DEGREE THEORY

E.N. DANCER

References: N. Lloyd, Degree theory, Cambridge University Press.

K. Deimling, Nonlinear functional analysis, Springer Verlag.

L. Nirenberg, Topics in nonlinear functional analysis, Courant Institute lecture notes, New York University.

R.F. Brown, A topological introduction to nonlinear analysis, Birkhauser.

K.C. Chang, Infinite dimensional Morse theory and multiple solution problems, Birkhauser.

M. Matzeu and A. Vignoli, Topological nonlinear analysis, Birkhauser.

(The first three are general references on degree theory)

LECTURE 1

The idea of degree theory is to give a "count" of the number of solutions of nonlinear equations but to count solutions in a special way so that the count is stable to changes in the equations. To see why the obvious count does not work well consider a family of maps $f_t(x)$ on R defined by $f_t(x) = x^2 - t$. As we vary t, f_t changes smoothly. For $t < 0$, it is easy to see that $f_t(x) = 0$ has no solution, $f_0(x) = 0$ has zero as its only solution while for $t > 0$, there are two solutions $\pm\sqrt{t}$. Hence the numbers of solutions changes as we vary t. Hence, to obtain something useful, we need a more careful count. A clue is that $\dfrac{\partial f_t}{\partial x}$ has different signs at the

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

two solutions $\pm\sqrt{t}$.

Before, we proceed further, we need some notation. Assume D is a bounded open set in R^n and $f : \overline{D} \longrightarrow R^n$ is C^1. We say p is a regular value of f if det $f'(x) \neq 0$ (equivalently $f'(x)$ is invertible) whenever $x \in D$ and $f(x) = p$. One thinks of the regular points as the nice points. Note that p is a regular value if $p \notin f(D)$. Now suppose $f : \overline{D} \longrightarrow R^n$ is smooth, $p \notin f(\partial D)$ and p is a regular value of f. Since p is a regular value of f, the inverse function theorem implies that the solutions of $f(x) = p$ in D are isolated in D. On the other hand $\{x \in D : f(x) = p\}$ is compact since it is a closed subset of $\overline{D}$. Since a compact metric space which consists of isolated points is easily seen to be finite, it follows that $\{x \in D : f(x) = p\}$ is finite. We then define the degree of f

$$\deg(f, p, D) \text{ to be } \sum \text{sign det } f'(x_i)$$

where x_i are the solutions of $f(x) = p$ in D. I stress that we are assuming f is smooth, $p \notin f(\partial D)$ and p is a regular value of f. Here

$$\text{sign } y = \begin{cases} 1 & \text{if } y > 0 \\ -1 & \text{if } y < 0 \end{cases}$$

and sign 0 is not defined.

You might ask why we assume $p \notin f(\partial D)$. The reason is that, otherwise, a solution might move from inside D to out of D as we make small perturbations of f or p. In this case, we would expect that any "count" of the number of solutions might well change.

Suppose p is not a regular value of f (but f is smooth and $p \notin f(\partial D)$). We try to define

$$\deg(f, p, D) = \lim_{i \longrightarrow \infty} \deg(f, p_i, D)$$

where p_i are regular values of f approaching p. There are two problems with this. Firstly, we need to know that there are such regular values and secondly that the limit exists (and is independent of the choice of p_i). The first problem is resolved by the following.

Sard's Theorem. Assume $f : D \longrightarrow R^n$ is C^1 where D is open in R^n (not necessarily bounded). Then the set of regular values of f are dense in R^n.

This is proved by showing that the complement of the set of regular values of f has measure zero. We only sketch the main idea. The main idea in the proof is to show that if $x_0 \notin D$ is and $f'(x_0)$ is not invertible, then, for $\widetilde{B}$ a small ball center x_0, $f(\widetilde{B})$ is squashed close to the hyperplane $f(x_0)+f'(x_0)R^n$ (since f is well approximated by its derivative near x_0). Note that a hyperplane has zero measure. One uses this to show $f(\widetilde{B})$ has much smaller measure than $\widetilde{B}$ if $\widetilde{B}$ has small radius.

To overcome the second problem is more difficult. (Note that the set of regular values of f need not be connected). It turns out that the shortest proof is indirect. We consider the integral

$$d(\phi) = \int_{\overline{D}} \phi(f(x))J_f(x)dx$$

where J_f denotes the Jacobian of f (where f is as above), ϕ in C^∞ on R^n, support of ϕ is contained in an n- dimensional cube $C, p \in$ int $C, C \cap f(\partial D)$ is empty and $\int_C \phi dx = 1$. Such a ϕ is said to be admissible. It is easy to find at least one admissible ϕ exists by choosing C a small cube center p. The advantage of using integrals is that they are much more regular under perturbations. We will show (rather sketch) that $d(\phi)$ is in fact the same as deg. To see this, note that, if ϕ_1 and ϕ_2 are admissible with support in the same cube C, then

$$d(\phi_1) - d(\phi_2) = \int_{\overline{D}} (\phi_1 - \phi_2)(f(x))J_f(x)dx$$

One proves the right hand side is zero by proving that the integrand is the divergence of a (vector) function vanishing near ∂D and applying the divergence theorem. This is the messy part of the proof. This implies that $d(\phi)$ is independent of admissible ϕ.

Secondly, note that p does not appear explicitly in the integral defining $d(\phi)$ and hence we deduce that $d(\phi)$ is locally constant in p.

Thirdly, if p is a regular value of f, then $d(\phi) = \deg(f, p, D)$. To see this note that, by the implicit function theorem, $f(x) = p$ has only a finite number of solutions $\{x_i\}_{i=1}^k$ in D and we can choose disjoint neighbourhoods D_i of x_i in D

such that f maps D_i diffeomorphically on to a neighbourhood of p and $f'(x)$ has fixed sign on D_i (by shrinking D_i is necessary). We than choose admissible ϕ such that the cube $C \subseteq \cap f(D_i)$. By the change of variable theorem for integrals

$$\begin{aligned}\int_{D_i} \phi(f(x))J_f(x)dx &= \text{sign } J_f(x_i)\int_{f(D_i)} \phi(z)dz \\ &= \text{sign } J_f(x_i) \quad \text{since support } \phi \subseteq f(D_i)\end{aligned}$$

Thus

$$\begin{aligned}d(\phi) = \int_D \phi(f(x))J_f(x)dx &= \sum_{i=1}^{h} \int_{D_i} \phi(f(x)J_f(x)dx \\ &\qquad \text{since the integrand is zero outside } \cup_{i=1}^{k} D_i \\ &= \sum_{i=1}^{k} \text{sign } J_f(x_i) \\ &= \deg\,(f, p, D)\end{aligned}$$

Note that we have used that $d(\phi)$ is independent of ϕ to be able to choose a special ϕ and the reason the argument works is the close relation between $d(\phi)$ and the change of variable theorem.

We now easily complete the proof that the degree is defined. If p is not a regular value for f. Then $d(\phi)$ is the same for all q near p. However, if q is a regular value, $d(\phi) = \deg(f, q, D)$. Hence we see that for all regular values p_i of f near p, $\deg\,(f, p_i, D) = d(\phi)$ and hence $\lim_{i \longrightarrow \infty} \deg\,(f, p_i, D)$ exists, as required.

Note that we could use the interval $d(\phi)$ to define the degree but it is harder to deduce properties of the degree from this definition.

Assume $f : R^n \longrightarrow R^n$ is smooth, D is bounded open in R^n and $p \notin f(\partial D)$. Our proof that $\lim_{i \longrightarrow \infty} \deg(f, p_i, D)$ has a limit if p_i are regular values of f approaching p was rather indirect (by way of $d(\phi)$). There is another proof which considers two regular values p_i, p_j of f near p and studies how the solutions of $f(x) = tp_i + (1-t)p_j$ changes as t varies from 0 to 1. This proof while more intuitive is rather more technical. Note that the degree can also be constructed by more topological arguments.

Note that by its construction $\deg(f, p, D)$ is integer valued.

We now consider properties of the degree we have constructed. The basic properties are the following where we assume D is bounded open in R^n, $f : \overline{D} \longrightarrow R^n$ is continuous and $p \notin f(\partial D)$.

(i) If $\deg(f, p, D) \neq 0$, there exists $x \in D$ such that $f(x) = p$.

(ii) (excision) If $D_i, i = 1, \cdots, m$, are disjoint open subsets of D and $f(x) \neq p$ if $x \in \overline{D} \setminus \cup_{i=1}^m D_i$, then

$$\deg(f, p, D) = \sum_{i=1}^{m} \deg(f, p, D_i)$$

(iii) (products) If D_1 is bounded open in R^m, $g : \overline{D}_1 \longrightarrow R^m$ is continuous and $q \notin g(\partial D_1)$, then $\deg((f,g),(p,q), D \times D_1) = \deg(f, p, D)\deg(g, q, D_1)$. Here (f, g) is the function on R^{m+n} defined by $(f,g)(x,y) = (f(x), g(y))$ for $x \in R^n, y \in R^m$.

(iv) homotopy invariance. If $F : \overline{D} \times [a,b] \longrightarrow R^n$ is continuous and if $F(x,t) \neq p$ for $x \in \partial D$ and $t \in [a,b]$, then $\deg(F_t, p, D)$ is defined and independent of t for $t \in [a,b]$. Here F_t is the map of $\overline{D}$ into R^n defined by $F_t(x) = F(x,t)$.

Note that we have not actually defined the degree of maps f which are only continuous. The above 4 properties are first proved for smooth functions. Properties (i) – (iii) are proved first for p a regular value and the general case is proved by approximating p by regular values. In the smooth case, (iv) is proved by using the integral formula for the degree (i.e. $d(\phi)$) to show that $\deg(F_t, p, D)$ is continuous in t and then using that a continuous integer valued function on $[a,b]$ is constant. Finally, if f is only continuous, deg (f,p,D) is defined to be deg $(\widetilde{f},p,D)$ where $\widetilde{f}$ is a smooth function uniformly close to f on $\overline{D}$. To show that this is independent of the choice of $\widetilde{f}$ one uses that, if $\widetilde{f}_1$ is another smooth approximation to f, then one can apply homotopy invariance to the smooth homotopy $t\widetilde{f}(x)+(1-t)\widetilde{f}_1(x)$ for $x\in\overline{D}, t\in[0,1]$. Lastly, one uses an approximation argument to extend properties (i) – (iv) from the smooth case to the continuous case. This is actually all straightforward, albeit a little tedious. This completes the construction of the degree.

It is easy to deduce from the definition a number of other properties of the degree. In particular,

$$\deg\,(f,p,D) = \deg\,(f(x)-p,0,D)$$

whenever deg (f,p,D) is defined and that deg $(A,0,D)$ = sign det A if A is an invertible $n\times n$ matrix and D is a bounded open set containing zero (and where we are identifying a matrix and the corresponding linear map).

In general, one can prove many properties of the degree by first proving it for f smooth and p a regular value of f and then using limit arguments.

The homotopy invariance property is a very important property of the degree because it often enables us to calculate the degree by deforming our map to a much simpler map. This enables us to calculate the degree of some complicated maps.

As a simple application, we prove the very useful Brouwer fixed point theorem. Assume S is closed bounded and convex in R^n and $f: S\longrightarrow S$ is continuous. Then f has a fixed point i.e. there exists $x\in S$ such than $x=f(x)$. This theorem is used in many places. For example, it is frequently used to prove the existence of periodic solutions of ordinary differential equations. To prove the theorem, we assume that int S is non-empty and $0\in$ int S. (It is not difficult to reduce the general case to this case using the geometry of convex sets). We assume $A(x)\neq x$ if $x\in\partial S$. (Otherwise we are finished). We use the homotopy $(x,t)\longrightarrow x-tA(x)$. The convexity and that $0\in$ int S imply that $x\neq tA(x)$ if $x\in S$ and $t\in[0,1]$ (since if $x-tA(x)=0, x=(1-t)0+tA(x)\in$ int S). Thus, by homotopy invariance, deg $(I - A, 0, \text{ int } S) = $ deg $(I, 0 \text{ int } S) = 1 \neq 0$ and hence there exists $x \in$ int S such that $x - A(x) = 0$ i.e. x is a fixed point.

I mention one more useful property of the degree. Firstly if $D \subseteq R^n$ is bounded open and symmetric (that is $x \in D \longleftrightarrow -x \in D$), $0 \in D, f : \overline{D} \longrightarrow R^n$ is continuous and odd and if $f(x) \neq 0$ for $x \in \partial D$, then deg $(f, 0, D)$ is odd. (The most important point is that it is non-zero.) If f is smooth and zero is a regular value of f, then this is easy to prove because ($f(0) = 0$ and if $f(\widetilde{x}) = 0$ then $f(-\widetilde{x}) = 0$ and $J_f(\widetilde{x}) = J_f(-\widetilde{x})$ (since f is odd). The main difficulty in the general case is to prove that we can approximate a smooth odd map by a smooth odd map which has zero as a regular value. This is rather technical. This result has many uses. for example, it easily implies that a continuous odd mapping of R^m into R^n with $m > n$ has a zero on each sphere $\|x\| = r$. We will return to uses for differential equations later. It has many other uses on the geometry of R^n. For example, it can be used to prove that if $f : R^n \longrightarrow R^n$ is $1-1$ and continuous then it is an open map (that is, f maps open sets to open sets).

There are many other results on the computation of the degree but do not assume the degree is always easy to evaluate!

In this lecture, we extend the degree to infinite dimensions. It turns out that this is important for many applications.

First note that we cannot expect to be able to do this for all maps. The simplest way to see this is to note that the analogue of Brouwer's fixed point theorem fails in some infinite dimensional Banach spaces. We consider the Banach space c_0 of sequences $(x_i)_{i=1}^{\infty}$ with norm $\|(x_i)\| = \sup_{i\geq 1}|x_i|$. We then consider the map $T : c_0 \longrightarrow c_0$ defined by $T(x_1, x_2, \cdots) = (1, x_1, x_2 \cdots)$. Note that T is an affine map. It is easy to check that T is continuous and that T maps the closed unit ball in c_0 into itself. However, T has no fixed point in c_0 because if $T(x_i) = (x_i)$, then $x_1 = 1$ and $x_{i+1} = x_i$ for $i \geq 1$. Thus $x_i = 1$ for $i \geq 1$, which contradicts that $(x_i) \in c_0$. It turns out that in every infinite- dimensional Banach space there is always a closed bounded set S and a continuous map of S into itself without a fixed point.

Hence to proceed further, we need a restricted class of maps. Assume that W is a closed subset of a Banach space K. We say that $A : W \longrightarrow E$ is completely continuous if A is continuous and if $\overline{A(S)}$ is a compact subset of E for each bounded subset S of W. We will construct a degree for maps $I - A$ where A is completely continuous. It turns out that many (but far from all) of the mappings occurring in applications are completely continuous.

The reason that we can construct a degree for such maps is that we can approximate completely continuous maps by maps whose range lie in a finite- dimensional space.

Lemma. If K is a compact convex subset of a Banach space E and $\epsilon > 0$, there is a continuous map $P_\epsilon : K \longrightarrow K$ such that $\|P_\epsilon(x) - x\| \leq \epsilon$ for $x \in K$ and the range of P_ϵ is contained in a finite-dimensional subspace of E.

This is proved by noting that a compact set has a finite $\frac{1}{2}\epsilon$ net and using some form of partition of unity. The details are not difficult. In using this in our applications, it is useful to note that the closed convex hull of a compact set is compact.

If D is bounded and open in E and $A : \overline{D} \longrightarrow E$ is completely continuous, then

$P_\epsilon A = P_\epsilon o A$ is a continuous map such that $\|P_\epsilon A(x) - A(x)\| \leq \epsilon$ on $\overline{D}$ and the range of $P_\epsilon A$ is contained in a finite-dimensional subspace of E. (We define K to be the closed convex hull of $A(\overline{D})$. If $p \notin (I - A)(\partial D)$ and ϵ is small, we define deg $(I - A, p, D) =$ deg $((I - A_\epsilon)|_M, p, D \cap M)$ where $A_\epsilon = P_\epsilon A$, M is a finite-dimensional subspace of E containing p and the range of A_ϵ. Note that $I - A_\epsilon$ will map $\overline{D} \cap M$ into M. There is quite a bit to be checked here. The right hand side is defined by our earlier construction. We need to prove that the finite-dimensional degree is defined and is independent of ϵ and the choice of M. The details are tedious but not difficult. Note that a simple compactness argument ensures that there is an $\alpha > 0$ such that $\|x - A(x)\| \geq \alpha$ if $x \in \partial D$, that our finite-dimensional degree is a degree on finite dimensional normed spaces rather than just R^n because it is easy to check that our original degree on R^n is independent of the choice of basis in R^n and that we need to prove a lemma on the finite- dimensional degree to prove that the right hand side of the definition is independent of the choice of M.

By using finite-dimensional approximations and compactness arguments, it is not difficult to check that the four basic properties of the finite-dimensional degree have analogues here. Assume that D is bounded and open in E and $A : \overline{D} \longrightarrow E$ is completely continuous such that $x \neq A(x) + p$ for $x \in \partial D$.

The following hold.

(i) If deg $(I - A, p, D) \neq 0$, there exists $x \in D$ such that $x = A(x) + p$.

(ii) If $D_i, i = 1, \cdots, m$, are disjoint open subsets of D such that $x \neq A(x) + p$ if $x \in \overline{D} \setminus \cup_{i=1}^m D_i$, then deg $(I - A, p, D) = \sum_{i=1}^{m}$ deg $(I - A, p, D_i)$.

(iii) products. If D_1 is bounded open in a Banach space F, $G : \overline{D}_i \longrightarrow F$ is completely continuous and $q \in F$ such that $x \neq G(x) + q$ for $x \in \partial D_1$, then deg $(I - (A, G), (p, q), D \times D_1) =$ deg $(I - A, p, D)$ deg $(I - G, q, D_1)$.

(iv) homotopy invariance. If $H : \overline{D} \times [a, b] \longrightarrow E$ is completely continuous and $x - H(x, t) \neq p$ for $x \in \partial D, t \in [a, b]$, then deg $(I - H_t, p, D)$ is independent of t for $t \in [a, b]$.

Note that in (iv) it is not sufficient to assume that H is continuous and each H_ϵ is completely continuous. However this is sufficient if we also assume that H_t is uniformly continuous in T (uniformly for $x \in \overline{D}$).

There is analogue of Brouwer's fixed point theorem, known as Schauder's fixed point theorem. Assume that D is closed and convex in a Banach space E, A is continuous, $A(D) \subseteq D$ and $\overline{A(D)}$ is compact. Then A has a fixed point. The easiest way to prove this to use the lemma to find a finite dimensional approximation to which we can apply Brouwer's theorem.

Nearly all the extra properties of the degree in finite dimensions have natural analogues here. The only one that is a little different is the formula for the degree of a linear map.

Assume that $B : E \longrightarrow E$ is linear and compact and $I - B$ has trivial kernel. (Note that for linear maps complete continuity is equivalent to compactness.) The spectrum of B consists of zero plus a finite or countable sequence of eigenvalues with 0 as it only limit point. If λ_i is a non-zero eigenvalue of B, the algebraic multiplicity $m(\lambda_i)$ of λ_i is defined to be the dimension of the kernel of $(\lambda_i I - B)^q$ for large enough q. It can be shown that $m(\lambda_i)$ is finite and independent of q for large q. The result is then that

$$\deg(I - B, 0, E_\delta) = (-1)^{\sum m(\lambda_i)}$$

where the summation is over the (real) eigenvalues of B in $(1, \infty)$ and E_δ is the open ball of radius δ in E. Note that the sum is finite. The proof of this is by using linear operator theory and constructing homotopies to reduce to the finite dimensional case.

As a final comment, there has been a good deal of work on extending the degree to mappings only defined on closed convex subsets of Banach spaces and to evaluating degree of critical points obtained by variational methods (as in Chabrowski's lectures or Chang's book). One major difference in the extension to closed convex sets is that the formula for the degree of isolated solutions becomes rather more complicated. This is discussed in my chapter in the book of Matzeu and Vignoli.

LECTURE 4

In this lecture, we consider some very simple applications of the degree. We do not try to obtain the most general results.

Firstly, we consider an application of Brouwer's fixed point theorem to prove the existence of periodic solutions of ordinary differential equations. Assume $f : R^{n+1} \longrightarrow R^n$ is C^1 and T periodic in the last variable, i.e., $f(x, t + T) = f(x, t)$ if $x \in R^n, t \in R$. We are interested in the existence of T periodic solutions of the equation

$$x'(t) = f(x(t), t) \tag{1}$$

To do this, we let $U(t, x_0)$ denote the solution of (1) with $U(0, x_0) = x_0$ (for $x_0 \in R^n$). Standard results ensure that (1) has a unique solution satisfying the initial condition. We need to place an assumption on f which ensures that $U(t, x_0)$ is defined for $0 \leq t \leq T$. (The only way this could fail is that the solution blows up before $t = T$). A sufficient condition ensuring this is true is that there is a $K > 0$ such that $\|f(x, t)\| \leq K\|x\|$ for $t \in [0, T]$ and for $\|x\|$ large (where $\| \quad \|$ is one of the standard norms on R^n). In this case $U(t, x_0)$ is continuous in t and x_0. Since f is T periodic in t, it is not difficult to show that $U(t, x_0)$ is a T periodic solution of (1) if and only if $U(T, x_0) = x_0$ i.e. x_0 is a fixed point of the map $x \longrightarrow U(T, x)$. Hence we can hope to our earlier degree results (to the map $f(x) = x - U(T, x)$). The simplest rest of the this type is that if we can find a closed ball $\overline{B}_r$ in R^n such that $U(T, x) \in \overline{B}_r$ if $x \in \overline{B}_r$ then Brouwer's theorem (applied on $\overline{B}_r$) implies that the map $x \longrightarrow U(T, x)$ has a fixed point and hence the original equation has a T periodic solution. For example, the inclusion condition on $\overline{B}_r$ holds if $< f(x, t), x > < 0$ for $\|x\| = r$ and $0 \leq t \leq T$ (where $<, >$ is the usual scalar product on R^n and we must use the norm induced by the scalar product).

Before we look at the second example, I need one very useful consequence of degree theory. Assume that E is a Banach space and $H : E \times [0, 1] \longrightarrow E$ is completely continuous, the map $x \longrightarrow H(x, 0)$ is odd and there is an $R > 0$ such that $x \neq H(x, t)$ if $\|x\| = R$ and $t \in [0, 1]$. Then the equation $x = H(x, 1)$ has a solution. This is proved by noting that by homotopy invariance

$$\deg\,(I - H(\ ,1), 0, E_R) = \deg\,(I - H(\ ,0), 0, E_R) \neq 0$$

by the result on the degree of odd mappings. This is a very nice theorem because the statement is very simple and it is very widely applied. In practice, one usually establishes for a large R that $x \neq H(x,t)$ if $t \in [0,1]$ and $\|x\| \geq R$. In other words, one is proving that any solution of $x = H(x,t)$ satisfies $\|x\| \leq R$. In practice, this is usually by far the most difficult step in verifying the assumption of the result above. Such a result is called an a priori bound. In some of the lectures on partial differential equations, I am sure you have discussed the problem of obtaining a priori bounds. In fact, the above result often reduces the proof of the existence of a solution of a partial differential equation to the proof of a priori bounds in a suitable Banach space. (The choice of a suitable Banach space is not always simple).

We give a very simple application of the result above which requires few technicalities. Assume Ω is a bounded domain in R^n with smooth boundary and $g : R \longrightarrow R$ is continuous such that $y^{-1}g(y) \longrightarrow a$ as $|y| \longrightarrow \infty$ where a is not an eigenvalue of $-\Delta$ under Dirichlet boundary conditions on Ω. We prove that the equation

$$\begin{aligned} -\Delta u(x) &= g(u(x)) \text{ in } \Omega \\ u(x) &= 0 \text{ on } \partial\Omega \end{aligned} \tag{2}$$

has a solution. By a solution we mean a function in $W^{2,p}(\Omega) \cap C(\overline{\Omega})$ which satisfies the equation almost everywhere. (It will be a classical solution if g is a little more regular). It is convenient to work in the space $W^{2,p}(\Omega)$ with $p > \frac{1}{2}n$ (and $p > 1$). By the Sobolev embedding theorem, this ensures that $W^{2,p}(\Omega) \subseteq C(\overline{\Omega})$. We write K to denote the inverse of $-\Delta$ under Dirichlet boundary conditions. We define $G : C(\overline{\Omega}) \longrightarrow C(\overline{\Omega})$ by $(G(u))(x) = g(u(x))$. Then (2) is equivalent to the equation $u = KG(u)$. Thus (2) becomes a fixed point problem. KG is easily seen to be completely continuous since we can think of KG as the composite $K\ o\ g\ o\ i$ where i is the natural inclusion of $W^{2,p}(\Omega)$ into $C(\overline{\Omega})$, since i is compact, since G is continuous and since K is a continuous map of $L^p(\Omega)$ into $W^{2,p}(\Omega)$. (The latter is a regularity result for the Laplacian.) Lastly, if we define $H : W^{2,p}(\Omega) \times [0,1] \longrightarrow W^{2,p}(\Omega)$ by $H(u,t) = K(tG(u) + (1-t)au)$ it is fairly easy to prove that the assumption of the above result hold and thus (2) has a solution. (The a priori bound corresponds to proving a bound for solutions of

$$-\Delta u = tg(u) + (1-t)au \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega.)$$

We omit the details.

In general in partial differential equations, one often has to choose the spaces carefully especially if the equations are nonlinear in derivatives in u. (In fact, these methods run into severe difficulties if the equations are highly nonlinear.) The methods also tend to run into difficulties if Ω is not bounded because the complete continuity tends to fail. There have been many attempts to define degrees for mappings which are not completely continuous to try to overcome this last problem. Some of these are discussed in Ize's chapter in the book of Matzeu and Vignoli.

Thirdly, I obtain a result on Banach spaces though it could be very easily applied to ordinary and partial differential equations. In many applications, there is a rather trivial solution of the problem and we want to look for other solutions. For simplicity, assume that the trivial solution is zero. Hence, we assume that E is a Banach space, $A : E \longrightarrow E$ is completely continuous, $A(0) = 0$ and we look at the equation $x = \lambda A(x)$, (λ might correspond to some physical parameter.) Our assumptions ensure 0 is a solution of the equation for all λ and we look for other solutions. We assume that there is a linear mapping B on E such that $\|A(x) - Bx\|/\|x\| \longrightarrow 0$ as $x \longrightarrow 0$. (B is in fact the derivative of A at zero in a certain sense. One can in fact define a calculus on Banach spaces). It is not difficult to prove that B is compact (since A is completely continuous). Assume μ^{-1} is a non-zero eigenvalue of B of odd algebraic multiplicity. One can then prove that there are solutions (x, λ) of $x = \lambda A(x)$ with $x \neq 0$ and $\|x\| + |\lambda - \mu|$ arbitrarily small. One usually says that $(0, \mu)$ is a bifurcation point. The interest here is that the main assumption (that μ has odd multiplicity) is purely on the linear part of A.

I omit the proof but the key ideas are the formula for the degree of a linear map and that if $\tau \neq 0$ is small and δ is very small compared with τ, deg $(I-(\mu+\tau)A, 0, E_\delta)$ $=$ deg $(I-(\mu+\tau)B, 0, E_\delta)$ and we can evaluate the right hand side. It turns out that much more is true. There is a connected set of solutions of $x = \lambda A(x)$ in $E \times R$ with $x \neq 0$ which starts at $(0, \mu)$ (as above) and continues rather globally (that is to points where x is not small or λ is not close to μ). This can be found in Brown's book.

Lastly, in many applications, the problem may require that we only look for non-negative solutions of our equation. Thus it might be natural to look for solutions u of an elliptic equation which satisfy $u(x) \geq 0$ on Ω (for example if u represents a population). Thus it is often natural to look at problems on convex sets such as $\{u \in C(\Omega) : u(x) \geq 0 \text{ on } \Omega\}$ rather than the whole space.