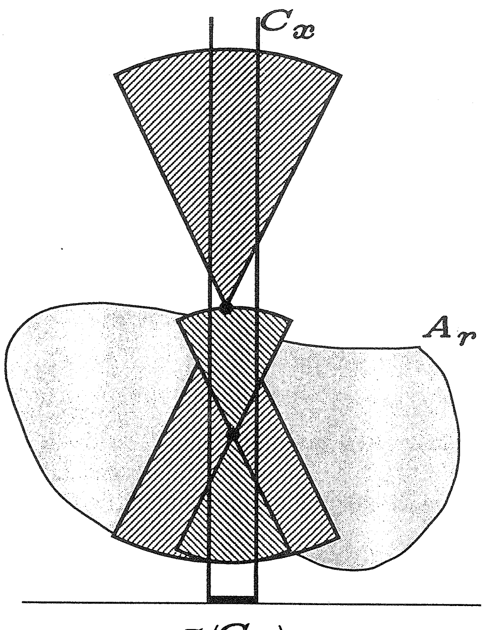
$$|x - x^*| = h(x) \equiv \max \left\{ |x - z| : z \in A_r \cap (\widehat{X} + x) \right\}.$$

Now define the cylinder

$$C_x = \mathbb{R}^p \cap \left\{ z : |\pi(z - x)| \le \frac{h(x)}{9} \right\} \qquad x \in A_r \sim R.$$

We shall show that

$$(\Delta) \quad A_r \cap C_x \subseteq B_{h(x)}(x^*) \cap B_{2h(x^*)}(x) \cap ((X + x) \cup (X + x^*)) \qquad x \in A_r \sim R.$$



To see this, consider $z \in A_r \cap C_x$. First, if $|z - x| > h(x)$ then $z - x \in \widehat{X}$, and then the definition of $h(x)$ ensures that $|z - x| \leq h(x)$ in any case. Thus $z \in B_{h(x)}(x)$, and then $z \in B_{2h(x)}(x^*)$ also, by the triangle inequality. Finally, suppose $z \notin X + x^*$. Then

$$\begin{aligned} |z - x^*| &\leq 3|\pi(z - x^*)| \\ &\leq 3\left(|\pi(z - x)| + |\pi(x - x^*)|\right) \\ &< \frac{2h(x)}{3} \qquad \text{(since } z \in C_x \text{ and } x^* \in \widehat{X}\text{)}. \end{aligned}$$

Thus, by the triangle inequality, $|z - x| > \frac{h(x)}{3}$. Then, since $z \in C_x$, we get $z - x \in X$, as desired.

★ *PART 5: VITALI COVERING ARGUMENT.*

Noting $h(x) \leq 2r < \frac{\delta}{2}$, (▲) and (△) imply

$$\mathcal{H}^n(A_r \cap C_x) \leq \frac{(1 + 2^n)\omega_n}{M}(h(x))^n \qquad x \in A_r \sim R.$$

We now establish (◀) by a Vitali argument. Define the thinner cylinder

$$\widehat{C}_x = \left\{ z : |\pi(z - x)| < \frac{h(x)}{45} \right\}.$$

Applying the Unsubtle Vitali Lemma (7) to $\pi(A_r \sim R)$, we obtain a countable $S \subset A_r \sim R$ such that $A_r \sim R \subseteq \cup_{x \in S} C_x$ and with $\{\widehat{C}_x\}_{x \in S}$ pairwise disjoint. Noting that each $\pi(\widehat{C}_x) \subseteq \pi(B_{2r}(a))$, we calculate

$$\begin{aligned} \mathcal{H}^n(A_r) &\leq \sum_{x \in S} \mathcal{H}^n(A_r \cap C_x) + \mathcal{H}^n(R) \\ &\leq \frac{45^n(1 + 2^n)\omega_n}{M} \sum_{x \in S} \left(\frac{h(x)}{45}\right)^n \\ &\leq \frac{N}{M} r^n, \end{aligned}$$

where $N = 90^n(1 + 2^n)\omega_n$. This establishes (◀), and we can rest.

34

Let M be the points in A where A has an approximate tangent plane. (Notice A may have tangent planes at points outside of A but, by 30 and Theorem 9(ii), these form a set of measure zero). Let $P = A \sim M$. We want to show that M is rectifiable and that P is purely unrectifiable.

By Theorem 9(ii) and Lemma 12, M has an approximate tangent plane at almost all points in M. Thus, by Theorem 14, M is rectifiable.

The proof that P is purely unrectifiable is similar. Let $R \subseteq P$ be rectifiable: we want to show that $\mathcal{H}^n(R) = 0$. By Theorem 13, R has a tangent plane at almost all points of R. Thus, by Theorem 9(ii) and Lemma 12, M has a tangent plane at almost all points of R. By the definition of P, $\mathcal{H}^n(R) = 0$, as desired.

(35) [Mo,p31], [Si,§§8,11].

We start with the Area Formula as given in (20) :

$$(\blacklozenge) \qquad \mathcal{H}^n(f(A)) = \int_A Jf \, d\mathcal{L}^n \qquad A \subseteq \mathbb{R}^n \ \mathcal{L}^n\text{-measurable.}$$

Here $f : \mathbb{R}^n \to \mathbb{R}^p$ is Lipschitz and either $f_{|A}$ is injective or $Jf_{|A} \equiv 0$. We now give generalizations of ($\blacklozenge$): for functions (Part 1); for manifolds (Part 2); and, finally, for rectifiable sets (Part 3).

★ *PART 1: FUNCTIONS.*

With f as above, we show that

$$(\blacktriangle) \qquad \int_{f(A)} g \, d\mathcal{H}^n = \int_A (g \circ f) Jf \, d\mathcal{L}^n .$$

Here $g : f(A) \to \mathbb{R}$ is nonnegative and measurable. To prove ($\blacktriangle$), first assume $g \leq M$. Then

$$g = \sum_{k=1}^{\infty} \frac{M}{2^k} \chi_{C_k}$$

where we inductively define

$$C_k = f(A) \cap \left\{ x : g(x) \geq \frac{M}{2^k} + \sum_{j=1}^{k-1} \frac{M}{2^j} \chi_{C_j}(x) \right\} .$$

Recalling, The Measurability Lemma of (12) , ($\blacktriangle$) now follows by summing in ($\blacklozenge$) and applying the Monotone Convergence Theorem ([EG,p20], [TheHutch1,p16]). By another application of the Monotone Convergence Theorem, we can dispense with the hypothesis that g is bounded.

★ *PART 2: MANIFOLDS.*

Now we consider $f : M^n \to \mathbb{R}^p$ Lipschitz where $M^n \subseteq \mathbb{R}^N$ is an n-dimensional C^1-submanifold. It is enough to focus upon a particular coordinate chart $\phi : U \to \mathbb{R}^n$. Writing $h = \phi^{-1}$, we note that $f \circ \phi$ is differentiable almost everywhere (Rademacher's Theorem), and thus we can define the derivative $D^M f(a) : T_a M \to \mathbb{R}^p$ by

$$D^M f(a) \cdot (D_j h(0)) \equiv D_j(f \circ h)(0) \qquad h(0) = a.$$

By the usual manifoldish calculations, $D^M f$ is independent of the chart ϕ and is a linear map ([Bartman,§3],[Bo,§4.1]). $T_a M$ also has the natural (induced) metric, and we can write

$$D^M f(a) = \rho \circ \sigma$$

where $\sigma : T_a M \to T_a M$ is linear and $\rho : T_a M \to \mathbb{R}^p$ is a linear and orthogonal injection. Then we define the Jacobian $J^M f(a)$ of f as

$$J^M f(a) \equiv |\det \sigma| .$$

We now prove the Area Formula: with the usual hypotheses on f,

$$(\blacktriangledown) \qquad \mathcal{H}^n(f(A)) = \int_A J^M f \, d\mathcal{H}^n \qquad A \subseteq M \;\; \mathcal{H}^n\text{-measurable.}$$

In fact, from (♦) and (▲), we have

$$\mathcal{H}^n(f(A)) = \int_{\phi(A)} J(f \circ h) \, d\mathcal{L}^n$$

and

$$\int_A J^M f \, d\mathcal{H}^n = \int_{\phi(A)} (J^M(f \circ h)) Jh \, d\mathcal{L}^n .$$

Thus we just have to show

$$J(f \circ h)(0) = J^M f(a) \cdot Jh(0) \qquad h(0) = a.$$

To see this, first note that $Jh = |\det Dh|$ (That is, considering $h : U \to \mathbb{R}^n$ or $h : U \to M$ gives rise to the same Jacobian). Therefore, since $D(f \circ h) = D^M f \circ Dh$, we have

$$J(f \circ h)(0) = |\det \sigma \circ Dh(0)| = |\det \sigma| \cdot |\det Dh(0)| = J^M f(a) \cdot Jh(0) ,$$

as desired.

★ *PART 3: RECTIFIABLE SETS.*

Finally, we want to prove (▼) continues to hold when M is a rectifiable set. The proof is similar to that for Corollary 8, except we use Theorem 13 in place of Whitney's Theorem. Writing $M \subseteq N_0 \cup \bigcup_j N_j$ as in Corollary 7, we can (almost everywhere) define

$$J^M f \equiv J^{N_j} f .$$

Then, applying (▼) to each $A \cap N_j$ and summing, (▼) for M follows.

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