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Lectures on Minimal Surfaces in $\mathbf{R}^3$

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Contents

1 Introduction

The theory of minimal submanifolds is a fascinating field in differential geometry. The simplest, one-dimensional minimal submanifold, the geodesic, has been studied quite exhaustively, yet there are still a lot of interesting open problems. In general, minimal submanifold theory deeply involves almost all major branches of mathematics; analysis, algebraic and differential topology, geometric measure theory, calculus of variations, and partial differential equations, to name just a few of them.

In these lecture notes our aim is quite modest. We discuss minimal surfaces in $\mathbf{R}^3$, and concentrate on the class of the embedded complete minimal surfaces of finite topological type.

I intend to introduce minimal surfaces with the minimum preliminary requirements. A student who has basic knowledge of differential geometry of curves and surfaces in $\mathbf{R}^3$ and of complex analysis will be able to understand and grasp the material supplied in these notes. I hope these notes will introduce one into a very old but still rapidly growing field of mathematics, and via it to go much further.

We begin with the definition of minimal surfaces in the setting of parametrised surfaces. We define minimal surfaces as conformal harmonic immersions from two dimensional manifolds to $\mathbf{R}^3$. Then we give the proof of the equivalence of this definition to that that the mean curvature of the surface is zero everywhere. After that, we introduce the first variation of surface area, also in the setting of parametrised surfaces, to show that a surface is minimal if and only if it is a stationary point of the area functional. Then we introduce the minimal surface equation and use it to prove several classical theorems of minimal surfaces, such as the maximum principle, the extension theorem, the reflection and rotation theorem, etc. One of the most important features of the theory of minimal surfaces in $\mathbf{R}^3$, which is quite different from the general case of minimal submanifolds in Riemannian manifolds (even in $\mathbf{R}^n$, $n > 3$), is the Enneper-Weierstrass representation. This representation connects minimal surfaces in $\mathbf{R}^3$ to one variable complex analysis. We introduce the Enneper-Weierstrass representation immediately after the necessary preparations and try to use it consistently throughout these notes.

The most interesting minimal surfaces in $\mathbf{R}^3$ are complete and are divided into two groups according to whether the total curvature is finite or infinite. We mainly discuss complete minimal surfaces of finite total curvature. We prove the classical theorem of Osserman (Theorem 10.8) about such surfaces. Then we further discuss the annular ends of such surfaces. After introducing the concept of flux (a formula based on Stokes' theorem), we prove a theorem of López and Ros about uniqueness of the catenoid.

A major part of these notes is devoted to the work of Hoffman and Meeks about global properties of complete minimal surfaces in $\mathbf{R}^3$. In particular, we introduce the Halfspace Theorem, the Cone Lemma, the standard barriers and the Annular End Theorem, and the partial classification of the conformal type of such surfaces.

An annular end of a complete minimal surface is a minimal annulus with compact boundary. In the last part of these notes we discuss minimal annuli. We first introduce results of Osserman and Schiffer, including the isoperimetric inequality for minimal annuli. Then we concentrate on minimal annuli in a slab, proving Shiffman's theorems and some generalisations. For this we first introduce the second variation of area functional and the concept of stability of minimal surfaces. We finish these notes with Nitsche's conjecture and two partial results. Recently, Pascal Collin [6] gives a proof of Nitsche's conjecture, I am regret that I cannot add it to these notes since the proof is quite involved and Collin's paper has not been published yet.

To help readers not familiar with PDE, we include an appendix on the eigenvalue problem of linear second order elliptic differential operators.

In these notes, we emphasize the close relation between minimal surfaces in $\mathbf{R}^3$ and complex analysis. This makes the theory of minimal surfaces in $\mathbf{R}^3$ both much simpler and more beautiful. But the draw back is that the methods are hardly generalisable to the study of general minimal submanifolds in Riemannian manifolds. Nevertheless, by its simplicity and beauty, the complex analysis method, via the Enneper-Weierstrass representation, deserves to be emphasized. Thus we work with isothermal coordinates and whenever possible, we try to express and analyse geometric quantities via the Enneper-Weierstrass representation. Using the Enneper-Weierstrass representation, we are able to give new proofs of the total curvature formula of a complete minimal surface of finite total curvature, and of Shiffman's second theorem and its generalisations.

A very active part of the theory of minimal surfaces in $\mathbf{R}^3$ is the construction of new embedded complete minimal surfaces. Minimal surface theory is among the oldest branches in mathematics. For over two hundred years, the only known embedded complete minimal surfaces of finite topology were the plane, the catenoid, and the helicoid. In 1984, Hoffman and Meeks started a new wave of discovery. Infinite embedded complete minimal surfaces were constructed via the Enneper-Weierstrass representation and with the aid of computer graphics. These discoveries stimulated a new wave of active researches in the theory of minimal surfaces in $\mathbf{R}^3$. It is a regret that we cannot discuss in detail the techniques of construction of minimal surfaces in these notes. The interested reader is recommoned to works such as [26], [27], [31], [39], [40], [41], [80].

Some classical topics such as the Plateau problem are not discussed here since there are already many excellent books available, for example, [9], [46], [77], [61], [37], [12]. We also do not discuss the regularity problem, which requires tools from the theory of partial differential equations, see [12].

I would like to express the most sincere thanks to Dr. John Hutchinson, without whose encouragement and support, careful reading and correcting my English expressions in the first several drafts, and wise observations on the mathematical material, these notes could never have been published.

These notes are based on lectures given at the ANU for a one-semester fourth year honours course in 1994. I appreciate all of the participants for their enthusiasm in this topic. I would like to thank Dr. John Urbas for pointing out an improved proof of Shiffman's third theorem.

I am much obliged to Prof. Fusheng Wei for supplying the pictures of complete minimal surfaces in these notes.

I learned minimal surfaces from Professor David Hoffman. I will never forget his guidance, encouragement and support.

Special thanks also go to Professor Neil Trudinger for his support and encouragement.

Last, but not least, I would like to thank my wife Lin Han, without whose love, patience and understanding, I could never have finished this job.

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2 Definition of Minimal Surfaces

Definition 2.1 A *minimal surface* in $\mathbf{R}^3$ is a *conformal harmonic immersion* $X : M \hookrightarrow \mathbf{R}^3$, where M is a 2-dimensional smooth manifold, with or without boundary. Here *conformal* means that for any point $p \in M$ there is a local coordinate neighbourhood $(U, (u, v))$ on M, such that in U the vectors

$$X_1 := X_u = \frac{\partial X}{\partial u} = \left(\frac{\partial X^1}{\partial u}, \frac{\partial X^2}{\partial u}, \frac{\partial X^3}{\partial u}\right) = (X_u^1, X_u^2, X_u^3) = (X_1^1, X_1^2, X_1^3)$$

and

$$X_2 := X_v = \frac{\partial X}{\partial v} = \left(\frac{\partial X^1}{\partial v}, \frac{\partial X^2}{\partial v}, \frac{\partial X^3}{\partial v}\right) = (X_v^1, X_v^2, X_v^3) = (X_2^1, X_2^2, X_2^3)$$

are perpendicular to each other and have the same length. Thus

$$\Lambda^2 := |X_u|^2 = |X_v|^2 > 0, \quad X_u \bullet X_v \equiv 0.$$

Here $\bullet$ is the Euclidean inner product. Such a coordinate neighbourhood $(U, (u, v))$ is called an *isothermal neighbourhood,* its coordinates (u, v) are called *isothermal coordinates.*

The word *immersion* means that for any $p \in M$, $X_* := dX : T_pM \to T_{X(p)}\mathbf{R}^3$ is a linear embedding. In the case X is conformal, it means simply that $\Lambda > 0$ on M.

The word *harmonic* means that

$$\triangle X = \frac{\partial^2 X}{\partial u^2} + \frac{\partial^2 X}{\partial v^2} = X_{uu} + X_{vv} = X_{11} + X_{22} \equiv \vec{0}.$$

If M is connected, then we say that the surface X is *connected.* We will only consider connected surfaces. Furthermore, since any non-orientable surface has an orientable double covering, we will only consider *oriented minimal surfaces.*

A homothety of $\mathbf{R}^3$ is the composition of a rigid motion and a dilation or a shrinking. Let T be a homothety of $\mathbf{R}^3$, $X : M \hookrightarrow \mathbf{R}^3$ be a surface. It is easy to see that X is a conformal harmonic immersion if and only $T \circ X$ is. Thus we consider all surfaces in $\mathbf{R}^3$ up to a homothety. That is, we do not distinguish the surfaces $X : M \hookrightarrow \mathbf{R}^3$ and $T \circ X : M \hookrightarrow \mathbf{R}^3$.

A classical theorem says that any C^k immersion, $2 \le k \le \infty$, can have an atlas of isothermal coordinate charts, so that X being conformal is not a special property of minimal surfaces. The important fact which distinguishes minimal surfaces is that under these isothermal charts, X is harmonic.

For an orientable surface $X : M \hookrightarrow \mathbf{R}^3$, let $\{(U_\alpha, z_\alpha = u_\alpha + iv_\alpha)\}_{\alpha \in A}$ be an atlas of isothermal coordinates of the same orientation, then $\{(U_\alpha, z_\alpha)\}_{\alpha \in A}$ defines a *complex (conformal) structure* on M.

Precisely, we will prove that if V is any isothermal coordinate neighborhood, with the coordinates $w = x + iy$ having the same orientation as $z = u + iv$ on $U \bigcap V$, then $z o w^{-1} : w(U \cap V) \to z(U \cap V)$ is a holomorphic function. Which is equivalent to saying that the functions $u(x, y)$ and $v(x, y)$ satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

To see this, compute

$$\frac{\partial X}{\partial x} = \frac{\partial X}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial X}{\partial v}\frac{\partial v}{\partial x}, \quad \frac{\partial X}{\partial y} = \frac{\partial X}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial X}{\partial v}\frac{\partial v}{\partial y}.$$

Since both coordinates are conformal, we get that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2, \quad \text{and} \quad \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}.$$

Thus we have that

$$\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 - \left(\frac{\partial u}{\partial y}\right)^2 + 2i\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}$$

$$= \left(\frac{\partial v}{\partial y}\right)^2 - \left(\frac{\partial v}{\partial x}\right)^2 - 2i\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = \left(\frac{\partial v}{\partial y} - i\frac{\partial v}{\partial x}\right)^2.$$

Hence

$$\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) = \pm\left(\frac{\partial v}{\partial y} - i\frac{\partial v}{\partial x}\right).$$

But if

$$\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) = -\left(\frac{\partial v}{\partial y} - i\frac{\partial v}{\partial x}\right),$$

then

$$\det\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = -\left(\frac{\partial v}{\partial y}\right)^2 - \left(\frac{\partial v}{\partial x}\right)^2 < 0,$$

contradicting the fact that U and V have the same orientation. So

$$\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) = \left(\frac{\partial v}{\partial y} - i\frac{\partial v}{\partial x}\right),$$

which is the complex form of the Cauchy-Riemann equations.

Since M is orientable, we get a complex analytic atlas $\{(U_\alpha, z_\alpha)\}_{\alpha \in A}$ on M, and M is diffeomorphic to a one-dimensional complex manifold. A one-dimensional complex manifold is usually called a *Riemann surface*.

Since any smooth orientable 2-dimensional manifold can be conformly embedded in $\mathbf{R}^3$, we see that any 2-dimensional smooth orientable surface M is diffeomorphic to a Riemann surface.

Moreover, if X is minimal, under this complex structure on M, X is harmonic, hence locally is the real part of a holomorphic mapping. It is here that complex function theory enters and plays an important role in the study of minimal surfaces.

Thus when we consider a minimal surface $X : M \hookrightarrow \mathbf{R}^3$, we can always assume that M is a Riemann surface with a *conformal structure* given as above.

The easiest global property of minimal surfaces is that if M is a *closed Riemann surface* (compact manifold without boundary), then there is no minimal immersion $X : M \to \mathbf{R}^3$. In fact, since M is compact, each component of X is a bounded harmonic function, and hence must have a maximum value on M. Thus X is a constant by the maximum principle, since M has no boundary. But then X is not an immersion.

Another definition of minimal surfaces is that the *mean curvature* of $X : M \hookrightarrow \mathbf{R}^3$ vanishes.

Remember that the mean curvature H of X is defined by

$$2H = g^{11}h_{11} + 2g^{12}h_{12} + g^{22}h_{22},$$

where $g_{ij} = X_i \bullet X_j$, $h_{ij} = X_{ij} \bullet N$ (N is the Gauss map, i.e., the unit normal vector $X_1 \wedge X_2/|X_1 \wedge X_2|$, where $\wedge$ is the cross product in $\mathbf{R}^3$), $(g^{ij}) = (g_{ij})^{-1}$, see any differential geometry textbook.

In case X is conformal, $g_{11} = g_{22} = \Lambda^2$, $g^{11} = g^{22} = \Lambda^{-2}$, $g_{12} = g^{12} = 0$. Thus

$$H = \frac{\triangle X \bullet N}{2\Lambda^2} = \frac{1}{2} \triangle_X X \bullet N,$$

where $\triangle_X$ is the Laplace-Beltrami operator under the metric (g_{ij}). Remember that $\triangle_X$ is given by

$$\triangle_X := \sum_{i=1}^{2} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sum_{j=1}^{2} \sqrt{g} g^{ij} \frac{\partial}{\partial x^j} \right) = \frac{1}{\Lambda^2} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) = \frac{4}{\Lambda^2} \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}},$$

where $g = \det(g_{ij})$, $(x^1, x^2) = (x, y)$, $z = x + iy$, and

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \right).$$

Thus in our case (conformal immersion), X is minimal (hence harmonic) implies that $H \equiv 0$, which is essentially an equivalent definition of minimal surface. In fact, this definition is easier to generalise to define minimal submanifolds in arbitrary Riemannian manifolds.

More precisely, $H \equiv 0$ implies that X is conformal harmonic under a certain complex structure. To see this, let us recall that for any immersion $X : M \hookrightarrow \mathbf{R}^3$,

$$\triangle_X X = 2HN. \tag{2.1}$$

Since we can always make X conformal, (2.1) shows that X is a minimal surface if and only if the mean curvature is zero.

Let us give the proof of (2.1) as a short review of differential geometry. Let us first recall that from the Gauss equation we have

$$X_{ij} = \sum_{k=1}^{2} \Gamma_{ij}^k X_k + h_{ij} N,$$

where

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^{2} g^{kl} \left(\frac{\partial g_{il}}{\partial u^j} + \frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l} \right).$$

We calculate

$$\begin{aligned}
\triangle_X X &= \frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial u^i} \left(g^{ij} \sqrt{g} X_j \right) \\
&= \sum_{i,j} g^{ij} X_{ij} + \sum_{i,j} \frac{\partial g^{ij}}{\partial u^i} X_j + \frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial \sqrt{g}}{\partial u^i} g^{ij} X_j \\
&= \sum_{i,j} g^{ij} X_{ij} + \sum_{i,j} \frac{\partial g^{ij}}{\partial u^i} X_j + \frac{1}{2g} \sum_{i,j} \frac{\partial g}{\partial u^i} g^{ij} X_j.
\end{aligned}$$

Now we have an identity

$$\frac{1}{g} \frac{\partial g}{\partial u^i} = \textbf{Trace} \left((g^{kl}) \left(\frac{\partial g_{kl}}{\partial u^i} \right) \right) = \sum_{k,l} g^{kl} \frac{\partial g_{kl}}{\partial u^i},$$

see the proof in the next section. Thus we have

$$\triangle_X X = \sum_{i,j} g^{ij} X_{ij} + \sum_{i,j} \frac{\partial g^{ij}}{\partial u^i} X_j + \frac{1}{2} \sum_{i,j,k,l} g^{ij} g^{kl} \frac{\partial g_{kl}}{\partial u^i} X_j.$$

We claim that $\triangle_X X$ is perpendicular to the *tagent planes*, i.e, planes generated by (X_1, X_2). In fact, since $\sum_j g_{ij} g^{jk} = \delta_{ik}$, we have

$$\begin{aligned}
\triangle_X X \bullet X_m &= \sum_{i,j} g^{ij} X_{ij} \bullet X_m + \sum_{i,j} \frac{\partial g^{ij}}{\partial u^i} X_j \bullet X_m + \frac{1}{2} \sum_{i,j,k,l} g^{ij} g^{kl} \frac{\partial g_{kl}}{\partial u^i} X_j \bullet X_m \\
&= \sum_{i,j,k} g^{ij} \Gamma_{ij}^k g_{km} + \sum_{i,j} \frac{\partial g^{ij}}{\partial u^i} g_{jm} + \frac{1}{2} \sum_{i,j,k,l} g^{ij} g^{kl} \frac{\partial g_{kl}}{\partial u^i} g_{jm} \\
&= \frac{1}{2} \sum_{i,j,k,l} g^{ij} g_{km} g^{kl} \left(\frac{\partial g_{il}}{\partial u^j} + \frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l} \right) - \sum_{i,j} g^{ij} \frac{\partial g_{jm}}{\partial u^i} + \frac{1}{2} \sum_{k,l} g^{kl} \frac{\partial g_{kl}}{\partial u^m} \\
&= \frac{1}{2} \sum_{i,j} g^{ij} \left(\frac{\partial g_{im}}{\partial u^j} + \frac{\partial g_{jm}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^m} \right) - \sum_{i,j} g^{ij} \frac{\partial g_{jm}}{\partial u^i} + \frac{1}{2} \sum_{i,j} g^{ij} \frac{\partial g_{ij}}{\partial u^m} \\
&= 0.
\end{aligned}$$

Thus $\triangle_X X$ is in the direction of N, and

$$\triangle_X X = (\triangle_X X \bullet N)N = \left(\sum_{i,j} g^{ij} X_{ij} \bullet N\right) N = \left(\sum_{i,j} g^{ij} h_{ij}\right) N = 2HN.$$

Equation (2.1) also tells us that if X is conformal, then $\triangle X$ is always perpendicular to the corresponding tangent plane of X.

Note that equation (2.1) holds for hypersurfaces in $\mathbf{R}^n$, $n \geq 3$, our proof is valid in the general case.

3 The First Variation

Let $X : M \hookrightarrow \mathbf{R}^3$ be a regular surface and $(U, (x, y))$ be a coordinate neighbourhood. Let $X_1 = X_x$, $X_2 = X_y$, $g_{ij} = X_i \bullet X_j$, and $g = \det(g_{ij})$. Then

$$dA := \sqrt{g}\, dx \wedge dy$$

is a well defined two form on M and $dA \neq 0$ everywhere.

Let $f : M \to \mathbf{R}$ be a continuous function of compact support, or suppose f does not change sign on M, then the integral of f on M is defined by

$$\int_M f := \int_M f\, dA.$$

When M is precompact and $f \equiv 1$, $\int_M dA$ is the area of the surface $X : M \hookrightarrow \mathbf{R}^3$.

The adjective "minimal" of minimal surfaces comes from the fact that at any point of the surface there exists a neighbourhood such that the surface in that neighbourhood has the least area among all surfaces with the same boundary.

To be precise, let $\Omega \subset M$ be a precompact domain and $X : \Omega \to \mathbf{R}^3$ be a surface. Let $X(t) : \Omega \to \mathbf{R}^3$, $-1 < t < 1$ and $X(0) = X$, such that $X(t)|_{\partial\Omega} = X|_{\partial\Omega}$, and $X(t, p) = X(t)(p)$ is C^2 on $\Omega \times (-1, 1)$. Such a family of surfaces is called a *variation of* X.

Consider the *area functional*

$$A(t) = \int_\Omega dA_t,$$

where dA_t is the area form induced by $X(t)$. The definition of minimal surface from the point view of the calculus of variations is that for any variation family $X(t)$,

$$\frac{dA(t)}{dt}\Big|_{t=0} = 0. \tag{3.2}$$

We will prove that this is another equivalent definition of minimal surface.

Without loss of generality, we may assume that X is conformal. Let $p \in \Omega$ and $p \in U \subset \Omega$ be an isothermal coordinate neighbourhood of p for X. On U, dA_t is expressed as

$$dA_t = \sqrt{\det[g_{ij}(t)]}\, dx \wedge dy,$$

where $z = x + iy$ is the isothermal coordinate and $g_{ij}(t) = X_i(t) \bullet X_j(t)$ (note that z may not be an isothermal coordinate for $X(t)$). Hence

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} \int_U dA_t &= \int_U \frac{d}{dt}\Big|_{t=0} dA_t = \int_U \frac{d\sqrt{\det[g_{ij}(t)]}}{dt}\Big|_{t=0} dx \wedge dy \\ &= \frac{1}{2} \int_U \frac{d \det[g_{ij}(t)]}{dt}\Big|_{t=0} \{\det[g_{ij}(0)]\}^{-1/2} dx \wedge dy.\end{aligned}$$

We need the formula

$$\frac{d\det(g_{ij}(t))}{dt} = \det(g_{ij}(t))\,\textbf{Trace}\left(\left(\frac{dg_{ij}(t)}{dt}\right)(g^{ij}(t))\right), \tag{3.3}$$

where $(g^{ij}(t)) = (g_{ij}(t))^{-1}$. To see this, let $(e_1, \ldots, e_n)$ be the standard orthonormal basis of $\mathbf{R}^n$. For any $n \times n$ matrix $A(t)$, we can write

$$A(t) = (A_1(t), \cdots, A_n(t)) = (A(t)e_1, \cdots, A(t)e_n),$$

where $A_i(t)$ is the i-th column of $A(t)$. If $\det A(t) \neq 0$, then

$$\begin{aligned}
\frac{d\det A(t)}{dt} &= \frac{d}{dt}\det(A(t)e_1, \cdots, A(t)e_n) \\
&= \sum_{i=1}^{n}\det\left(A(t)e_1, \cdots, \frac{dA(t)}{dt}e_i, \cdots, A(t)e_n\right) \\
&= \det A(t)\sum_{i=1}^{n}\det A^{-1}(t)\det\left(A(t)e_1, \cdots, \frac{dA(t)}{dt}e_i, \cdots, A(t)e_n\right) \\
&= \det A(t)\sum_{i=1}^{n}\det\left[A^{-1}(t)\left(A(t)e_1, \cdots, \frac{dA(t)}{dt}e_i, \cdots, A(t)e_n\right)\right] \\
&= \det A(t)\sum_{i=1}^{n}\det\left(e_1, \cdots, A^{-1}(t)\frac{dA(t)}{dt}e_i, \cdots, e_n\right) \\
&= \det A(t)\sum_{i=1}^{n}\det\left(e_1, \cdots, \sum_{j=1}^{n}\left(A^{-1}(t)\frac{dA(t)}{dt}\right)_{ji} e_j, \cdots, e_n\right) \\
&= \det A(t)\sum_{i=1}^{n}\det\left(e_1, \cdots, \left(A^{-1}(t)\frac{dA(t)}{dt}\right)_{ii} e_i, \cdots, e_n\right) \\
&= \det A(t)\sum_{i=1}^{n}\left(A^{-1}(t)\frac{dA(t)}{dt}\right)_{ii} = \det A(t)\,\textbf{Trace}\left(A^{-1}(t)\frac{dA(t)}{dt}\right) \\
&= \det A(t)\,\textbf{Trace}\left(\frac{dA(t)}{dt}A^{-1}(t)\right).
\end{aligned}$$

This establishes (3.3).

Thus we have

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0}\int_U dA_t &= \frac{1}{2}\int_U \frac{d\det(g_{ij}(t))}{dt}\Big|_{t=0}[\det(g_{ij}(0))]^{-1/2}dx\wedge dy \\
&= \frac{1}{2}\int_U \textbf{Trace}\left[\left(\frac{dg_{ij}(t)}{dt}\right)(g^{ij}(t))\right]\Big|_{t=0}\sqrt{\det(g_{ij}(0))}dx\wedge dy.
\end{aligned}$$

Since X is conformal, we have $g^{ij}(0) = \Lambda^{-2}\delta_{ij}$. Thus

$$\textbf{Trace}\left(\left(\frac{dg_{ij}(t)}{dt}\right)(g^{ij}(t))\right)\Big|_{t=0} = \sum_{ij}\frac{d\,g_{ij}(t)}{dt}g^{ij}(t)\Big|_{t=0} = \Lambda^{-2}\sum_{i=1}^{2}\frac{d\,g_{ii}(t)}{dt}\Big|_{t=0}.$$

Define the *variation field* E as

$$E(p) := \frac{dX(t)(p)}{dt}\Big|_{t=0}, \quad p \in \Omega.$$

Then

$$\frac{d\,g_{ii}(t)}{dt}\Big|_{t=0} = \frac{d(X_i \bullet X_i)}{dt}\Big|_{t=0} = 2E_i \bullet X_i.$$

Since (X_1, X_2, N) is a basis of $\mathbf{R}^3$, where N is the unit normal, we can write $E = \alpha X_1 + \beta X_2 + \gamma N$, where α, β, and γ are C^1 functions defined in Ω. Using $N \bullet X_i = 0$, $\gamma = E \bullet N$, and

$$\gamma \Lambda^{-2} \sum_{i=1}^{2} X_{ii} \bullet N = (E \bullet N)(\triangle_X X \bullet N) = 2(E \bullet N)(HN \bullet N) = 2H(E \bullet N),$$

we have

$$\begin{aligned} \mathbf{Trace}\left(\left(\frac{dg_{ij}(t)}{dt} \right) (g^{ij}(t)) \right) \Big|_{t=0} &= 2\Lambda^{-2} \sum_{i=1}^{2} E_i \bullet X_i \\ &= 2(\alpha_1 + \beta_2) + 2\Lambda^{-2}(\alpha \Lambda_1^2 + \beta \Lambda_2^2) - 2\gamma \Lambda^{-2} \sum_{i=1}^{2} X_{ii} \bullet N \\ &= 2(\alpha_1 + \beta_2) + 2\Lambda^{-2}(\alpha \Lambda_1^2 + \beta \Lambda_2^2) - 4H(E \bullet N). \end{aligned}$$

Again since X is conformal, $\sqrt{\det(g_{ij}(0))} = |X_1|^2 = |X_2|^2 = \Lambda^2$, we have

$$\frac{d}{dt}\Big|_{t=0} \int_U dA_t = \frac{1}{2} \int_U \mathbf{Trace}\left(\left(\frac{dg_{ij}(t)}{dt} \right) (g^{ij})(t) \right) \Big|_{t=0} \Lambda^2 dx \wedge dy$$

$$= \int_U \mathbf{Div}(\Lambda^2(\alpha, \beta)) dx \wedge dy - 2 \int_U H(E \bullet N) dA_0 = \int_{\partial U} \Lambda^2(\alpha, \beta) \bullet n \, ds - 2 \int_U H(E \bullet N) dA_0,$$

where n and ds are the outward unit normal vector field and the line element of ∂U in the Euclidean metric respectively. Dividing Ω into a finite number of disjoint isothermal coordinate neighbourhoods U_i,

$$\sum_i \int_{\partial U_i \cap \Omega} \Lambda^2(\alpha, \beta) \bullet n_i \, ds_i = 0$$

since each arc in $\partial U_i \cap \Omega$ appears twice in the summation and with opposite unit normal. Moreover, because $\alpha = \beta = 0$ on $\partial \Omega$, we have

$$\sum_i \int_{\partial U_i} \Lambda^2(\alpha, \beta) \bullet n_i \, ds_i = \sum_i \int_{\partial U_i \cap \Omega} \Lambda^2(\alpha, \beta) \bullet n_i \, ds_i + \int_{\partial \Omega} \Lambda^2(\alpha, \beta) \bullet n \, ds = 0,$$

where in the last integral n and ds are the outward unit normal vector field and the line element of $\partial\Omega$ in the Euclidean metric. Thus we finally have the *first variational formula* for the surface area functional:

$$\left.\frac{dA}{dt}\right|_{t=0} = -2\int_\Omega H(E \bullet N)dA_0. \tag{3.4}$$

If X is minimal, then $H = 0$, so $\left.\frac{dA}{dt}\right|_{t=0} = 0$. On the other hand, if X is a stationary point for the area functional $A(t)$ (for example, if X has minimal area among all surfaces with the same boundary), then $\left.\frac{dA}{dt}\right|_{t=0} = 0$ for any variation of X. Since E can be any vector field, $\left.\frac{dA}{dt}\right|_{t=0} = 0$ forces that $H \equiv 0$, that is, X is a minimal surface.

Finally we will give an area formula for surfaces in $\mathbf{R}^3$. Suppose $X : \Omega \hookrightarrow \mathbf{R}^3$ is an immersion; without loss of generality, we may assume that X is conformal. Let $\vec{n}$ be the unit *conormal* on $X(\partial\Omega)$, i.e., $\vec{n}$ is tangent to $X(\Omega)$ and is perpendicular to $X(\partial\Omega)$. Let ds be the line element of $X(\partial\Omega)$, (e_1, e_2) be the standard orthonormal basis on U_i in the Euclidean metric. Let $n_i = ae_1 + be_2$. The integral

$$\int_{\partial U_i \cap \partial\Omega} \Lambda^2(\alpha, \beta) \bullet n_i \, ds_i$$

can be rewritten as

$$\begin{aligned}
&\int_{\partial U_i \cap \partial\Omega} \Lambda^2(\alpha e_1 + \beta e_2) \bullet (ae_1 + be_2)\, ds_i \\
&= \int_{\partial U_i \cap \partial\Omega} \Lambda^2(a\alpha + b\beta) ds_i = \int_{\partial U_i \cap \partial\Omega} \Lambda^{-1}[E \bullet dX(n_i)] X^*(ds) \\
&= \int_{X(\partial U_i \cap \partial\Omega)} \Lambda^{-1}[E \bullet dX(n_i)] ds = \int_{X(\partial U_i \cap \partial\Omega)} (E \bullet \vec{n}) ds,
\end{aligned}$$

since $E = \alpha X_1 + \beta X_2 + \gamma N$, $dX(n_i) = aX_1 + bX_2$, $X^*(ds) = \Lambda ds_i$, and $\vec{n} = \Lambda^{-1} dX(n_i)$. Thus if we do not assume that α and β vanish on $\partial\Omega$, we have the first variation formula

$$\left.\frac{dA}{dt}\right|_{t=0} = -2\int H(E \bullet N) dA_0 + \int_{X(\partial\Omega)} (E \bullet \vec{n}) ds. \tag{3.5}$$

Now let $a \in \mathbf{R}^3$ be any fixed vector; then $X(t)(p) = t(X(p) - a)$ is a variation of X, not fixed on boundary. Clearly $E(t)(p) = X(p) - a$ is the variation vector field independent of t. An easy calculation shows that

$$g_{ij}(t) = t^2 g_{ij}, \quad g^{ij}(t) = t^{-2} g^{ij}, \quad h_{ij}(t) = t\, h_{ij}.$$

Hence

$$dA_t = t^2 dA_1 = t^2 dA, \quad H(t) = t^{-1} H,$$

where $H = H(1)$, etc. Note that

$$A := \textbf{Area of } X(\Omega) = \int_\Omega dA,$$

and

$$A(t) := \textbf{Area} \text{ of } X(t)(\Omega) = \int_\Omega dA_t = t^2 A.$$

Since $E(t) = X - a$, by (3.5)

$$2A = -2\int H[(X-a)\bullet N]dA + \int_{X(\partial\Omega)}[(X-a)\bullet\vec{n}]ds. \tag{3.6}$$

This formula is useful when we derive the isoperimetric inequalities for minimal surfaces.

4 The Minimal Surface Equation

Sometimes our surface is a graph over a domain $\Omega \subset \mathbf{R}^2$, i.e., $(x, y, z) \in X(M)$ is expressed as $z = z(x, y)$, $(x, y) \in \Omega$. Moreover, locally we can always treat a "small piece" of surface as a graph. Thus we need know the differential equation governing z, the *minimal surface equation*, in order to derive more information.

To derive the minimal surface equation we use the following equivalent form of $\triangle_X$,

$$\triangle_X X = \sum_{i=1}^{2}[\tau_i\tau_i X - (\nabla_{\tau_i}\tau_i)X], \tag{4.7}$$

where $(\tau_1, \tau_2)(p)$ is an orthonormal frame of T_pM in the induced metric by X and $\nabla_{\tau_i}\tau_i = (D_{\tau_i}\tau_i)^T$ is the covariant differential, in our case, namely the tangent part of $D_{\tau_i}\tau_i$.

Our surface can be written as

$$X(x, y) = (x, y, z(x, y)), \quad (x, y) \in \Omega.$$

Thus $X_x = (1, 0, z_x)$ and $X_y = (0, 1, z_y)$. We will take the upward normal

$$N = \frac{1}{(1 + z_x^2 + z_y^2)^{1/2}}(-z_x, -z_y, 1).$$

We take (τ_1, τ_2) as

$$\tau_1 = dX^{-1}\left(\frac{1}{(1 + z_x^2)^{1/2}}X_x\right) = \frac{1}{(1 + z_x^2)^{1/2}}\frac{\partial}{\partial x},$$

$$\tau_2 = dX^{-1}\left[\left(\frac{1 + z_x^2}{1 + z_x^2 + z_y^2}\right)^{1/2}\left(X_y - \frac{z_x z_y}{1 + z_x^2}X_x\right)\right] = \left(\frac{1 + z_x^2}{1 + z_x^2 + z_y^2}\right)^{1/2}\left(\frac{\partial}{\partial y} - \frac{z_x z_y}{1 + z_x^2}\frac{\partial}{\partial x}\right).$$

By (4.7) and (2.1),

$$\begin{aligned}
2H &= (D_{\tau_1}\frac{1}{(1 + z_x^2)^{1/2}}\frac{\partial}{\partial x} + D_{\tau_2}\left[\left(\frac{1 + z_x^2}{1 + z_x^2 + z_y^2}\right)^{1/2}\left(\frac{\partial}{\partial y} - \frac{z_x z_y}{1 + z_x^2}\frac{\partial}{\partial x}\right)\right] \bullet N \\
&= \left[\frac{X_{xx}}{1 + z_x^2} + \frac{z_x^2 z_y^2 X_{xx}}{(1 + z_x^2)(1 + z_x^2 + z_y^2)} - \frac{2z_x z_y X_{xy}}{1 + z_x^2 + z_y^2} + \frac{(1 + z_x^2)X_{yy}}{1 + z_x^2 + z_y^2}\right] \bullet N \\
&= \left[\frac{1 + z_y^2}{1 + z_x^2 + z_y^2}X_{xx} - \frac{2z_x z_y}{1 + z_x^2 + z_y^2}X_{xy} + \frac{1 + z_x^2}{1 + z_x^2 + z_y^2}X_{yy}\right] \bullet N.
\end{aligned}$$

Since $X_{xx} = (0, 0, z_{xx})$, $X_{xy} = (0, 0, z_{xy})$, and $X_{yy} = (0, 0, z_{yy})$, we have

$$2H = \frac{1}{(1 + z_x^2 + z_y^2)^{3/2}}\left[(1 + z_y^2)z_{xx} - 2z_x z_y z_{xy} + (1 + z_x^2)z_{yy}\right].$$

This can be written as

$$2H = \mathbf{Div}\,\frac{Dz}{(1+|Dz|^2)^{1/2}} = \frac{\partial}{\partial x}\frac{z_x}{(1+z_x^2+z_y^2)^{1/2}} + \frac{\partial}{\partial y}\frac{z_y}{(1+z_x^2+z_y^2)^{1/2}}.$$

We get the minimal surface equation

$$(1+z_y^2)z_{xx} - 2z_x z_y z_{xy} + (1+z_x^2)z_{yy} = 0, \tag{4.8}$$

or

$$\mathbf{Div}\,\frac{Dz}{(1+|Dz|^2)^{1/2}} = 0. \tag{4.9}$$

In general, if $H = H(x,y)$ is a given function, then the *prescribed mean curvature equation* is defined as

$$(1+z_y^2)z_{xx} - 2z_x z_y z_{xy} + (1+z_x^2)z_{yy} = 2H(1+z_x^2+z_y^2)^{3/2}, \tag{4.10}$$

or

$$\mathbf{Div}\,\frac{Dz}{(1+|Dz|^2)^{1/2}} = 2H. \tag{4.11}$$

Equations (4.8) and (4.10) are second order elliptic equations. We will see that they play an important role in the study of minimal, or more generally, constant mean curvature surfaces.

For example, let $\Omega \subset \mathbf{R}^2$ be a C^2 simply connected domain, $\phi \in C^0(\partial\Omega)$. Then $(x, \phi(x))$ defines a Jordan curve (continuously embedded closed curve) Γ in $\mathbf{R}^3$, where $x \in \partial\Omega$. We want to find a minimal surface bounded by Γ. So consider the Dirichlet problem

$$\begin{cases} (1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)u_{yy} = 0, & \text{in } \Omega; \\ u|_{\partial\Omega} = \phi, & \text{on } \partial\Omega. \end{cases} \tag{4.12}$$

A solution of (4.12) will give us a minimal graph, which is a minimal surface bounded by Γ. From the theory of PDE we know that

Theorem 4.1 *The Dirichlet problem (4.12) is solvable for arbitrary $\phi \in C^0(\partial\Omega)$ if and only if Ω is convex.*

See for example, [21], Theorem 16.8.

A very important problem in minimal surface theory is the *Plateau problem* which asks: is there a simply connected minimal surface bounded by a given Jordan curve Γ? In general there are alway solutions to the Plateau problem as long as Γ is *rectifiable*, that is, has finite arc length. We are not going to discuss the Plateau problem in these notes.

There is a general theorem which says that for certain elliptic equations (including the minimal surface equation) the solution is real analytic. A simple proof of this fact for the minimal surface equation (2-dimensional) can be found in [61], §131 on page 125. The proof there uses special isothermal coordinates (see the next section), which shows that for the minimal surface, we do not need to call on the classical isothermal coordinate theorem.

One application of real analyticity is that if two minimal surfaces coincide in a piece of surface, then they must be essentially the same.

Theorem 4.2 (Extension Theorem) *Suppose $X : M \hookrightarrow \mathbf{R}^3$ and $Y : N \hookrightarrow \mathbf{R}^3$ are two connected minimal surfaces. If there are open sets $U \subset M$ and $V \subset N$ such that $X(U) = Y(V)$, then $X(M) \cup Y(N)$ is contained in a (perhaps larger) minimal surface.*

Proof. We prove that $X(M) \cup Y(N)$ is an immersed surface. To prove this we define $A \subset \overline{X(M)} \cap \overline{Y(N)}$ such that $x \in A$ if and only if there is a small ball B in $\mathbf{R}^3$ centred at x and either $X(M) \cap B \subset Y(N) \cap B$ or $Y(N) \cap B \subset X(M) \cap B$. By our hypothesis, $A \neq \emptyset$. We need only prove that A is closed in $\overline{X(M)} \cap \overline{Y(N)}$, since then clearly $A \cup X(M)$ and $A \cup Y(N)$ are both immersed surfaces.

First assume that X and Y are embedded.

If A is not closed, then there is a point $p \in (\overline{A} - A) \cap \overline{X(M)} \cap \overline{Y(N)}$. Thus there is a sequence $\{x_n\} \subset A$ such that $\lim_{n\to\infty} x_n = p$ and $p \in \overline{X(M)} \cap \overline{Y(N)}$. By definition of A, locally $X(M)$ and $Y(N)$ coincide at x_n, hence $X(M)$ and $Y(N)$ have the same tangent plane at x_n. Taking limits, we know that $X(M)$ and $Y(N)$ have the same limit tangent plane at $p \in \overline{X(M)} \cap \overline{Y(N)}$. After a rotation and translation if necessary, we can assume that $p = (0,0,0)$ and the common tangent plane of $X(M)$ and $Y(N)$ at p is the xy-plane. Then in a small disk D in the xy-plane centred at $(0,0)$, $X(M)$ and $Y(N)$ are graphs over domains $\Omega_1 \subset D$ and $\Omega_2 \subset D$ such that $(0,0) \in \overline{\Omega_1 \cap \Omega_2}$. Thus there are u and v satisfying the minimal surface equation on Ω_1 and Ω_2 respectively, such that $(x, y, u(x,y))$ represents $X(M)$ and $(x, y, v(x,y))$ represents $Y(N)$. By definition of p, we know that there is an open subset $Q \subset$ of $\Omega_1 \cap \Omega_2$ on which $u \equiv v$. But u and v are real analytic, so $u \equiv v$ on $\Omega_1 \cap \Omega_2$. Hence both u and v can be extended to $\Omega_1 \cup \Omega_2$, and represent the same surface. This is a contradiction to the assumption $p \notin A$. Thus A is closed in $\overline{X(M)} \cap \overline{Y(N)}$.

If X or Y is not an embedding, first consider the local version of the proof, then modify the definition of A at multiple points of $\mathbf{R}^3$, i.e., at points which are images of more than one point of M or of N.

The proof then is complete. □

Definition 4.3 An *equiangular system of order k* at a point $q \in \mathbf{C}$ consists of k curved rays $\gamma_1, \gamma_2, \cdots, \gamma_k$ emitting from q such that any two adjacent rays intersect at q with angle $2\pi/k$.

Theorem 4.4 *Let $X : M \hookrightarrow \mathbf{R}^3$ and $Y : N \hookrightarrow \mathbf{R}^3$ be two minimal surfaces and $x \in X(M) \cap Y(N)$ be such that $X(M)$ and $Y(N)$ at x have the same tangent plane P. Then* *either $X(M)$ and $Y(N)$ are part of a (maybe larger) minimal surface or the orthogonal projection of $X(M) \cap Y(N)$ on P forms an equiangular system of even order $k \geq 4$.*

Proof. By a rotation and translation, we may assume that $x = (0,0,0)$ and P is the xy-plane. Then there is a disk $D \subset P$ centred at $(0,0)$ such that $X(M)$ and $Y(N)$ are graphs given by $u : D \to \mathbf{R}$ and $v : D \to \mathbf{R}$ respectively. Moreover, since P is the common tangent plane, $Du = Dv = (0,0)$ at $(0,0)$.

Let $w = v - u$, then by real analyticity, w satisfies

$$w = \sum_{n=k}^{\infty} P^{(n)}(x,y), \quad k \geq 2,$$

where

$$P^{(n)}(x,y) = \sum_{i=0}^{n} \frac{1}{i!(n-i)!} \frac{\partial^n w}{\partial^i x \partial^{n-i} y}(0,0) x^i y^{n-i}$$

is a homogeneous polynomial of degree n. If $P^{(n)}(x,y) \equiv 0$ for $n \geq 2$, then $u \equiv v$ in D. By Theorem 4.2, $X(M)$ and $Y(N)$ are part of a (maybe larger) minimal surface.

If $P^{(n)}(x,y) \not\equiv 0$ for some $n \geq 2$, then let k be the smallest n such that $P^{(n)} \not\equiv 0$. In this case, we say that $X(M)$ and $Y(N)$ has $k-1$ *contact.*

Now since u and v satisfy the minimal surface equation, we have

$$\begin{aligned}
\triangle w &= \triangle v - \triangle u \\
&= 2v_x v_y v_{xy} - 2u_x u_y u_{xy} - v_y^2 v_{xx} + u_y^2 u_{xx} - v_x^2 v_{yy} + u_x^2 u_{yy} \\
&= -u_y^2 w_{xx} + (u_y^2 - v_y^2) v_{xx} - u_x^2 w_{yy} + (u_x^2 - v_x^2) v_{yy} + 2u_x u_y w_{xy} - 2(u_x u_y - v_x v_y) v_{xy} \\
&= -u_y^2 w_{xx} + (u_y + v_y)(u_y - v_y) v_{xx} - u_x^2 w_{yy} + (u_x + v_x)(u_x - v_x) v_{yy} \\
&\quad + 2u_x u_y w_{xy} - 2[v_x(u_y - v_y) + (u_x - v_x) u_y] v_{xy} \\
&= -u_y^2 w_{xx} - (u_y + v_y) v_{xx} w_y - u_x^2 w_{yy} - (u_x + v_x) v_{yy} w_x \\
&\quad + 2u_x u_y w_{xy} + 2v_x v_{xy} w_y + 2u_y v_{xy} w_x = O(r^k),
\end{aligned}$$

where $r = (x^2 + y^2)^{1/2}$. The last equality comes from the fact that $Du = Dv = (0,0)$ at $(0,0)$ and $w = O(r^k)$. By

$$\triangle P^{(n)} = O(r^{n-2}) \quad \text{and} \quad \triangle\, w = O(r^k),$$

we have that

$$\triangle P^{(k)} = O(r^{k-1}).$$

Since $\triangle P^{(k)}$ is a polynomial of degree at most $k-2$, it must be the case that $\triangle P^{(k)} = 0$, that is, $P^{(k)}$ is a harmonic polynomial.

Now $P^{(k)}(x,y) = \Re H(z)$, where H is a holomorphic function, $\Re$ denotes the real part, and $z = x + iy$. Since $P^{(k)} = O(r^k)$, we can choose H such that $H(z) = z^k F(z)$, where $F(0) \neq 0$. In a smaller disk contained in D, $(F(z))^{1/k}$ is well defined, hence let

$\zeta = z(F(z))^{1/k}$, then $H(z) = \zeta^k$. Let $\zeta = \rho e^{i\psi} = \xi + i\eta$, we have $P^{(k)} = \Re H(z) = \rho^k \cos(k\psi)$. Thus the zero set of $P^{(k)}$ is an equiangular system of even order $2k \geq 4$.

Since $w = P^{(k)}(x,y) + \sum_{n=k+1}^{\infty} P^{(n)}(x,y)$ is analytic and $\sum_{n=k+1}^{\infty} P^{(n)}(x,y) = o(r^k)$, the zero set of w also consists of an equiangular system.

The projection of $X(M) \cap Y(N)$ around $(0,0,0)$ on P is exactly the zero set of w. The proof of the theorem is complete. □

Corollary 4.5 *Let $X(M)$ be a non-planar minimal surface, $p \in X(M)$ and $P = T_pM \subset T_{X(p)}\mathbf{R}^3$. Then $X(M) \cap P$ consists of an equiangular system of even order at least* 4.

Proof. This is the special case that P is the minimal surface $Y(N)$. □

Remark 4.6 Theorem 4.4 and Corollary 4.5 are called *maximum* (or *comparison*) *principle for minimal surface.* Together with Theorem 4.2 it follows that two minimal surfaces cannot touch each other at isolated interior points.

5 Isothermal Coordinates for Minimal Surfaces

There is a direct construction of isothermal coordinates for minimal surfaces. Let $X : M \hookrightarrow \mathbf{R}^3$ be a minimal surface and $p \in M$. Without of loss generality we can assume that $X(p) = (0,0,0)$ and $N(p) = (0,0,1)$, and there is a simply connected domain $(0,0) \in \Omega \subset \mathbf{R}^2$ such that near $(0,0,0)$, $X(M)$ can be written as a graph $(x, y, u(x,y))$, with $u : \Omega \to \mathbf{R}$ a solution to the minimal surface equation. Writing $p = u_x$, $q = u_y$ and $W = (1+p^2+q^2)^{1/2}$, we see that $pdx + qdy$ is a closed form, i.e., $d(pdx+qdy) = 0$ on Ω. Furthermore, it is also easy to check that the two 1-forms

$$\eta_1 := \frac{1}{W}\left((1+p^2)dx + pq\,dy\right), \quad \eta_2 := \frac{1}{W}\left(pq\,dx + (1+q^2)dy\right),$$

are closed. Since Ω is simply connected,

$$\xi(x,y) := x + \int_{(0,0)}^{(x,y)} \eta_1 = x + F(x,y), \quad \eta(x,y) := y + \int_{(0,0)}^{(x,y)} \eta_2 = y + G(x,y),$$

are well defined. Thus

$$\frac{\partial \xi}{\partial x} = 1 + \frac{1+p^2}{W}, \quad \frac{\partial \xi}{\partial y} = \frac{pq}{W},$$

$$\frac{\partial \eta}{\partial x} = \frac{pq}{W}, \quad \frac{\partial \eta}{\partial y} = 1 + \frac{1+q^2}{W},$$

and

$$J = \frac{\partial(\xi,\eta)}{\partial(x,y)} = 2 + \frac{2+p^2+q^2}{W} = \frac{(W+1)^2}{W} > 0.$$

Thus the transformation $(x,y) \to (\xi,\eta)$ has a local inverse $(\xi,\eta) \to (x,y)$ and setting $x = x(\xi,\eta)$, $y = y(\xi,\eta)$, $z(\xi,\eta) = u(x(\xi,\eta), y(\xi,\eta))$, we find

$$\frac{\partial x}{\partial \xi} = \frac{W+1+q^2}{(W+1)^2}, \quad \frac{\partial x}{\partial \eta} = -\frac{pq}{(W+1)^2},$$

$$\frac{\partial y}{\partial \xi} = -\frac{pq}{(W+1)^2}, \quad \frac{\partial x}{\partial \eta} = \frac{W+1+p^2}{(W+1)^2},$$

$$\frac{\partial z}{\partial \xi} = p\frac{\partial x}{\partial \xi} + q\frac{\partial y}{\partial \xi}, \quad \frac{\partial z}{\partial \eta} = p\frac{\partial x}{\partial \eta} + q\frac{\partial y}{\partial \eta}.$$

Calculation shows that

$$|X_\xi|^2 = |X_\eta|^2 = \frac{W}{J} = \frac{W^2}{(W+1)^2}, \qquad X_\xi \bullet X_\eta = 0.$$

Thus (ξ,η) is an isothermal coordinate. Furthermore, (ξ,η) has the property that

$$|(\xi,\eta)|^2 > |(x,y)|^2. \tag{5.13}$$

To see this, note that

$$\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x};$$

thus there is a function E such that

$$\frac{\partial E}{\partial x} = F, \quad \frac{\partial E}{\partial y} = G,$$

and

$$\left(\frac{\partial^2 E}{\partial x \partial y}\right) = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{1+p^2}{W} & \frac{pq}{W} \\ \frac{pq}{W} & \frac{1+q^2}{W} \end{pmatrix}$$

is positive.

Lemma 5.1 *Let $E \in C^2$ such that the Hessian of E is positive. Then the mapping $x = (x_1, x_2) \to (u_1, u_2) = (E_{x_1}, E_{x_2}) = u(x)$ satisfies*

$$(v - u) \bullet (y - x) > 0, \tag{5.14}$$

for $y \neq x$ in Ω and $v = u(y)$, $u = u(x)$.

Proof. Let $G(t) = E(ty + (1-t)x)$, $0 \leq t \leq 1$. Then

$$G'(t) = \sum_{i=1}^{2} \left[\frac{\partial E}{\partial x_i}(ty + (1-t)x)\right](y_i - x_i),$$

$$G''(t) = \sum_{i,j=1}^{2} \left[\frac{\partial^2 E}{\partial x_i \partial x_j}(ty + (1-t)x)\right](y_i - x_i)(y_j - x_j) > 0,$$

for $0 \leq t \leq 1$. Hence $G'(1) > G'(0)$, or

$$\sum v_i(y_i - x_i) > \sum u_i(y_i - x_i),$$

which is (5.14). □

Lemma 5.2 *Under the hypotheses of Lemma 5.1, define a map*

$$(x_1, x_2) \to (\tau_1, \tau_2) = \tau,$$

where $\tau_i = x_i + u_i(x_1, x_2)$. Then for $x \neq y$,

$$(\tau(y) - \tau(x)) \bullet (y - x) > |y - x|^2.$$

Proof. Since $\tau(y) - \tau(x) = (y - x) + (v - u)$, this comes from (5.14). □

Now by the Cauchy-Schwarz inequality,

$$|\tau(y) - \tau(x)| > |y - x|.$$

Note that our transformation $(x, y) \to (\xi, \eta)$ is the form defined in Lemma 5.2. Taking $x = (0, 0)$ we have $|\tau(y)| > |y|$ since $\tau(0) = 0$. If $\Omega = \mathbf{R}^2$, then the map $(x, y) \to (\xi, \eta)$ is a diffeomorphism from $\mathbf{R}^2$ to $\mathbf{R}^2$.

6 The Enneper-Weierstrass Representation

Suppose that $X : M \hookrightarrow \mathbf{R}^3$ is minimal. Since X is harmonic, on an isothermal neighbourhood $(U, (x, y))$,

$$\phi = (\phi_1, \phi_2, \phi_3) = \frac{\partial X}{\partial x} - i\frac{\partial X}{\partial y} = 2\frac{\partial X}{\partial z} \tag{6.15}$$

is holomorphic. In fact,

$$\frac{\partial \phi}{\partial \bar{z}} = 2\frac{\partial^2 X}{\partial \bar{z}\partial z} = \frac{1}{2} \triangle X = \vec{0}.$$

Let V be another isothermal neighborhood with coordinate $w = u + iv$, and let

$$\tilde{\phi} = \frac{\partial X}{\partial u} - i\frac{\partial X}{\partial v}.$$

On $U \cap V$

$$\begin{aligned}\phi = \frac{\partial X}{\partial x} - i\frac{\partial X}{\partial y} &= \frac{\partial X}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial X}{\partial v}\frac{\partial v}{\partial x} - i\left(\frac{\partial X}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial X}{\partial v}\frac{\partial v}{\partial y}\right) \\ &= \left(\frac{\partial X}{\partial u} - i\frac{\partial X}{\partial v}\right)\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right) = \tilde{\phi}\frac{dw}{dz}.\end{aligned} \tag{6.16}$$

Hence

$$\tilde{\phi}\, dw = \phi\, dz, \tag{6.17}$$

which means that $\phi\, dz$ gives a globally defined vector valued holomorphic 1-form. Write

$$\omega = (\omega_1, \omega_2, \omega_3) = (\phi_1, \phi_2, \phi_3)dz = \phi\, dz. \tag{6.18}$$

By the definition of ϕ, X being conformal is equivalent to

$$\sum_{i=1}^{3} \omega_i^2 = \sum_{i=1}^{3} \phi_i^2 (dz)^2 = 0. \tag{6.19}$$

The condition that X is an immersion is equivalent to

$$\infty > \sum_{i=1}^{3} |\omega_i|^2 = \sum_{i=1}^{3} |\phi_i|^2 |dz|^2 = \left(\left|\frac{\partial X}{\partial x}\right|^2 + \left|\frac{\partial X}{\partial y}\right|^2\right) |dz|^2 = 2\Lambda^2 |dz|^2 > 0. \tag{6.20}$$

Remark 6.1 When $\sum_{i=1}^{3} |\omega_i|^2 = 0$ at some point $p \in M$, we call p a *branch point* of the surface $X : M \to \mathbf{R}^3$. At such a point, X ceases to be an immersion. At times we want to study minimal surfaces with branch points, called *branched minimal surfaces.* For branched minimal surface, since our data ϕ is holomorphic, we see that branch points are isolated. Thus in any precompact domain there are at most a finite number of branch points.

Our main interest is in minimal surfaces without branch points. All minimal surfaces in these notes are branch point free, unless specified otherwise.

The immersion X can be expressed as

$$X(p) = X(p_0) + \Re \int_{p_0}^{p} \omega, \tag{6.21}$$

where p_0 is a fixed point of M. For any closed path γ on M,

$$\Re \int_{\gamma} \omega = (0, 0, 0), \tag{6.22}$$

since X is well defined.

On the other hand, if we have three holomorphic 1-forms ω_i on M satisfying (6.19), (6.20), and (6.22) for any closed path γ in M, then (6.21) gives a minimal surface. This is because as the real part of a holomorphic mapping, X is harmonic; (6.19) is equivalent to X being conformal; (6.20) says that X is an immersion; and (6.22) guarantees that X is well defined.

So far everything we discussed in these notes is true in case $X : M \hookrightarrow \mathbf{R}^n$, $n \geq 3$, except the minimal surface equation should be a system of equations for $n > 3$ and the theorem about equiangular systems. Here is something special to the case $n = 3$. Let us write (6.19) as

$$(\omega_1 - i\omega_2)(\omega_1 + i\omega_2) + \omega_3^2 = 0. \tag{6.23}$$

We can assume that $\omega_3 \not\equiv 0$, as otherwise the surface lies in a plane parallel to the xy-plane, and by rotation we can get an equivalent surface such that $\omega_3 \not\equiv 0$. We define a meromorphic function g on M by

$$g = \frac{\omega_3}{\omega_1 - i\omega_2} \not\equiv 0.$$

By (6.23),

$$g^2 = \frac{\omega_3^2}{(\omega_1 - i\omega_2)^2} = -\frac{\omega_1 + i\omega_2}{\omega_1 - i\omega_2}.$$

Writing $\eta = \omega_1 - i\omega_2$, after a little calculation we have

$$\begin{cases} \omega_1 &= \frac{1}{2}(1 - g^2)\eta, \\ \omega_2 &= \frac{i}{2}(1 + g^2)\eta, \\ \omega_3 &= g\eta. \end{cases} \tag{6.24}$$

Then (6.21) can be written as

$$X(p) = X(p_0) + \Re \int_{p_0}^{p} \left(\frac{1}{2}(1 - g^2)\eta,\ \frac{i}{2}(1 + g^2)\eta,\ g\eta \right). \tag{6.25}$$

The formula (6.25) is called the *Enneper-Weierstrass representation* of the minimal surface $X : M \hookrightarrow \mathbf{R}^3$.

The meromorphic function g and the holomorphic 1-form η are called the *Enneper-Weierstrass data* of the minimal surface X, or shortly the *data* of X.

It is convenient in local coordinates to write $\eta = f(z)dz$, where $z = x + iy$ and f is a holomorphic function. Thus (6.24) can be written as

$$\begin{cases} \omega_1 &= \frac{1}{2} f(1-g^2)dz \\ \omega_2 &= \frac{i}{2} f(1+g^2)dz \\ \omega_3 &= fg\,dz. \end{cases} \tag{6.26}$$

Since g is a meromorphic function, if $dg \neq 0$ and g is not a pole at $p \in M$, then g is a holomorphic diffeomorphism in a neighbourhood U of p. Suppose U is a coordinate neighbourhood, with coordinate $z = x + iy$. Then $w = u(z) + iv(z) = g(z)$ is a local coordinate as well, and $dw = g'(z)dz = g' \circ g^{-1}(w)dz$. We define

$$F(w) = \frac{f \circ g^{-1}(w)}{g' \circ g^{-1}(w)}, \quad F(w)dw = f \circ g^{-1}(w)dz = f(z)dz = \eta.$$

Hence in the w coordinate, (6.26) becomes

$$\begin{cases} \omega_1 &= \frac{1}{2}(1-w^2)F(w)dw \\ \omega_2 &= \frac{i}{2}(1+w^2)F(w)dw \\ \omega_3 &= F(w)w\,dw. \end{cases} \tag{6.27}$$

The function F is called the *Weierstrass function* of the minimal surface $X \circ g^{-1} : g(U) \hookrightarrow \mathbb{R}^3$, where $g(U) \subset \mathbb{C}$ is a domain in $\mathbb{C}$. Notice that this is only a local representation which holds as long as g is a holomorphic diffeomorphism on U.

Now let us analyse (6.20). By (6.24), (6.20) is true if and only if whenever g has a pole of order m at $p \in M$, then η has a zero of order $2m$ at $p \in M$. Moreover, this is the only case where η can vanish.

In summary, if we have a meromorphic function g and a holomorphic 1-form η on M, such that (6.24) defines three holomorphic 1-forms which satisfy (6.20) and (6.22), then (6.25) defines a minimal surface. An important fact is that recently many interesting minimal surfaces were discovered via the Enneper-Weierstrass representation by specifying g and η on certain Riemann surfaces. See, for example, [31], [39], [41], and [80].

7 The Geometry of the Enneper-Weierstrass Representation

Let $X: M \hookrightarrow \mathbf{R}^3$ be a minimal surface. We will give the geometric data, such as the Gauss map, the first and second fundamental forms, the principal and Gauss curvatures, etc., of a minimal surface via its Enneper-Weierstrass representation.

One important fact is that the meromorphic function g in the Enneper-Weierstrass representation corresponds to the *Gauss map* N. For this we first recall that the Gauss map $N: M \to \Sigma = S^2$ of an immersion $X: M \hookrightarrow \mathbf{R}^3$ is defined as

$$N = |X_u \wedge X_v|^{-1}(X_u \wedge X_v) : M \to \Sigma.$$

Let $\tau : S^2 - \{\mathcal{N}\} \to \mathbf{C}$ be stereographic projection, where $\mathcal{N}$ is the north pole. Then

$$\tau(x, y, z) = \frac{x + iy}{1 - z}, \quad \tau^{-1}(w) = \frac{1}{1 + |w|^2}(2\Re w, 2\Im w, |w|^2 - 1),$$

where $\Re$ and $\Im$ are the real and imaginary parts. We claim that

$$g = \tau \circ N : M \to \mathbf{C}.$$

In fact,

$$\tau^{-1} \circ g = \frac{1}{1 + |g|^2}(2\Re g, \ 2\Im g, \ |g|^2 - 1).$$

By (6.15), (6.18), and (6.26)

$$\begin{aligned} X_u &= \Re\left(\frac{1}{2}f(1 - g^2), \ \frac{i}{2}f(1 + g^2), \ fg\right), \\ X_v &= -\Im\left(\frac{1}{2}f(1 - g^2), \ \frac{i}{2}f(1 + g^2), \ fg\right), \end{aligned}$$

thus

$$X_u \wedge X_v = \begin{pmatrix} -\Re\frac{i}{2}f(1 + g^2)\Im fg + \Re fg\Im\frac{i}{2}f(1 + g^2) \\ \Re\frac{1}{2}f(1 - g^2)\Im fg - \Re fg\Im\frac{1}{2}f(1 - g^2) \\ -\Re f(1 - g^2)\Im\frac{i}{4}f(1 + g^2) + \Re\frac{i}{4}f(1 + g^2)\Im f(1 - g^2) \end{pmatrix}$$

$$= \begin{pmatrix} \Im[\frac{i}{2}f(1 + g^2)\overline{fg}] \\ \Im[\frac{1}{2}\overline{f(1 - g^2)}fg] \\ \Im\left[\frac{-i}{4}\overline{f(1 + g^2)}f(1 - g^2)\right] \end{pmatrix} = \begin{pmatrix} \frac{1}{2}|f|^2\Re(\overline{g} + |g|^2 g) \\ \frac{1}{2}|f|^2\Im(g - |g|^2\overline{g}) \\ \frac{1}{4}|f|^2\Re(|g|^4 - 1 - \overline{g}^2 + g^2) \end{pmatrix}$$

$$= \frac{|f|^2(1+|g|^2)^2}{4(1+|g|^2)} \begin{pmatrix} 2\Re g \\ 2\Im g \\ |g|^2 - 1 \end{pmatrix} = \frac{1}{4}|f|^2(1+|g|^2)^2 \tau^{-1} \circ g.$$

Since $\tau^{-1} \circ g \in S^2$, $|\tau^{-1} \circ g| = 1$. Since X is conformal, the first fundamental form is given by $g_{12} = 0$ and

$$g_{11} = g_{22} = \Lambda^2 = |X_u||X_v| = |X_u \wedge X_v| = \frac{1}{4}|f|^2(1+|g|^2)^2. \tag{7.28}$$

Thus

$$N = |X_u \wedge X_v|^{-1}(X_u \wedge X_v) = \frac{1}{1+|g|^2}\left(2\Re g, 2\Im g, |g|^2 - 1\right) = \tau^{-1} \circ g, \tag{7.29}$$

as we claimed.

Later we will *also* call the function $g = \tau \circ N$ the *Gauss map of the immersion* $X : M \hookrightarrow \mathbf{R}^3$. We have seen that if X is a minimal surface then g is a meromorphic function. The converse is also true, i.e., X is minimal if and only if $g = \tau \circ N$ is meromorphic. We give a sketch of the proof of the converse direction; the reader can fill in the details or see [34], pages 7 to 14.

Let $T_{X(p)}M \subset \mathbf{R}^3$ be the tangent space at $X(p)$, $p \in M$. $T_{X(p)}M$ is oriented by the basis (X_1, X_2). The orientation determined by (X_1, X_2) will be called the *positive orientation.* Thus we can regard $T_{X(p)}$ as a point in $G^+_{3,2}$, the Grasmann manifold of oriented two dimensional subspaces in $\mathbf{R}^3$. We want to embed $G^+_{3,2}$ in $\mathbf{CP}^2$, the two (complex) dimensional complex projective space.

One way to express $P \in G^+_{3,2}$ is to select a positive orthogonal basis (e_1, e_2). But if (e_1, e_2) is a positive orthogonal basis of P and A is a rotation in P, then $A(e_1, e_2)$ is also a positive orthogonal basis of P. If we consider $e_1 + ie_2$ as a vector in $\mathbf{C}^3$, then A corresponds to a unit complex number $e^{i\theta}$, and $(e_1, e_2)A$ corresponds to $e^{i\theta}(e_1 + ie_2) \in \mathbf{C}^3$. Moreover, $e^{i\theta}(e_1 + ie_2)/|e_1 + ie_2|$ corresponds to a positive orthonormal basis of P. Thus we find that given a positive orthogonal basis (e_1, e_2), all positive orthonormal bases can be written as $\Theta(e_1 + ie_2) \in \mathbf{C}^3$, where Θ is an nonzero complex number. Fixing a positive orthogonal basis (e_1, e_2) of P and identifying $\Theta(e_1 + ie_2) \in \mathbf{C}^3$ for all $\Theta \in \mathbf{C} - \{0\}$ gives us a point $[e_1 + ie_2] \in \mathbf{CP^2}$. Thus P corresponds to a unique point in $\mathbf{CP}^2$. This is our embedding $E : G^+_{3,2} \to \mathbf{CP}^2$. By local coordinates it is easy to verify that E is C^∞.

Now remember that for any conformal immersion $X : M \hookrightarrow \mathbf{R}^3$, the 1-forms $\overline{\phi} = X_1 + iX_2$ are well defined in a coordinate neighbourhood U. Since (X_1, X_2) is a positive orthogonal basis of $T_{X(p)}M \subset \mathbf{R}^3$, we can define $\overline{\phi} : U \to \mathbf{CP}^2$ by $\overline{\phi}(p) = E(T_{X(p)}) = [(X_1 + iX_2)(p)]$. X is conformal implies that (6.19) is true, thus the image of $\overline{\phi}$ is contained in the submanifold $Q_1 := \{[z_1, z_2, z_3] \in \mathbf{CP}^2 \,\big|\, z_1^2 + z_2^2 + z_3^2 = 0\}$. We claim that Q_1 is conformally homeomorphic to S^2. In fact, let (z_1, z_2, z_3) be a representative of $[z_1, z_2, z_3] \in Q_1$ and write $(z_1, z_2, z_3) = e_1 + ie_2$, where the e_i's are real vectors. Then $[z_1, z_2, z_3] \in Q_1$ implies that (e_1, e_2) is orthogonal, therefore there is a unique

$e_3 \in S^2$ such that (e_1, e_2, e_3) is a orthogonal basis of $\mathbf{R}^3$ with positive orientation. Define $\sigma([z_1, z_2, z_3]) = e_3$; clearly σ is a homeomorphism from Q_1 to S^2. A little calculation shows that σ is conformal. Clearly, $\sigma \circ \overline{\phi}(p) = N(p)$, where N is the Gauss map. Now $g(p) = \tau \circ \sigma \circ \overline{\phi}(p)$, or $\overline{\phi} = \sigma^{-1} \circ \tau^{-1} \circ g$. Since τ reverses orientation, it is anti-conformal. If g is holomorphic, then g is conformal and thus $\overline{\phi}$ is anti-conformal or anti-holomorphic. This implies that $\phi = \overline{\overline{\phi}}$ is holomorphic. Thus

$$\frac{1}{2}\left(\frac{\partial^2 X}{\partial x^2} + \frac{\partial^2 X}{\partial y^2}\right) = \frac{1}{2}\left[\frac{\partial^2 X}{\partial x^2} + i\frac{\partial^2 X}{\partial x \partial y} - i\left(\frac{\partial^2 X}{\partial x \partial y} + i\frac{\partial^2 X}{\partial y^2}\right)\right] = \frac{\partial \phi}{\partial \overline{z}} = 0.$$

Hence X is harmonic and therefore minimal. This ends the sketch of the proof.

Remark 7.1 Note that if $p \in M$ is a branch point of a branched minimal surface and (U, z) is an isothermal neighbourhood of p such that $z(p) = 0$, then we can write $\phi = z^m \psi$, where ψ is a holomorphic vector function and $\psi(0) \neq 0$. Since ϕ satisfies (6.19), ψ also satisfies (6.19). We can use $[\psi] \in \mathbf{CP}^2$ to define the tangent space $T_{X(p)}M$. Thus for a branched minimal surface, the tangent space is well defined even at branch points.

We next give a Gauss curvature formula of the minimal surface $X : M \hookrightarrow \mathbf{R}^3$ via the Enneper-Weierstrass representation, namely

$$K = -\left[\frac{4|g'|}{|f|(1+|g|^2)^2}\right]^2. \tag{7.30}$$

To prove this, remember that for a surface with conformal metric $ds^2 = \Lambda^2 |dz|^2$, where $dz = dx + i\,dy$ and $|dz|^2 = (dx)^2 + (dy)^2$, the Gauss curvature is given by

$$K = -\frac{1}{2\Lambda^2} \triangle \log \Lambda^2 = -\frac{2}{\Lambda^2}\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z} \log \Lambda^2.$$

By (7.28), since $\log |f|$ is harmonic, we have

$$\begin{aligned}
\frac{2}{\Lambda^2}\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z} \log \Lambda^2 &= \frac{4}{\Lambda^2}\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z} \log |f| + \frac{4}{\Lambda^2}\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z} \log(1+|g|^2) \\
&= \frac{4}{\Lambda^2}\frac{\partial}{\partial \overline{z}}\frac{g'\overline{g}}{1+|g|^2} = \frac{4}{\Lambda^2}\frac{g'\overline{g'}(1+|g|^2) - g'\overline{g}g\overline{g'}}{(1+|g|^2)^2} \\
&= \frac{4}{\Lambda^2}\frac{|g'|^2}{(1+|g|^2)^2} = \frac{16|g'|^2}{|f|^2(1+|g|^2)^4}.
\end{aligned}$$

We can also calculate the second fundamental form of X via the Enneper-Weierstrass representation. Recall that

$$X_1 - iX_2 = X_x - iX_y = (\phi_1, \phi_2, \phi_3)$$

are holomorphic functions of $z = x + iy$. Hence

$$X_{11} - iX_{12} = X_{xx} - iX_{xy} = (\phi_1', \phi_2', \phi_3').$$

Because X is harmonic, the data of the second fundamental form then must be

$$h_{11} = X_{11} \bullet N = \Re(\phi_1', \phi_2', \phi_3') \bullet N, \qquad h_{22} = -h_{11},$$

$$h_{12} = X_{12} \bullet N = -\Im(\phi_1', \phi_2', \phi_3') \bullet N.$$

By (6.15), (6.18), and (6.26),

$$\begin{aligned}
X_{11} \bullet N &= \Re(\phi_1', \phi_2', \phi_3') \bullet N \\
&= \Re\left[\left(\frac{1}{2}f'(1-g^2), \frac{i}{2}f'(1+g^2), f'g\right) + (-fgg', ifgg', fg')\right] \bullet N \\
&= \frac{1}{1+|g|^2}\Big(\Re f'(1-g^2)\Re g - \Im f'(1+g^2)\Im g + \Re f'g(|g|^2-1) \\
&\quad -2\Re fgg'\Re g - 2\Im fgg'\Im g + \Re fg'(|g|^2-1)\Big) \\
&= \frac{1}{1+|g|^2}\Big(\Re f'\Re g - \Re f'g^2\Re g - \Im f'\Im g - \Im f'g^2\Im g \\
&\quad +\Re f'g(|g|^2-1) - 2\Re fgg'\overline{g} + \Re fg'(|g|^2-1)\Big) \\
&= \frac{1}{1+|g|^2}\Big(\Re f'g - \Re f'g^2\overline{g} + \Re f'g(|g|^2-1) - 2|g|^2\Re fg' + \Re fg'(|g|^2-1)\Big) \\
&= \frac{1}{1+|g|^2}\Big(-\Re fg'(|g|^2+1)\Big) = -\Re fg'.
\end{aligned}$$

Similarly, we have $h_{12} = \Im fg'$. From these we see that for a minimal surface,

$$h_{11} - ih_{12} = -fg' \tag{7.31}$$

is a holomorphic function.

Again let $dz = dx + i\,dy$ and $(dz)^2 = (dx)^2 - (dy)^2 + 2i\,dx\,dy$. The second fundamental form of X can be written as

$$h_{11}(dx)^2 + 2h_{12}\,dx\,dy + h_{22}(dy)^2 = -\Re(fg')((dx)^2-(dy)^2) + 2\Im(fg')\,dx\,dy$$

$$= -\Re(fg')\Re(dz)^2 + \Im(fg')\Im(dz)^2 = -\Re(fg'(dz)^2) = -\Re(f\,dg\,dz).$$

Let $V \in T_pM$ be a unit tangent vector and write

$$V = \Lambda^{-1}\left(\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial x}\right) = 2\Re\Lambda^{-1}e^{i\theta}\frac{\partial}{\partial z} = \Lambda^{-1}e^{i\theta}\frac{\partial}{\partial z}\Lambda^{-1}e^{-i\theta}\frac{\partial}{\partial \overline{z}}$$

in complex form; then

$$II(V,V) = -\Lambda^{-2}\Re(fg'e^{2i\theta})$$

by the previous formulae. Thus the two principal curvatures are

$$\kappa_1 = \max_{0\le\theta\le2\pi} -\Lambda^{-2}\Re(fg'e^{2i\theta}) = \Lambda^{-2}|fg'| = \frac{4|g'|}{|f|(1+|g|^2)^2}, \tag{7.32}$$

$$\kappa_2 = \min_{0\le\theta\le2\pi} -\Lambda^{-2}\Re(fg'e^{2i\theta}) = -\Lambda^{-2}|fg'| = -\frac{4|g'|}{|f|(1+|g|^2)^2}. \tag{7.33}$$

Then from $K = \kappa_1\kappa_2$ we recover formula (7.30).

Now let $r(t) = r_1(t) + ir_2(t)$ be a curve on M and $r'(t) = r_1'(t) + ir_2'(t)$; then

$$\begin{aligned} II(r'(t), r'(t)) &= -\Re\{f[r(t)]\, g'[(r(t)]\, [r'(t)]^2\}(dt)^2 \\ &= -\Re\{d[g(r(t)]f[r(t)]dr(t)\} \\ &= -\Re\{d[g(r(t)]\eta[r(t)]\}, \end{aligned} \tag{7.34}$$

since $\eta = fdz$. Remember that a regular curve r is an *asymptotic line* on a surface M if $II(r'(t), r'(t)) \equiv 0$; a curve r is a *curvature line* if and only if $r'(t)$ is in a principal direction, if and only if $|r'(t)|^{-2}II(r'(t), r'(t))$ takes either maximum or minimum value of $II(v, v)$ for all unit tangent vectors in $T_{r(t)}M$. We have the following criteria:

1. A regular curve r is an asymptotic line if and only if $f[r(t)]\, g'[r(t)]\, [r'(t)]^2 \in i\mathbf{R}$.

2. A regular curve r is a curvature line if and only if $f[r(t)]\, g'[r(t)]\, [r'(t)]^2 \in \mathbf{R}$.

The last assertion comes from the fact that $-\Re\{f[r(t)]g'([(t)][r'(t)]^2\}$ achieves its maximum or minimum for all unit vectors $r'(t)$ at $r(t)$ only if $f[r(t)]g'[r(t)][r'(t)]^2$ is real.

8 Some Applications of the Enneper-Weierstrass Representation

Given a minimal surface $X : M \hookrightarrow \mathbf{R}^3$ and its Enneper-Weierstrass representation, fix a simply connected open set $U \subset M$. Fixing $p_0 \in U$ we can define a family of isometric minimal surfaces associated to $X_\theta : U \to \mathbf{R}^3$, $0 \le \theta < 2\pi$, by

$$X_\theta(p) = \Re e^{i\theta} \int_{p_0}^{p} (\omega_1, \omega_2, \omega_3) + C, \quad p \in U, \tag{8.35}$$

where the ω_i's are the 1-forms in the Enneper-Weierstrass representation of X and C is a constant vector. The Enneper-Weierstrass data for X_θ are $g_\theta = g$ and $\eta_\theta = e^{i\theta}\eta$. When $\theta = \pi/2$, $X_{\pi/2}$ is called the *conjugate surface* of X.

Let $I \subset \mathbf{R}$ be an interval and $r : I \to U$ be a geodesic such that $X \circ r$ is a plane curve. Then we know that r must be a curvature line, thus by our criterion in the previous section,

$$d(g \circ r)\eta \circ r \in \mathbf{R}.$$

Since X and $X_{\pi/2}$ are isometric, r is also a geodesic of $X_{\pi/2}$. Moreover,

$$d(g_{\pi/2} \circ r)\eta_{\pi/2} \circ r = id(g \circ r)\eta \circ r \in i\mathbf{R},$$

and hence r is an asymptotic line of $X_{\pi/2}$. Since the space curve $X_{\pi/2} \circ r$ is both a geodesic and an asymptotic line of $X_{\pi/2}$, it must be a straight line segment on $X_{\pi/2}$ (in fact, the normal and geodesic curvatures of $X_{\pi/2} \circ r$ are both zero, and so its curvature is zero everywhere). Since X and $X_{\pi/2}$ are conjugate to each other (up to sign), we have

Proposition 8.1 *$X \circ r$ is a plane geodesic (straight line segment) if and only if $X_{\pi/2} \circ r$ is a straight line segment (a plane geodesic).*

In fact, we have more information whenever we have a plane geodesic or a straight line segment on X. Namely, the surface X must have some symmetry.

Theorem 8.2 (Reflection and Rotation Theorems) *If a plane geodesic which is not a straight line segment lies on a minimal surface, then reflection in the plane of the geodesic is a congruence of the minimal surface.*

If a straight line segment lies on a minimal surface, then 180°*-rotation around the straight line is a a congruence of the minimal surface.*

Proof. Let $X \circ r$ be a plane geodesic but not a straight line segment on X. By a rotation in $\mathbf{R}^3$ we can assume that $X \circ r$ is in the xz-plane. Since $X \circ r$ is a geodesic and is not a straight line segment, the Gauss map N of X along r must be in the xz-plane. Thus $g = \tau \circ N$ is real along r. Select a point $r(t_0)$ such that $g'(r(t_0)) \neq 0$; then in a simply connected neighbourhood U of $r(t_0)$, $w = g(z)$ is a well defined coordinate of M. We can use the representation (6.27) and consider the holomorphic mapping on U,

$$(G^1, G^2, G^3) = G := \int_{p_0}^{p} (\omega_1,\, \omega_2,\, \omega_3) + C,$$

where $p_0 = r(t_0)$ and $C \in \mathbf{C}^3$ is a constant complex vector. Remember that our surface $X = \Re G$ and $X_{\pi/2} = -\Im G$. By Proposition 8.1, $X_{\pi/2} \circ r$ is a straight line segment. Since the Gauss map of $X_{\pi/2}$ is the same as that of X, the Gauss map of $X_{\pi/2}$ is in the xz-plane along r, so the straight line segment $X_{\pi/2} \circ r$ is parallel to the y-axis. Thus $\Im G^1 \circ r$ and $\Im G^3 \circ r$ are constants. By adjusting C we may assume that the constants are zeros. Remember that along r, $w \in \mathbf{R}$. Now let $U_+ := \{w \in U \,\big|\, \Re w \geq 0\}$ and $U_- := \{w \in U \,\big|\, \Re w \leq 0\}$. We can extend $G^1\big|_{U_+}$ and $G^3\big|_{U_+}$ to U by $\tilde{G}^i(w) = \overline{G^i(\overline{w})}$, for $i = 1, 3$, $w \in U_-$ and $\overline{w} \in U_+$. Since $\Re G^2 \circ r = 0$, we can extend $G^2\big|_{U_+}$ to U_- by $\tilde{G}^2(w) = -\overline{G^2(\overline{w})}$, for $w \in U_-$ and $\overline{w} \in U_+$. Since G is holomorphic, we know that $\tilde{G}$ is holomorphic and $\tilde{G} = G$ on U. Choose a small disk $D \subset U_- \cup U_+$ such that $\overline{D} = D$, then $Y = \Re \tilde{G}$ is a minimal surface on D. Since $X = Y$ on $D \cap U_-$, by the Extension Theorem (Theorem 4.2), $X = Y$ on D. Looking at the real part, we have for any $w \in D$,

$$(X^1(w),\, X^2(w),\, X^3(w)) = \Re G(w) = \Re \tilde{G}(w) = \left(X^1(\overline{w}),\, -X^2(\overline{w}),\, X^3(\overline{w})\right) = X(\overline{w}),$$

which is a reflection in the xz-plane. By the Extension Theorem (Theorem 4.2) again, this reflection is a congruence of X.

Similarly we can prove that if $X \circ r$ is a straight line segment, then the rotation by $180°$ around $X \circ r$ is a congruence of X. □

Exercise : Prove that if $X \circ r$ is a straight line segment, then rotation by $180°$ around $X \circ r$ is a congruence of X.

Finally, we show that each component of the Gauss map N is an eigenvector of the Laplace operator $\triangle_X$. First remember that for a conformal representation of a minimal surface, $\triangle_X = \Lambda^{-2}\triangle$.

Proposition 8.3 *The Gauss map N satisfies*

$$\triangle_X N = 2KN, \tag{8.36}$$

where K is the Gauss curvature.

Proof. Let g and η be the Enneper-Weierstrass data for X. On an isothermal neighbourhood (U, z) we have

$$\begin{aligned}
\triangle N &= 4\frac{\partial^2}{\partial z \partial \overline{z}} N = 4\frac{\partial^2}{\partial z \partial \overline{z}} \left[\frac{1}{1+|g|^2}\left(2\Re g, 2\Im g, |g|^2 - 1\right)\right] \\
&= \left[4\frac{\partial^2}{\partial z \partial \overline{z}}(1+|g|^2)^{-1}\right]\left(2\Re g, 2\Im g, |g|^2 - 1\right) + 4(1+|g|^2)^{-1}\frac{\partial^2}{\partial z \partial \overline{z}}\left(2\Re g, 2\Im g, |g|^2 - 1\right) \\
&\quad + 8\Re\left\{\left[\frac{\partial}{\partial z}(1+|g|^2)^{-1}\right]\frac{\partial}{\partial \overline{z}}\left(2\Re g, 2\Im g, |g|^2 - 1\right)\right\}.
\end{aligned}$$

Since g is holomorphic,

$$\frac{\partial}{\partial z}(1+|g|^2)^{-1} = \frac{-g'\overline{g}}{(1+|g|^2)^2},$$

$$4\frac{\partial^2}{\partial z\partial\overline{z}}(1+|g|^2)^{-1} = \frac{4|g'|^2(|g|^2-1)}{(|g|^2+1)^3}.$$

Using the Cauchy-Riemann equations we have

$$\frac{\partial}{\partial\overline{z}}\left(2\Re g, 2\Im g, |g|^2-1\right) = (\overline{g'}, i\overline{g'}, g\overline{g'}).$$

Sincce $\Re g$ and $\Im g$ are harmonic,

$$\frac{\partial^2}{\partial z\partial\overline{z}}\left(2\Re g, 2\Im g, |g|^2-1\right) = (0,0,|g'|^2).$$

Hence

$$\begin{aligned}
\triangle N &= \frac{4|g'|^2(|g|^2-1)}{(1+|g|^2)^3}\left(2\Re g, 2\Im g, |g|^2-1\right) + 4(1+|g|^2)^{-1}(0,0,|g'|^2) \\
&\quad +8\Re\left[\frac{-g'\overline{g}}{(1+|g|^2)^2}\overline{g'}\,(1,i,g)\right] \\
&= \frac{4|g'|^2(|g|^2-1)}{(1+|g|^2)^3}\left(2\Re g, 2\Im g, |g|^2-1\right) + 4(1+|g|^2)^{-1}(0,0,|g'|^2) \\
&\quad +8\left[\frac{-|g'|^2\overline{g}}{(1+|g|^2)^2}\left(\Re g, \Im g, |g|^2\right)\right] \\
&= \frac{4|g'|^2}{(|g|^2+1)^2}\frac{|g|^2}{1+|g|^2}\left(2\Re g, 2\Im g, |g|^2-1\right) + \frac{4|g'|^2}{(1+|g|^2)^2}\frac{1}{1+|g|^2}(0,0,(1+|g|^2)^2) \\
&\quad -\frac{4|g'|^2}{(1+|g|^2)^2}\frac{1+|g|^2}{1+|g|^2}\left(2\Re g, 2\Im g, 2|g|^2\right)] \\
&= \frac{-8|g'|^2}{(1+|g|^2)^2}\left[\frac{1}{1+|g|^2}\left(2\Re g, 2\Im g, |g|^2-1\right)\right] = \frac{-8|g'|^2}{(1+|g|^2)^2}N.
\end{aligned}$$

Now by (7.28) and (7.30),

$$K\Lambda^2 = \frac{-4|g'|^2}{(1+|g|^2)^2},$$

and thus

$$\triangle N = 2K\Lambda^2 N.$$

□

9 Conformal Types of Riemann Surfaces

We will discuss complete minimal surfaces of finite topological type and their annular ends. We need first consider a little of the conformal type of such surfaces.

All *closed* (compact without boundary) 2-dimensional manifolds are classified topologically by their genus and orientability. For example, the topological classification of closed orientable 2-dimensional manifolds is as follows:

The simplest surface is the sphere S^2. Then we can do "surgery" on S^2; by deleting two disjoint disks on S^2 and gluing the boundary of a cylinder along the two circular boundaries we obtain a torus. We say a torus has genus one and is a sphere plus one handle, while S^2 has genus zero, and is a sphere without handle. Thus we obtain genus k surfaces S_k for all integers $k \geq 0$ by adding k handles to a sphere. These are all possible topological types of closed orientable 2-dimensional manifolds. An important topological invariant is Euler's characteristic $\chi(S_k) = 2(1-k)$. Two closed 2-dimensional manifolds are homeomorphic if and only if they have the same Euler's characteristic. Euler's characteristic can be calculated by Gauss-Bonnet Formula, if we have a Riemannian metric on the manifold.

As we have seen before, any smooth orientable 2-dimensional manifold is diffeomorphic to a Riemann surface. Let M and N be two Riemann surfaces. We say that $f : M \to N$ is holomorphic if for any $p \in M$ there is an isothermal coordinate neighbourhood $U \ni p$ with complex coordinate z and an isothermal coordinate $V \ni f(p)$ with complex coordinate w, such that $w \circ f(z)$ is holomorphic.

In the category of Riemann surfaces, $M \cong N$ (*have equivalent conformal type*) if and only if there is a diffeomorphism $f : M \to N$ such that f and f^{-1} are both holomorphic. Such an f is called a *conformal diffeomorphism*.

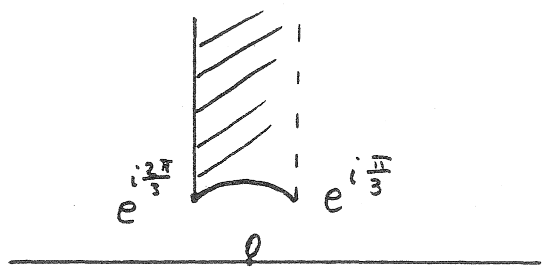

Figure 1

There is considerable interest in classifying the conformal type of closed Riemann surfaces. Although the topological classification is quite simple, the conformal classification is still not clear. In general, S^2 has only one conformal type, i.e., any two *closed* (without boundary) orientable Riemann surfaces of genus zero are conformally diffeomorphic to each other. A typical coordinate system on S^2 is given by stereographic projection from the north and south poles. The conformal structure of genus-one Riemann surfaces corresponds to a region in $\mathbf{C}$ as in the picture above. Such a representation is called a *Riemann moduli space*, hence the Riemann moduli space of the torus is one complex dimension. For genus $k > 1$, it is known that the Riemann moduli space is an algebraic variety of complex dimension $3k - 3$. So far, the Riemann moduli spaces are not well understood.

We are interested in surfaces obtained by making a finite number of punctures and/or removing a finite number of closed disks from a closed Riemann surface M_k of genus k. Suppose that we make n punctures and remove l closed disks, then $M = M_k - (\{p_1, \dots, p_n\} \cup \bigcup_{i=1}^{l} D_i)$ is said to have *finite topological type*, or just *finite type*, since the topologically (even homotopically) invariant Euler's characteristic is $\chi(M) = 2(1-k) - (n+l)$, which is finite. Thus topologically we cannot tell how many closed disks were removed or how many punctures were made, although we know the sum of them.

When we come to consider the conformal type, removing a closed disk or making a puncture are quite different. For example, making one puncture on S^2 we get the complex plane $\mathbf{C}$, but removing a closed disk on S^2 we get (conformally) the open unit disk $\mathbf{D} := \{z \in \mathbf{C}, \big| |z| < 1\}$. Although $\mathbf{D}$ and $\mathbf{C}$ are C^∞ diffeomorphic to each other, they do have different conformal type. We can see this by Liouville's theorem: were there a conformal diffeomorphism $f : \mathbf{C} \to \mathbf{D}$ then f would be a bounded entire function and hence a constant.

For any connected Riemann surface M without boundary, there is a universal covering Riemann surface $\tilde{M}$ and a holomorphic covering map $f : \tilde{M} \to M$. That f is a covering means for any $p \in M$ there is an open set $U \ni p$ such that $f^{-1}(U)$ consists of disjoint open subsets of $\tilde{M}$ and each component V of $f^{-1}(U)$ is conformally diffeomorphic to U under $f|_V$. Being a universal covering, $\tilde{M}$ must be simply connected. There are only 3 different simply connected Riemann surfaces without boundary, up to conformal diffeomorphism; they are the unit disk $\mathbf{D}$, the unit sphere $\Sigma := S^2 \cong \mathbf{C} \cup \{\infty\}$, and $\mathbf{C}$. It turns out that unless $M = \Sigma$, then $\tilde{M} \neq \Sigma$ (see for example, [1], III, 11G). Hence in general, $\tilde{M}$ is non-compact. An open (non-compact) Riemann surface M is called *parabolic* if there are no non-constant negative subharmonic functions defined on M, otherwise M is *hyperbolic* (see for example, [1], IV, 6). Clearly $\mathbf{D}$ is hyperbolic, since $\Re z - 1$ is a negative subharmonic function. If M is closed then M is called *elliptic*. By maximum principle for subharmonic function, we know that a hyperbolic surface cannot be closed. Thus all Riemann surfaces are divided into three mutually exclusive families.

If $M \subset \mathbf{C}$ is a plane domain with more than one boundary point, then the universal covering is $\mathbf{D}$. In other words, among the planar domains, only $\mathbf{C}$ and $\mathbf{C} - \{p\}$ having $\mathbf{C}$ as universal covering, where $p \in \mathbf{C}$.

An equivalent definition of hyperbolicity of M is that there is a Green's function $G(\zeta, z)$ on M for any $\zeta \in$M which is positive except at ζ and such that in any local coordinate U of ζ, $G(\zeta, z) + \log |z - \zeta|$ is a harmonic function on U. See, for example, [1], IV, 6.

Now return to our $M = M_k - (\{p_1, \dots, p_n\} \cup \bigcup_{i=1}^{l} D_i)$. At any puncture p or removed closed disk U of M_k, take a larger open disk $D \subset M_k$ such that $p \in D$ or $U \subset D$. Since such (topological) annuli ($D - U$ or $D - \{p\}$) determine the global properties of our complete minimal surfaces we should understand more about their conformal structure. The rest of this section is devoted to the description of the conformal structures of annuli.

Among all doubly connected domains (annuli) in $\mathbf{C}$ we consider the special case that the doubly connected domains D are bounded by two Jordan curves (embedded closed curves), such domains are called *annuli with Jordan curve boundaries.*

Without loss of generality, we can assume that $\partial D = C_1 \cup C_2$, where C_1 and C_2 are analytic Jordan curves. Perhaps the best way to see that this is true is by the following picture, of course we assume the Riemann Mapping Theorem.

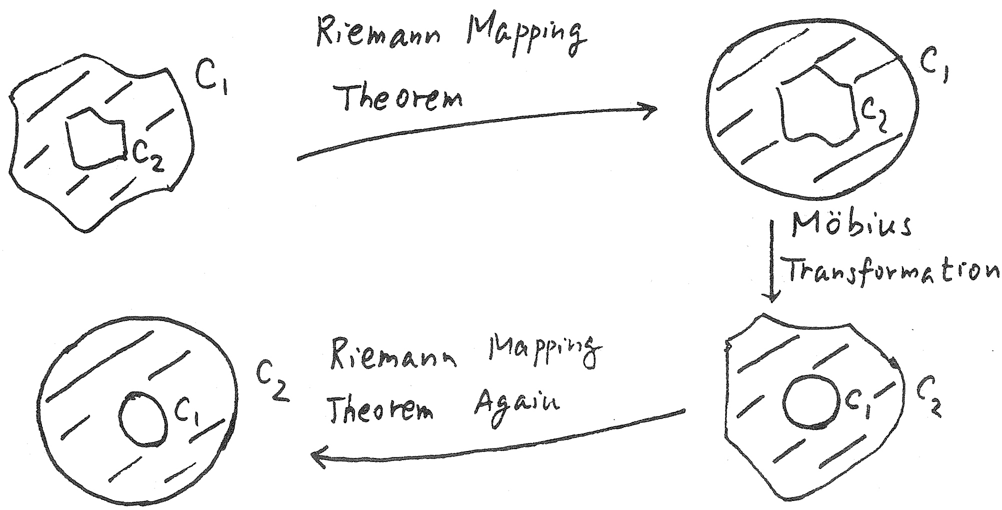

Figure 2

Let $\exp(z) = e^z$. Consider the universal covering $f = \exp : \mathbf{C} \to \mathbf{C} - \{0\} \supset \overline{D}$; then $f^{-1}(\overline{D})$ is a simply connected domain in $\mathbf{C}$ with analytic boundary. $\Re f^{-1}(\overline{D})$ is bounded and for any $w \in f^{-1}(\overline{D})$, $w + 2n\pi i \in f^{-1}(\overline{D})$ for any $n \in \mathbf{Z}$. In fact, $f^{-1}(z) = \log z$, a multivalued holomorphic function. By the Riemann Mapping Theorem, for any $b > 0$ there is a conformal diffeomorphism

$$\phi : f^{-1}(\overline{D}) \to S_b = \{z \in \mathbf{C} : 0 \leq \Re z \leq b\}$$

such that $\phi[f^{-1}(C_1)] =$ the y-axis. Any two such maps ϕ_1 and ϕ_2 induce a conformal diffeomorphism

$$h = \phi_2 \circ \phi_1^{-1} : S_b \to S_b$$

which maps the y-axis and the straight line $\Re z = b$ onto themselves. By Schwarz's reflection principle, h can be extended to a conformal diffeomorphism from $\mathbf{C}$ to $\mathbf{C}$, hence $h(z) = az + e$ since it maps ∞ to ∞. We have $a(i\,y) + e = i\,u$, $a(b + i\,y) + e = b + i\,d$, where y is any real number and u, d are real numbers. It must be that $a = 1$ and e is a pure imaginary number $e = i\,c$. So we have $\phi_2 = \phi_1 + i\,c$. Now take

$$\psi(w) = \phi(w + 2\pi i).$$

Then ψ is a conformal diffeomorphism from $f^{-1}(\overline{D})$ to S_b, so $\psi = \phi + i\,c$ for some number $c \in \mathbf{R}$. If $c \neq 0$, then take

$$\frac{b'}{b}\psi = \frac{b'}{b}\phi + i\,\frac{b'}{b}c,$$

which is a conformal diffeomorphism from $f^{-1}(\overline{D})$ to $S_{b'}$. By adjusting b' we can get $b'c/b = \pm 2\pi$, then

$$\frac{b'}{b}\phi(w + 2\pi i) = \frac{b'}{b}\phi(w) \pm 2\pi i.$$

Denoting $\frac{b'}{b}\phi$ by ϕ, we can define a conformal diffeomorphism from $\overline{D}$ to the annular ring

$$\{z \in \mathbf{C} : 1 \leq |z| \leq e^{b'}\}$$

by $g(z) = e^{\phi(f^{-1}(z))}$. If $c = 0$, we can do this directly. Thus we have proved:

Lemma 9.1 *Each annulus with Jordan curve boundaries is conformally equivalent to an annular ring*

$$\triangle = \{z \in \mathbf{C} : r \leq |z| \leq R\}.$$

Let

$$D = \{z \in \mathbf{C} : \rho \leq |z| \leq P\}$$

be another annular ring which is conformally diffeomorphic to $\triangle$, i.e., there is a holomorphic hemeomophism $h : D \to \triangle$ such that h maps $\{|z| = \rho\}$ to $\{|z| = r\}$ and $\{|z| = P\}$ to $\{|z| = R\}$. By repeatedly using Schwarz's reflection principle, we can extend h to a conformal diffeomorphism $h : \mathbf{C} \to \mathbf{C}$ such that $h(0) = 0$ and $h(\infty) = \infty$. Hence $h(z) = az$. Thus we will have $|a| = r/\rho = R/P$, and

$$M := \frac{R}{r} = \frac{P}{\rho}. \tag{9.37}$$

The number M defined in (9.37) is called *modulus*. For an annulus with Jordan curve boundaries we can define its modulus by Lemma 9.1. We have just proved that if D and $\triangle$ are conformally equivalent, then they have the same moduli. On the other hand, if D and $\triangle$ are annular rings which have the same moduli, then $h(z) = \frac{r}{\rho}z$ is a conformal diffeomorphism from D to $\triangle$. Thus we have:

Proposition 9.2 *Two annuli with Jordan curve boundaries are conformally equivalent if and only if they have the same moduli.*

Thus the interior of any annulus with Jordan curve boundary is conformally equivalent to

$$A_R = \{z \in \mathbf{C} : \frac{1}{R} < |z| < R\}, \tag{9.38}$$

for some $R > 1$.

Letting $R \to \infty$, we get $\mathbf{C} - \{0\}$ which topologically is an annulus. Its universal covering space is $\mathbf{C}$, as we have seen. Therefore it is parabolic, and different from the A_R for $1 < R < \infty$, which are hyperbolic.

The remaining case is the punctured disk $\mathbf{D}^* := \mathbf{D} - \{0\}$ which is conformally equivalent to $\{z \in \mathbf{C}, \big| \rho \leq |z| < \infty\}$ for any $\rho > 0$. Since $\Re z - 2$ is a non-constant negative harmonic function on $\mathbf{D}^*$, $\mathbf{D}^*$ is hyperbolic. We can naturally think of D^* as having ∞ modulus. This suggests that D^* is different from annuli with Jordan curve boundaries. Later we will see that this is indeed true. Our proof actually uses minimal surfaces, see the next section.

10 Complete Minimal Surfaces, Osserman's Theorem

Let $X : M \hookrightarrow \mathbf{R}^3$ be a surface, $\Lambda^2 = |X_1|^2 = |X_2|^2$, and $\gamma : I \to M$ be a differentiable curve. The arc length of γ is $\Gamma := \int_I |(X \circ r)'(t)|\, dt$. A *divergent path* on M is a piecewise differentiable curve $\gamma : [0, \infty) \to M$ such that for every compact set $V \subset M$ there is a $T > 0$ such that $\gamma(t) \notin V$ for every $t > T$. If γ is piecewise differentiable, we define its arc length as

$$\Gamma := \int_0^\infty |(X \circ r)'(t)|\, dt = \int_0^\infty \Lambda(\gamma(t))\, |r'(t)|\, dt.$$

Note that Γ could be ∞.

Definition 10.1 We say that X is *complete* if for any divergent curve γ, $\Gamma = \infty$.

Remark 10.2 The use of a divergent curve instead of boundary to describe completeness is because that if $M = \mathbf{D}^*$, $\{0\}$ is not a boundary point of M, but is the limit point of a divergent curve.

Note that in case that (M, g) is a non-compact Riemannian manifold and $\partial M = \emptyset$, according to the Hopf-Reno theorem, this definition of completeness is equivalent to each of the following:

1. Any geodesic $\gamma : I \subset \mathbf{R} \hookrightarrow M$ can be extended to a geodesic $\gamma : \mathbf{R} \hookrightarrow M$,
2. (M, d) is a complete metric space, where d is the induced distance from the Riemannian metric g (roughly speaking, $d(p, q)$ = the arc length of the shortest geodesic segment connecting p and q),
3. in (M, d), any bounded closed set is compact.

In general, there are many examples of closed minimal submanifolds $M \hookrightarrow (N, g)$ where (N, g) is a Riemannian manifold. For example, $S^2 \subset S^3$ is minimal. But we have seen that there are no closed minimal surfaces in $\mathbf{R}^3$. Hence in some sense a complete minimal surface without boundary is the closest analogue to a "closed minimal surface in $\mathbf{R}^3$".

Definition 10.3 Let $X : M \hookrightarrow \mathbf{R}^3$ be a complete minimal surface. Remember that the Gauss curvature K is a non-positive function on M, hence the integral of K has a meaning. We define

$$K(M) := \int_M K dA \tag{10.39}$$

to be the *total Gauss curvature* of M.

Let $X : M \hookrightarrow \mathbf{R}^3$ be a surface and K be the Gauss curvature. Let $K^- = \max\{-K, 0\}$, $K^+ = \max\{K, 0\}$, then $K = K^+ - K^-$, $|K| = K^+ + K^-$. We first prove a theorem of A. Huber, the proof shown here belongs to B. White [82].

Theorem 10.4 (Huber [35]) *Let $X : M \hookrightarrow \mathbf{R}^3$ be a non-compact, complete surface. If $\int_M |K^-| dA < \infty$, then $\int_M |K^+| dA < \infty$, and M is homeomorphic to $\bar{M} - \{p_1, \cdots, p_k\}$, where $\bar{M}$ is a compact 2-manifold.*

Proof. Fix $x_0 \in M$, and let

$$\Omega_r = \Omega(r) = \{x \in M \mid d(x, x_0) < r\},$$

where $d(x, y)$ is the geodesic distance from x to y. P. Hartman [23] has shown that $\partial\Omega_r$ is, for almost all r, a piecewise smooth, embedded closed curve. Let θ_i, $i = 1, \cdots, n$ be the exterior angles of $\partial\Omega_r$. By Gauss-Bonnet theorem,

$$\begin{aligned}\int_{\partial\Omega_r} \kappa_g \, ds + \sum_i \theta_i &= 2\pi\chi(r) - \int_{\Omega_r} K \, dA \\ &= 2\pi(2 - 2h(r) - c(r)) - \int_{\Omega_r} K \, dA, \end{aligned} \qquad (10.40)$$

where $\chi(r)$, $h(r)$, and $c(r)$ are the Euler characteristic, number of handles, and the number of boundary components, respectively, of Ω_r.

Let $L(r)$ denote the length of $\partial\Omega_r$. P. Hartman [23] has proved that $L(r)$ is absolutely continuous. As proved in [83],

$$L'(r) = 2\pi(2 - 2h(r) - c(r)) - \int_{\Omega_r} K \, dA + \sum_{\theta_i < 0} (2\tan(\theta_i/2) - \theta_i),$$

when $-\pi/2 < \theta_i < 0$, $2\tan(\theta_i/2) - \theta_i < 0$, so

$$L'(r) \le 2\pi(2 - 2h(r) - c(r)) - \int_{\Omega_r} K \, dA.$$

Since M is complete and noncompact, $L(r) > 0$ for all $r > 0$; so

$$\begin{aligned} 0 &\le \limsup L'(r) \\ &\le 2\pi(2 - 2\liminf h(r) - \liminf c(r)) - \int_M |K^+| \, dA + \int_M |K^-| \, dA. \end{aligned} \qquad (10.41)$$

Thus the negative quantities on the right-hand side are all finite. Since $h(r)$ is a non-decreasing, integer valued function of r,

$$h(r) = \text{some constant } h \ \text{ for } r > R.$$

Also, $c(r)$ is integer valued, so we can find a sequence $r_i \to \infty$ with

$$c(r_i) = c = \liminf c(r) < \infty.$$

Let A_i be the union of Ω_{r_i} with those connected components of $M - \Omega_{r_i}$ which happen to be compact. (There may not be any, in which case $A_i = \Omega_{r_i}$.) Let $h(A_i)$ and $c(A_i)$ denote the number of handles and boundary components, respectively, of A_i. Then

$$h = h(\Omega_{r_i}) \le h(A_i) \le h(\Omega_{r_{i+j}}) = h$$

provided j is large enough that $A_i \subset \Omega_{r_{i+j}}$: so

$$h(A_i) \equiv h \tag{10.42}$$

and clearly $c(A_i) \leq c(\Omega_{r_i})$. By passing to a subsequence we may assume

$$c(A_i) \equiv c' \; (\leq c). \tag{10.43}$$

By (10.42) and (10.43), the A_i are homeomorphic, with A_{i+1} obtained from A_i by attaching annuli. The result follows immediately. □

Since for minimal surfaces $K \leq 0$ on M, we know that a complete minimal surface of finite total curvature has finite topology. We are interested in the conformal structure of M. Now since M has finite topology, $M = S_k - (\{p_1, \cdots, p_n\} \cup \bigcup_{i=1}^{l} U_i)$ as a Riemann surface, where $U_i \subset S_k$ is conformally a closed disk. Furthermore, there are disjoint conformal open disks $D_i \subset S_k$, $i = 1, \cdots, n+l$, such that $p_i \in D_i$, $i = 1, \cdots, n$, and $U_i \subset D_{i+n}$, $i = 1, \cdots, l$, and the boundaries ∂D_i are mutually disjoint analytic Jordan curves. See, for example, [1], I 44D and II 3B. Hence each $\overline{D}_{i+n} - U_i$ is conformally a doubly connected plane domain, which must be conformally equivalent to some

$$\tilde{A}_R := \{z \in \mathbf{C} \,|\, 1/R \leq |z| < R\}$$

with $1 < R < \infty$.

Let $\phi : \overline{D}_{i+n} - U_i \to \tilde{A}_R$ be a conformal diffeomorphism. Since X is complete with finite total curvature, $X \circ \phi^{-1}$ is a complete minimal annulus with finite total curvature, where completeness of $X \circ \phi^{-1}$ means for any curve $\gamma : I \to A_R$ diverging to $|z| = R$, the arc length Γ of γ is infinity. We will prove that such a complete minimal annulus does not exist and hence $M = S_k - \{p_1, \cdots, p_n\}$.

Actually, we will prove $M = S_k - \{p_1, \cdots, p_n\}$ by showing that for any $\tilde{A}_R$ there is no complete Riemannian metric which is conformal to the Euclidean metric and has non-positive Gauss curvature and finite total curvature. If there were a closed disk U_i removed from S_k, the induced metric on some $\tilde{A}_R$ by $X \circ \phi^{-1}$ would be a complete Riemannian metric which is conformal to the Euclidean metric and has non-positive Gauss curvature and finite total curvature. Thus we know that it must be the case that $M = S_k - \{p_1, \cdots, p_n\}$.

By the way, since there do exist complete minimal annuli $Y : D^* \hookrightarrow \mathbf{R}^3$, we see that D^* is not conformally equivalent ot any A_R or $\tilde{A}_R$ as mentioned in the last section.

First we give an easy lemma which uses the special structure of $\tilde{A}_R$. The proof is left as an exercise.

Lemma 10.5 *If $g_{ij} = \lambda^2 \delta_{ij}$ is a complete Riemannian metric on $\tilde{A}_R$, then $\tilde{g}_{ij}(z) := \lambda^2(z)\lambda(1/z)\delta_{ij}$ is a complete Riemannian metric on $\hat{A}_R := \{1/R < |z| < R\}$.*

Next we prove that if $\lambda = e^h$ and h is harmonic, then $\lambda^2 \delta_{ij}$ on $\tilde{A}_R$ cannot be complete.

Proposition 10.6 *Suppose $D \subset \mathbf{C}$ and $g_{ij} = e^{2h}\delta_{ij}$ is a complete Riemannian metric on D. If $\triangle h = 0$, then conformally D is either $\mathbf{C} - \{0\}$ or $\mathbf{C}$.*

Proof. Consider the conformal universal covering $\pi : \tilde{D} \subset \mathbf{C} \to D$. Since π is holomorphic, $\tilde{h}(z) = h(\pi(z))$ is harmonic. Furthermore, $\tilde{g}_{ij} = e^{2\tilde{h}}\delta_{ij}$ is a complete Riemannian metric on $\tilde{D}$. Since $\tilde{D}$ is simply connected, there is a harmonic function $\tilde{k}$ conjugate to $\tilde{h}$. Define a holomorphic function $w : \tilde{D} \to \mathbf{C}$ by

$$w(z) = \int_0^z e^{\tilde{h}(\zeta) + i\tilde{k}(\zeta)} d\zeta.$$

Since $\tilde{D}$ is simply connected, w is well defined.

First we claim that w sends a geodesic into a straight line. In fact, the induced metric by w from the Euclidean metric on $\mathbf{C}$ is $|w'|^2\delta_{ij} = e^{2\tilde{h}}\delta_{ij} = \tilde{g}_{ij}$. Hence the metric $\tilde{g}$ is the first fundamental form of the surface $w : \tilde{D} \to \mathbf{C} \subset \mathbf{R}^3$ and $w : (\tilde{D}, \tilde{g}) \to (\mathbf{C}, \bullet)$ is an isometry. Let γ be a geodesic on $\tilde{D}$, then $w \circ \gamma$ is a plane geodesic of $w(\tilde{D}) \subset \mathbf{C}$, and thus $w \circ \gamma$ must be a straight line segment in $\mathbf{C}$.

Next we prove that w is one to one and onto. Let $\gamma : [0, \infty) \hookrightarrow \tilde{D}$ be a geodesic ray of unit speed on $\tilde{D}$. Then $(w \circ \gamma)'(t) = w'(\gamma(t))\, \gamma'(t) = \rho(t)e^{i\theta(t)} \neq 0$, since w' and γ' are both non-zero. Since γ is unit speed, $1 = |\gamma'(t))|_{\tilde{g}} = |(w \circ \gamma)'(t)| = \rho(t)$. Since $w \circ \gamma$ is a straight line segment, $\theta(t)$ must be a constant, say θ_0. Thus we can write

$$w(\gamma(t)) = w(\gamma(0)) + te^{i\theta_0}.$$

This proves that w sends any geodesic ray one to one and onto a ray in $\mathbf{C}$.

Now by completeness, $\tilde{D}$ is the union of all geodesic rays starting from 0. Since w is locally a conformal diffeomorphism, different geodesic rays starting from 0 are mapped by w one to one and onto different rays starting from $w(\gamma(0)) \in \mathbf{C}$, thus w must be one to one and onto $\mathbf{C}$.

Now $w : \tilde{D} \to \mathbf{C}$ is a conformal diffeomorphism, so $\tilde{D} = \mathbf{C}$. Since the conformal universal covering of D is $\mathbf{C}$, conformally D must be either $\mathbf{C}$ or $\mathbf{C} - \{0\}$. □

Now we have to use the facts that $X : M \hookrightarrow \mathbf{R}^3$ has finite total curvature and $\tilde{A}_R$ is hyperbolic in order to construct a complete metric $e^{2h}\delta_{ij}$ on A_R such that h is harmonic.

Proposition 10.7 *Let $D \subset \mathbf{C}$ be a hyperbolic domain and $g_{ij} = \lambda^2\delta_{ij}$ a Riemannian metric on D, such that $\triangle \log \lambda \geq 0$ and*

$$\int_D \triangle \log \lambda \, dx\, dy < \infty. \tag{10.44}$$

Then there is a harmonic function h such that $\log \lambda \leq h$.

Proof. Since D is a hyperbolic domain, there is a Green's function $G(\zeta, z)$ on D for any $\zeta \in D$ which is positive except at ζ and such that $G(\zeta, z) + \log|z - \zeta|$ is a harmonic function on D. Since $\triangle \log \lambda \in L^1(D)$,

$$u(\zeta) := \frac{1}{2\pi} \int_D G(\zeta, z) \,\triangle \log \lambda \, dx\, dy$$

solves the Poisson equation $\triangle u = -\triangle \log\lambda$. Note that $u \geq 0$, $h = u + \log\lambda \geq \log\lambda$ is the desired harmonic function. $\square$

Now by Lemma 10.5 the induced metric by $X\circ\phi^{-1}$ on $\tilde{A}_R$ is $\Lambda^2 g_{ij}$. Then the Gauss curvature is

$$K = -\frac{\triangle \log\Lambda}{\Lambda^2}.$$

Since $K \leq 0$, we know that $\triangle \log\Lambda \geq 0$ is non-negative. Moreover, since

$$K(\tilde{A}_R) = \int_{\tilde{A}_R} K dA = -\int_{\tilde{A}_R} \frac{\triangle \log\Lambda}{\Lambda^2}\Lambda^2\, dx\, dy = -\int_{\tilde{A}_R} \triangle \log\Lambda\, dx\, dy,$$

condition (10.44) is equivalent to $X\circ\phi^{-1}$ having finite total curvature. So we have a harmonic function $h \geq \log\Lambda$. Thus

$$\log\Lambda(z) + \log\Lambda(1/z) \leq H(z) := h(z) + h(1/z).$$

Obviously $e^{2H}\delta_{ij}$ is a complete Riemannian metric on $\hat{A}_R$. Since $H(z) = h(z) + h(1/z)$ is harmonic, by Proposition 10.6 we have $R = \infty$, a contradiction. This contradiction proves the first part of the following theorem due to Osserman:

Theorem 10.8 (Osserman, [66]) *Let M be a Riemann surface without boundary and $X : M \hookrightarrow \mathbf{R}^3$ a complete minimal surface such that the total curvature $K(M)$ is finite. Then*

1. *There exists a closed Riemann surface S_k and a finite number of points $p_1, \dots, p_r$ on S_k such that M is conformally $S_k - \{p_1, \dots, p_r\}$;*

2. *The Gauss map $g : M \to \Sigma$ can be extended to S_k such that the extension $\tilde{g} : S_k \to \Sigma$ is a holomorphic function. Moreover,*

$$K(M) = -4\pi \deg g. \tag{10.45}$$

Recall that if $g : S_k \to \mathbf{C}$ is a meromorphic function, where S_k is a closed Riemann surface, then there is a positive integer n such that for all but finitely many $p \in \mathbf{C}$, $g^{-1}(p) \subset S_k$ consists of n points. We say that g has *degree* n, and denote this by $\deg g = n$.

Proof. Since we have proved the first part, we only need prove the second part.

Recall that

$$K = -\frac{16|g'|^2}{|f|^2(1+|g|^2)^4} \quad \text{and} \quad \Lambda^2 = \frac{1}{4}|f|^2(1+|g|^2)^2.$$

Recall that $\tau^{-1} : \mathbf{C} \to S^2$ is a complex chart of S^2. In this chart the volume form of S^2 is

$$dS(w) = \frac{4}{(1+|w|^2)^2} du \wedge dv$$

where $w = u + iv \in \mathbf{C}$. Obviously

$$\int_{S^2} dS = \int_{\mathbf{C}} dS(w) = 4\pi.$$

Let U be a coordinate neighbourhood in M on which g has no pole. Since g is holomorphic, $|g'|^2 = \det Dg$, where we interpret g as $g : U \to \mathbf{R}^2$. The induced metric by $g : U \to (\mathbf{C}, dS)$ has the volume form

$$g^*(dS) = \frac{4 \det Dg}{(1+|g|^2)^2} dx\, dy.$$

Since $g^*(dS)$ is well defined on $M - g^{-1}(\infty)$ (in fact on M), by

$$\int_U K dA = \int_U K\Lambda^2 dx \wedge dy = -\int_U \frac{4|g'|^2}{(1+|g|^2)^2} dx \wedge dy$$

$$= -\int_U \frac{4 \det Dg}{(1+|g|^2)^2} dx \wedge dy = -\int_U g^*(dS)$$

we have

$$\int_M K dA = -\int_M g^*(dS).$$

Thus by the area formula we finally get

$$\int_M g^*(dS) = \int_{\mathbf{C}} \sharp\{g^{-1}(w)\} dS(w) = 4\pi \deg g,$$

where $\sharp\{g^{-1}(w)\}$ is the number of points in $g^{-1}(w)$. Since g is meromorphic, for almost every $w \in \mathbf{C}$, $\sharp\{g^{-1}(w)\} = \deg g$. The proof is complete. □

Corollary 10.9 *If $X : M \hookrightarrow \mathbf{R}^3$ is a non-planar complete minimal surface of finite total curvature, then the Gauss map $g : M \to \mathbf{C}$ can miss at most a finite number of points of $\mathbf{C}$.*

Proof. Since g can be extended to a closed Riemann surface S_k and g is not a constant, (otherwise X will be contained in a plane) we know that $g(S_k) = \mathbf{C} \cup \{\infty\}$. Now $M = S_k - \{p_1, \cdots, p_r\}$, so $\mathbf{C} - g(M)$ has at most a finite number of points. □

Corollary 10.10 (Bernstein's Theorem) *Let $u : \mathbf{R}^2 \to \mathbf{R}$ be a solution to the minimal surface equation. Then u is an affine function, i.e., $u(x, y) = ax + by + c$ where a, b, and c are constants.*

Proof. If u is not affine, then the graph of u is a non-planar complete minimal surface S of conformal type $\mathbf{C} = S^2 - \{(0,0,1)\}$. In fact, the special isothermal coordinate in Section 5, $(\xi, \eta) : \mathbf{R}^2 \to \mathbf{R}^2$, is one to one and onto $\mathbf{R}^2$. If S has infinite total curvature, then the Gauss map g of S has an essential singularity at ∞, and hence g misses at most one point in $\mathbf{C}$. If S has finite total curvature and is non-planar, then Corollary 10.9 tells us that g misses at most a finite number of points of $\mathbf{C}$. But since S is a graph,

$$N = \frac{1}{(1 + u_x^2 + u_y^2)^{1/2}} (-u_x, -u_y, 1)$$

misses the lower hemisphere of S^2. This contradiction shows that S must be planar and forces $g \equiv$ constant, and hence u_x and u_y must be constant and u must be affine. $\square$

11 Ends of Complete Minimal Surfaces

By Osserman's theorem, any complete minimal surface of finite total curvature is an immersion $X : M = S_k - \{p_1, \cdots, p_r\} \hookrightarrow \mathbf{R}^3$, where S_k is a closed Riemann surface of genus k. Consider conformal closed disks $D_i \subset S_k$ such that $p_i \in D_i$ and $p_j \notin D_i$ for $j \neq i$. Denote $D_i^* := D_i - \{p_i\}$. For any such D_i, the restriction $X : D_i^* \hookrightarrow \mathbf{R}^3$ is called a *representative of an end of* X *at* p_i or simply *an end.* When we say that some property holds at an end of X at p_i, for example embeddedness, we mean that there is a disk like domain D_i such that for any disk like domain $p_i \in U_i \subset D_i$, $X : U_i - \{p_i\}$ satisfies the property. Such a representative $X : U_i - \{p_i\} \to \mathbf{R}^3$ is called a *subend* of the end $X : D_i^* \hookrightarrow \mathbf{R}^3$.

Osserman's theorem says that the Gauss map g extends to p_i and the extended g is a meromorphic function. Since $N = \tau^{-1} \circ g$ we have a well defined normal vector $N(p_i)$ at p_i, which we call the *limit normal* at p_i. This also defines a *limit tangent plane* at the end E_i corresponding to p_i.

Intuitively, and we will prove it later (see Proposition 11.5), $E_i = X(D_i^*) \subset \mathbf{R}^3$ is an unbounded set. Moreover, since $M - \bigcup_{i=1}^r D_i^*$ is precompact, $X(M) - \bigcup_{i=1}^r E_i$ is bounded. Thus if X is an embedding, an end E_i is just a connected component of $X(M) - B$, where B is any sufficiently large ball in $\mathbf{R}^3$ centred at 0.

In this section, all ends considered are ends of some complete minimal surface of finite total curvature.

Now consider the Enneper-Weierstrass representation of the complete minimal surface $X : M \hookrightarrow \mathbf{R}^3$. By (6.20)

$$\Lambda^2 = \frac{1}{2}\left(|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2\right).$$

Now let $r : [0,1) \to D_i^*$ be a regular curve such that $|r'(t)| = 1$ and $\lim_{t\to 1} r(t) = p_i$. By completeness,

$$\int_0^1 \Lambda(r(t))\,|r'(t)|\,dt = \infty.$$

This implies that $\Lambda(q) \to \infty$ as $q \to p$. Since ϕ_i's are meromorphic, one of them must have a pole at p. Hence let z be the local coordinate of D_i such that $z(p_i) = 0$, we must have

$$\Lambda^2 = \frac{1}{2}\left(|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2\right) \sim \frac{c}{|z|^{2m}}, \tag{11.46}$$

where $c > 0$ and $m \geq 1$ is an integer.

Definition 11.1 If $\Lambda^2 \sim c/|z|^{2m}$ at an end, we say that Λ has order m at that end.

Remark 11.2 Since Λ^2 is the pull back metric of $X : M \to \mathbf{R}^3$, we see that the order of Λ is invariant under an isometry in $\mathbf{R}^3$. Precisely, if A is an isometry of $\mathbf{R}^3$ then AX and X has the same pull back metric Λ^2. Thus the order of Λ at an end is invariant.

X being complete requires that the order of Λ at an end is at least one. In fact, we can prove that the order of Λ at an end is at least 2.

Lemma 11.3 *Let $X : M = S_k - \{p_1, \cdots, p_r\} \hookrightarrow \mathbf{R}^3$ be a complete minimal immersion with finite total curvature, and $(\omega_1, \omega_2, \omega_3)$ its Enneper-Weierstrass representation. Then at each p_j, at least one of ω_1, ω_2, ω_3 has a pole of order at least 2.*

Proof. Let (D_j, z) be a coordinate neighbourhood such that $z(p_j) = 0$ and on D_j^*, $(\omega_1, \omega_2, \omega_3) = (\phi_1, \phi_2, \phi_3)dz$.

We have shown that at least one of ϕ_1, ϕ_2, ϕ_3 has a pole at p_j. So $m \geq 1$. If $m = 1$, there are complex constants c_1, c_2 and c_3, not all zero, such that $f_i := \phi_i - c_i/z$ is holomorphic in D_j. Now

$$\Re(c_i \log z) = \Re \int (\phi_i - f_i)dz = X_i - \Re \int f_i\, dz, \qquad i = 1,\ 2,\ 3,$$

are well defined harmonic functions on D_i^*. Since

$$\Re(c_i \log z) = (\Re c_i) \log |z| - (\Im c_i) \arg z,$$

c_i must be real. But

$$0 = \phi_1^2 + \phi_2^2 + \phi_3^2 = f_1^2 + f_2^2 + f_3^2 + (c_1^2 + c_2^2 + c_3^2)/z^2 + 2(c_1 f_1 + c_2 f_2 + c_3 f_3)/z.$$

Comparing the terms of the same order, it must be that $c_i = 0$ for $i = 1$, 2, 3. But then $\phi_i = f_i$ is holomorphic and bounded in D_j, contradicting the fact that X is complete. □

Now recall that by definition $X : S_k - \{p_1, \cdots, p_r\} \hookrightarrow \mathbf{R}^3$ is complete if and only if for any divergent curve γ the arc length of $X \circ \gamma$ is infinity. Thus either $X \circ \gamma$ goes to infinity in $\mathbf{R}^3$ or $X \circ \gamma$ stays in a compact set of $\mathbf{R}^3$ but has infinite arc length. To study these two cases, we introduce the concept of *properness*.

Definition 11.4 A mapping $X : M \to N$ between two topological spaces is *proper* if for any compact set $C \subset N$, $X^{-1}(C)$ is also compact.

Proposition 11.5 (Osserman) *If $X : M \to \mathbf{R}^3$ is a complete minimal surface of finite total curvature then X is proper.*

Proof. We know that $M = S_k - \{p_1, \cdots, p_r\}$ where S_k is a closed Riemann surface of genus k. Let $p \in \{p_1, \cdots, p_r\}$. Since the order of Λ is invariant under isometries of $\mathbf{R}^3$, after a rotation, we may assume that $g(p) = 0$. There is a coordinate disk $U \subset S_k$ at p such that $z(p) = 0$ and $|z| < 1$ on U. So we can write that $g(z) = z^n h(z)$, where $n > 0$ and $h(0) \neq 0$. On $U - \{p\}$, η must have a pole of order $m \geq 2$, hence we can write $\eta = f(z)dz$ where

$$f(z) = \sum_{i=-m}^{\infty} a_i z^i = \frac{1}{z^m} F(z),$$

where F is holomorphic and $a_{-m} = F(0) \neq 0$. We can write

$$f(z)g^2(z) = \sum_{2n-m}^{\infty} b_i z^i.$$

Recall that

$$\phi_1(z) = \frac{1}{2} f(z)\,(1 - g^2(z)), \quad \phi_2(z) = \frac{i}{2} f(z)\,(1 + g^2(z)).$$

Since on the loop $C := \{|z| = \rho < 1\}$,

$$\begin{aligned}
0 &= \Re \int_C \phi_1\, dz - i\,\Re \int_C \phi_2\, dz \\
&= \frac{1}{2}\Re \int_C (a_{-1} - b_{-1}) z^{-1} dz + i\frac{1}{2}\Im \int_C (a_{-1} + b_{-1}) z^{-1} dz \\
&= \pi i (a_{-1} + \overline{b_{-1}}) \quad \text{(by the residue theorem)},
\end{aligned}$$

we have

$$a_{-1} = -\overline{b_{-1}}. \tag{11.47}$$

Let $X(z) = (X^1, X^2, X^3)(z)$, then

$$\begin{aligned}
(X^1 - iX^2)(z) &= \Re \int_{z_0}^{z} \phi_1(\zeta) d\zeta - i\Re \int_{z_0}^{z} \phi_2(\zeta) d\zeta + (X^1 - iX^2)(z_0) \\
&= \Re \int_{z_0}^{z} \frac{1}{2} f(\zeta)(1 - g^2(\zeta)) d\zeta + i\Im \int_{z_0}^{z} \frac{1}{2} f(\zeta)(1 + g^2(\zeta)) d\zeta + (X^1 - iX^2)(z_0) \\
&= \frac{1}{2} \int_{z_0}^{z} f(\zeta) d\zeta - \frac{1}{2} \overline{\int_{z_0}^{z} f(\zeta) g^2(\zeta) d\zeta} + (X_1 - iX_2)(z_0) \\
&= \frac{1}{2} \sum_{\substack{i=-m \\ i \neq -1}}^{\infty} \frac{a_i}{1+i} z^{i+1} - \frac{1}{2} \overline{\sum_{\substack{i=2n-m \\ i \neq -1}}^{\infty} \frac{b_i}{1+i} z^{i+1}} + \frac{1}{2}(a_{-1} - \overline{b_{-1}}) \log |z| \\
&= \frac{1}{2} \frac{a_{-m}}{1-m} z^{1-m} + \frac{1}{2}(a_{-1} - \overline{b_{-1}}) \log |z| + O(|z|^{2-m}).
\end{aligned} \tag{11.48}$$

Since $a_{-m} \neq 0$ and $m \geq 2$, (11.48) shows that $|X|^2 \to \infty$ as $z \to 0$. Thus for any compact set $B \subset \mathbf{R}^3$, there are open disks $p_i \in D_i \subset S_k$ such that $X^{-1}(B) \subset S_k - \bigcup_{i=1}^{r} D_i$ is compact. $\square$

We want to know how to determine whether an end is embedded by looking at the Enneper-Weierstrass representation.

Lemma 11.6 *If the order of Λ at an end is $m = 2$, then there is an open conformal disk D such that $X : D - \{p\} \hookrightarrow \mathbf{R}^3$ is an embedding, where p is the puncture corresponding to the end.*

Proof. In the proof of Proposition 11.5, since $n \geq 1$ and $m = 2$ we see that $b_{-1} = 0$ and hence $a_{-1} = 0$ by (11.47). Now by the same calculation which led to (11.48),

$$(X^1 - iX^2)(z) = -\frac{1}{2}\frac{a_{-2}}{z} + O(|z|). \tag{11.49}$$

Obviously for some $0 < \rho < 1$ small enough, $X_1 - iX_2 : D - \{p\} := \{z \in U | 0 < |z| < \rho\} \to \mathbf{C}$ is one to one and $\lim_{|z| \to 0} |X_i - iX_2|(z) = \infty$. Hence $X\Big|_{D-\{p\}}$ is an embedding. □

When Λ has order 2 at an end, we can get more information about the behaviour of X at that end; in fact this end can be expressed as a minimal graph with a very nice growth property. To prove this, we first show:

Lemma 11.7 *Let $p \in \{p_1, \cdots, p_r\}$ and Λ have order 2 at p. Then there are $R > 0$ and $\rho > 0$ such that the mapping $X^1 - iX^2 : D - \{p\} \to \mathbf{C}$ defined in Lemma 11.6 is onto $\{\xi \in \mathbf{C} \,|\, |\xi| > R\}$.*

Proof. We have seen in Lemma 11.6 that for some $0 < \rho < 1$, $X^1 - iX^2 : D - \{p\} = \{0 < |z| \leq \rho\} \to \mathbf{C}$ is one to one and $\lim_{|z| \to 0} |X^1 - iX^2|(z) = \infty$. Let $R = \max_{|z|=\rho}\{|X^1 - iX^2|(z)\}$. Note that $\alpha := (X^1 - iX^2)(\{|z| = \rho\})$ is a Jordan curve in $\mathbf{C}$. If there is a $\xi \in \mathbf{C}$, $|\xi| > R$ and $\xi \notin (X^1 - iX^2)(D - \{p\})$, then there is a $0 < r < \rho$ such that $\min_{|z|=r}\{|X_1 - iX_2|(z)\} > |\xi|$. Let $\beta := (X^1 - iX^2)(\{|z| = r\})$, then β is a Jordan curve in $\mathbf{C}$ and $\alpha \cap \beta = \emptyset$. Let $\Omega := \mathbf{C} - \{0\} - \{\xi\}$, where α and β are not free homotopic to each other in Ω. But clearly $(X^1 - iX^2)(\{r < |z| < \rho\}) \subset \Omega$ and $\phi(\theta, t) := (X^1 - iX^2)[(r + t(\rho - r))e^{i\theta}]$, $0 \leq t \leq 1$, $0 \leq \theta \leq 2\pi$, is a homotopy from β to α in Ω. Thus we get a contradiction. This contradiction proves that $\xi \in (X^1 - iX^2)(D - \{p\})$. The lemma is proved. □

Theorem 11.8 *Let the notation be as in Lemmas 11.6 and 11.7. Then there is an $R > 0$ and an $\epsilon \in (0, 1)$ such that outside the solid cylinder $\{(x_1, x_2, x_3) \in \mathbf{R}^3 \,|\, x_1^2 + x_2^2 \leq R^2\}$, $X(0 < |z| < \epsilon)$ is a graph $(x_1, x_2, u(x_1, x_2))$ over the x_1x_2-plane. Furthermore, asymptotically,*

$$u(x_1, x_2) = \alpha \log r + \beta + r^{-2}(\gamma_1 x_1 + \gamma_2 x_2) + O(r^{-2}), \tag{11.50}$$

where $r = (x_1^2 + x_2^2)^{1/2}$, and α, β, γ_1 and γ_2 are real constants.

Proof. We have proved that there is an $\epsilon \in (0, 1)$ such that the mapping $X^1 - iX^2$: $D^* := \{z \,|\, 0 < |z| < \epsilon\} \to \mathbf{C}$ is one to one and onto $|\xi| > R$ for some $R > 0$. Let $\Omega = \{|\xi| > R\}$. For any $(x_1, x_2) \in \Omega$ there is a unique $z \in D^*$ such that $x_1 = X^1(z)$ and $x_2 = X^2(z)$. Define $u(x_1, x_2) = X^3(z)$ on $(X^1 - iX^2)^{-1}(\Omega)$, then u is a well defined function. Now use the data written down in the proof of Proposition 11.5, recalling that $g(z) = z^n h(z)$, $f(z) = a_{-2}z^{-2} + \sum_{i=0}^{\infty} a_i z^i$, and so $\phi_3(z) = a_{-2}h(0)z^{n-2} + a_{-2}h'(0)z^{n-1} + \sum_{i=n}^{\infty} b_i z^i$.

We consider the two cases of $n = 1$ or $n > 1$. If $n = 1$, let $C := \{|z| = \epsilon_1\}$ for some $0 < \epsilon_1 < \epsilon$. Since

$$0 = \Re \int_C \phi_3(z)dz = \Re(a_{-2}h(0)2\pi i),$$

we see that $\alpha := -a_{-2}h(0) \neq 0$ is real. Thus

$$\begin{aligned} u(x_1, x_2) &= X^3(z) = \Re \int_{z_0}^{z} \phi_3(\zeta)d\zeta + X^3(z_0) \\ &= -\alpha \log|z| + \Re(a_{-2}h'(0)z) + O(|z|^2) + X^3(z_0). \end{aligned}$$

By (11.49),

$$r^2 = |x_1 - ix_2|^2 = \frac{|a_{-2}|^2}{4|z|^2} + O(1) = \frac{1}{|z|^2}\left(\frac{|a_{-2}|^2}{4} + O(|z|^2)\right),$$

$$2\log r = -2\log|z| + \log\left(\frac{|a_{-2}|^2}{4} + O(|z|^2)\right) = -2\log|z| + 2\log\frac{|a_{-2}|}{2} + O(|z|^2).$$

Also by (11.49),

$$z = \frac{-a_{-2}}{2(x_1 - ix_2)} + O(r^{-2}) = \frac{-a_{-2}(x_1 + ix_2)}{2r^2} + O(r^{-2}).$$

Thus there are real constants γ_1 and γ_2 such that

$$\Re(a_{-2}h'(0)z) = \frac{\gamma_1 x_1 + \gamma_2 x_2}{r^2}.$$

Setting $\beta = -\alpha \log \frac{|a_{-2}|}{2} + X^3(z_0)$, we have

$$u(x_1, x_2) = \alpha \log r + \beta + r^{-2}(\gamma_1 x_1 + \gamma_2 x_2) + O(r^{-2}).$$

If $n > 1$ then ϕ_3 is bounded in D^*, hence $\alpha = 0$. In this case, the end approximates a plane. $\square$

We have shown that if Λ has order 2 at an end, then that end is embedded and is a minimal graph. Next we will show that if an end is embedded, then Λ must have order 2 at that end.

An outline of the proof is as follows: If $m > 2$ and $g(0) = 0$ then

$$(X^1 - iX^2)(z) = \frac{c}{z^k} + O(|z|^{1-k})$$

with $k > 1$. This shows that $(X^1 - iX^2)$ is not one to one, and $\lim_{|z|\to 0} |X_1 - iX_2|(z) = \infty$. But it is possible that the surface $X = (X^1, X^2, X^3)$ is embedded. However, intuitively we know that X is a graph over $\mathbb{C} - B$, where B is a large disk in $\mathbb{C}$, since our surface has a limit tangent plane corresponding to the puncture. It follows that X is embedded is equivalent to $X^1 - iX^2$ being one to one. The next lemma gives a rigorous proof of this fact.

Lemma 11.9 *Let D and p be as in Proposition 11.5. If $X: D-\{p\}$ is an embedding then there is an $R>0$ such that X is a graph over $\mathbf{R}^2 - B_R$, where $B_R := \{x \in \mathbf{R}^2 \,|\, |x| \leq R\}$. In particular, Λ has order 2 at p.*

Proof. We assume that the limit normal to X at p is $(0,0,-1)$. Let $P(x_1,x_2,x_3) = (x_1,x_2)$ be the perpendicular projection. Let $C_r := \{(x_1,x_2,x_3) \in \mathbf{R}^3 \,|\, x_1^2 + x_2^2 = r^2\}$, $V_r := \{x_1,x_2,x_3) \in \mathbf{R}^3 \,|\, x_1^2 + x_2^2 > r^2\}$.

We will prove that there is an $R > 0$ such that $P: X(D-\{p\}) \cap V_R \to \mathbf{R}^2 - B_R$ is one to one and onto $\mathbf{R}^2 - B_R$. Hence X is a graph over $\mathbf{R}^2 - B_R$. Moreover, $\partial[X^{-1}(V_R)]$ is a homotopically non-trivial Jordan curve $J_R \subset D - \{p\}$, hence $X^{-1}(V_R)$ is conformally a punctured disk.

Since the limit normal of X at p is $(0,0,-1)$, there is an $0 < \rho < 1$ such that $N_3(z) < -1/2$ for any $0 < |z| \leq \rho$. Let $D^*_\rho := \{z \,|\, 0 < |z| < \rho\}$. Since X is continuous, there is an $R > 0$ such that $|X^1 - iX^2|^2(z) < R^2$ for $|z| = \rho$. For any $r > R$, consider the set $X^{-1}(C_r) \subset D^*_\rho$. Since $N_3(z) < -1/2$ for any $0 < |z| < \rho$, X is transverse to C_r. (i.e., $X(D^*_\rho)$ and C_r have different tangent planes at common points.) This implies that $X^{-1}(C_r)$ is a family of one-dimensional submanifolds in D^*_ρ. From the expression for $X^1 - iX^2$ we know that $|X^1 - iX^2|(z) \to \infty$ when $|z| \to 0$, hence any component J_r of $X^{-1}(C_r)$ is a compact one-dimensional submanifold, i.e., it is a Jordan curve in D^*_ρ. If J_r is homotopically trivial, then it bounds a disk like domain $\Omega \subset D^*_\rho$. We will prove that $|X^1 - iX^2|^2(z) \equiv r^2$ on Ω. In fact, let $z \in \Omega$ be such that $|X^1 - iX^2|^2(z)$ achieves a maximum or minimum other than r^2 on $\overline{\Omega}$. Then z is an interior point of Ω and $D|X^1 - iX^2|^2(z) = (0,0)$. This says that

$$(X^1,X^2)_x \bullet (X^1,X^2) = 0, \quad (X^1,X^2)_y \bullet (X^1,X^2) = 0. \tag{11.51}$$

Since $(X^1,X^2)(z) \neq (0,0)$, (11.51) implies that $(X^1,X^2)_x$ and $(X^1,X^2)_y$ are linearly dependent. This then implies that $N_3(z) = 0$, contradicting $N_3(z) < -1/2$. But if $|X^1 - iX^2|^2 \equiv r^2$ on Ω, X maps Ω to C_r, another contradiction to the fact that $N_3(z) < -1/2$ in D^*_ρ. These contradictions prove that J_r is homotopically non-trivial. Now if $X^{-1}(C_r)$ has more than one component, say J_r^1 and J_r^2. The above argument shows that they are both homotopically non-trivial. Thus they are in the same $\mathbf{Z}_2$ homotopy class, and bound a compact doubly-connected domain $\Omega \subset D^*_\rho$. By the same argument we can prove that $X(\Omega) \subset C_r$, which is impossible. Thus we have shown that $J_r := (|X^1 - iX^2|^2)^{-1}(r^2) = X^{-1}(C_r)$ is a homotopically non-trivial Jordan curve in D^*_ρ.

Now $X : D^*_\rho \to \mathbf{R}^3$ is an embedding, so $\alpha := X(J_r)$ is a Jordan curve on C_r. Let $\beta : S^1 \to D^*_\rho$ be a parametrisation of J_r. If $\beta(t_i) = z_i \in J_r$ for $i = 1,2$ where $z_1 \neq z_2$ and $(X^1,X^2)(z_1) = (X^1,X^2)(z_2)$, then there is a $t \in S^1$ such that $\alpha'(t) = C(0,0,1)$ for some non-zero constant C. Since $\alpha'(t)$ is a tangent vector of X, we must have $N_3(\beta(t)) = 0$, a contradiction to $N_3(z) < -1/2$. This shows that $P: X(J_r) \to \partial B_r$ is one to one and onto for any $r > R$; hence (X^1,X^2) is one to one and onto $\mathbf{R}^2 - B_R$. $\square$

Remark 11.10 The fact that X is an embedding is used only when claiming that $\alpha = X(J_r)$ is a Jordan curve. Hence it is true that $(|X^1 - iX^2|^2)^{-1}(r^2) = X^{-1}(C_r)$ is a homotopically non-trivial Jordan curve when X is only an immersion. In general, $P: X(J_r) \to \partial B_r$ is an m to one projection except for a finite number of points in ∂B_r. The number m is the I_i in Theorem 12.1.

An immediate application of Theorem 11.8 and Lemma 11.9 is:

Corollary 11.11 *If $X: S_k - \{p_1, \ldots, p_n\} \hookrightarrow \mathbf{R}^3$ is a complete minimal embedding, then the limit normal must be parallel.*

Definition 11.12 An embedded end of a complete immersed minimal surface in $\mathbf{R}^3$ of finite total curvature is a *flat* (or *planar end*) if $\alpha = 0$ in (11.50), and is a *catenoid end* otherwise.

Remark 11.13 We have proved that X is embedded at an end E if and only if Λ has order 2. Let p be the puncture corresponding to E. From the proof of Theorem 11.8, we know that E is flat if and only if p is a branch point of the Gauss map g.

Finally, we give a description of the image of a flat end at the limit height.

Proposition 11.14 *Let $E = X(D - \{p\})$ be an embedded flat end and g have branch order $k > 0$. Let β be as in Theorem 11.8, and B be a large ball centre at $(0, 0, \beta)$. Then $(E - B) \cap \{(x, y, z) \in \mathbf{R}^3 \mid z = \beta\}$ has $2k$ components.*

Proof. Without loss of generality we may assume that $g(p) = 0$ and $g(z) = z^{k+1}$. Now $\eta = z^{-2}h(z)dz$, $h(0) \neq 0$, so

$$X_3(z) = \beta + \Re\left(\frac{1}{k}h(0)z^k\right) + o(|z|^k).$$

Thus $X_3^{-1}(\beta) \cap (D - \{p\})$ consists of k curves intersecting at $z = 0$. This is equivalent to $(E - B) \cap \{(x, y, z) \mid z = \beta\}$ consisting of $2k$ components. $\square$

12 Complete Minimal Surfaces of Finite Total Curvature

To have a better understanding of a complete immersed minimal surface of finite total curvature, we will prove a theorem due to Jorge and Meeks which says that if one looks at the surface from infinity, then the surface looks like a finite number of planes passing through the origin.

Let $X : M \cong S_k - \{p_1, \cdots, p_n\} \hookrightarrow \mathbf{R}^3$ be an immersed complete surface. Let $S^2(r)$ be the sphere centred at $(0,0,0)$ with radius r. Let $Y_r = X(M) \cap S^2(r)$ and

$$W_r = \frac{1}{r} Y_r \subset S^2.$$

Theorem 12.1 ([38]) *Suppose that the Gauss map on M extends continuously to S_k. Then*

1. *$X : M \cong S_k - \{p_1, \cdots, p_n\} \hookrightarrow \mathbf{R}^3$ is proper.*
2. *For large r, $W_r = \{\gamma_1^r, \cdots, \gamma_n^r\}$ consists of n immersed closed curves on S^2.*
3. *γ_i^r converges in the C^1 sense to a geodesic of S^2 with multiplicity $I_i \geq 1$ as r goes to infinity.*
4. *If X is a minimal surface then the convergence in 3 is C^∞.*
5. *X is embedded at an end corresponding to p_i if and only if $I_i = 1$.*

Proof. We need only consider a neighbourhood of a puncture p. Let $D^* = D - \{p\}$ be a punctured disk and ∂D be compact. Suppose that

$$N = \lim_{|z| \to 0} N(z),$$

and that

$$N \bullet N(z) = \cos\theta \geq \frac{\sqrt{3}}{2} \text{ for } 0 \leq \theta \leq \frac{\pi}{6} \tag{12.52}$$

for all $z \in D^*$. Let π be a plane containing the line generated by N and let $\Gamma = X^{-1}(\pi)$. Since $N \bullet N(z) \geq \sqrt{3}/2$, X is transversal to π. It follows that Γ consists of points in ∂D and connected curves (in fact, the interior of $X^{-1}(\pi)$ is a one-dimensional manifold). Let γ be a connected component of Γ that is a curve.

We will consider coordinates (t, y) in π such that the y-axis is the line generated by N. It follows from (12.52) that the tangent vector of $X(\gamma)$ is never collinear with N. Thus $X(\gamma)$ is the graph of a function $y(t)$. The angle between the normal vector $(-y', 1)$ of $X(\gamma)$ and N is less than or equal to θ. Therefore

$$\frac{1}{\sqrt{1 + y'(t)^2}} \geq \cos\theta,$$

which implies that

$$|y'(t)| \leq \tan\theta, \quad \text{for all } t. \tag{12.53}$$

If γ is compact it follows that the extremal points of $X(\gamma)$ are contained in $X(\partial D)$. Let $x_1 = X(\gamma(t_1)) \in X(\partial D)$ and $x_2 = X(\gamma(t_2)) \in X(\partial D)$, and $x = X(\gamma(t))$, $t \in (t_1, t_2)$. Then

$$|x| \leq |x_1| + |x - x_1| \leq |x_1| + \int_{t_1}^{t_2} \sqrt{1 + y'(s)^2}\, ds \leq |x_1| + \frac{2}{\sqrt{3}}|t_2 - t_1| \leq |x_1| + 2|x_2 - x_1|.$$

Thus

$$\sup_{x \in X(\gamma)} |x| \leq d_0 + 2d_1 \tag{12.54}$$

where

$$d_0 = \sup_{x \in X(\partial D)} |x|, \quad d_1 = \text{diameter of } X(\partial D).$$

If γ is non-compact in D^*, then it must be a divergent curve; hence $X(\gamma)$ has infinite arc length (X is complete). It follows from (12.53) that $y(t)$ is defined in the interval $(-\infty, a]$, $[a, \infty)$ or $(-\infty, \infty)$.

Let C_r be the solid cylinder of radius r whose axis is the line generated by N. Let $\tilde{A}$ be the annulus D^* with the metric induced by X so that $X : \tilde{A} \to \mathbf{R}^3$ is an isometric immersion. Note that $\partial\tilde{A} = \partial D$.

Claim : $X^{-1}(C_r)$ is a compact set of $\tilde{A}$. In particular, the immersion $X : D^* \hookrightarrow \mathbf{R}^3$ is proper.

Proof of the Claim : We will denote by $\tilde{\rho}$ the distance on $\tilde{A}$. Choose $r > 0$ such that $X(\partial D)$ is contanined in C_r. Let $\tilde{x} \in \tilde{A}$ be such that $X(\tilde{x}) = x$. Let π' be the plane passing through x and the line generated by N. Consider a connected curve γ in $X^{-1}(\pi')$ containing $\tilde{x}$. We know that $X(\gamma)$ is the graph of a function $y(t)$ in π' with $x = (t_0, y(t_0))$. Observe that $|t_0| \leq r$. If the domain of $y(t)$ is the interval $(-\infty, a]$ or $[a, \infty)$, then $(a, y(a)) \in X(\partial D)$ and

$$\tilde{\rho}(\tilde{x}, \partial\tilde{A}) \leq \left| \int_a^{t_0} \sqrt{1 + y'(t)^2}\, dt \right| \leq 2r\sec\theta \leq 4r. \tag{12.55}$$

Assume now that t varies from $-\infty$ to ∞. Let π_t be the plane passing through the point $(t, y(t))$ of $X(\gamma)$, orthogonal to π' and parallel to the line generated by N. Let γ_{t_0} be the connected curve in $X^{-1}(\pi_{t_0})$ that contains the point $\tilde{x}$. If γ_{t_0} intersects $\partial\tilde{A}$, then (12.55) holds. We assert that there exists $t \in (-r, r)$ such that $X^{-1}(\pi_t)$ contains some curve γ_t intersecting both γ and $\partial\tilde{A}$. If not, then $X(\gamma_{t_0})$ is a graph in π_{t_0} over the t-axis of π_{t_0}. As t_0 varies along the t-axis of π', $X(\gamma_{t_0})$ describes some surface M_0 that is a graph over the plane orthogonal to the vector N. Then $X^{-1}(M_0)$ contains some connected component of $\tilde{A}$ without boundary which contradicts the fact that $\tilde{A}$ is connected and has boundary. Thus for some t, γ_t intersects $\partial\tilde{A}$; if $|t| \geq r$ then $\pi_t \cap X(\partial\tilde{A}) = \emptyset$, hence $|t| < r$. This proves the assertion.

If γ_t is given by the assertion above, then in the same way as in (12.55), letting $x' \in \gamma_t$ be such that $X(x') = (t, y(t))$, we have

$$\tilde{\rho}(x', \partial\tilde{A}) \leq 4r.$$

Let t_1 be a point on the t-axis of π' such that $X(\gamma_t) \cap \pi' = (t, y(t))$. It follows easily from the triangle inequality that

$$\tilde{\rho}(\tilde{x}, \partial\tilde{A}) \leq 4r + \left|\int_{t_0}^{t_1} \sqrt{1 + y'(t)^2}\, dt\right| \leq 8r,$$

which proves the claim.

Now let $r_0 = d_0 + 2d_1$ where d_0 and d_1 are defined after (12.54). Then $X(\partial\tilde{A})$ is contained inside the solid cylinder C_{r_0}. By the above claim, the set $X^{-1}(C_{r_0})$ is compact. Set

$$r_1 = \sup_{z \in X^{-1}(C_{r_0})} \{|X(z)|\}$$

and $r_2 > \max\{r_0, r_1\}$ such that

$$\frac{r_0 + r_1}{r_2} + \tan\frac{\pi}{6} < \frac{\sqrt{3}}{2}.$$

Then $X(\partial D)$ is contained inside the sphere $S^2(r_2)$ of radius r_2 and centred at the origin. By the claim and by the fact that $\lim_{|z| \to 0} N(z) = N$, there exists a subannulus $A' \subset D^*$ such that

1. (12.52) holds for $z \in A'$,

2. $X(z)$ is outside C_{r_2} for $z \in A'$.

Let π be the plane containing $X(z)$ and the axis of C_{r_0} for $z \in A'$. Let γ be a connected component of $X^{-1}(\pi)$ containing z. The $X(\gamma)$ is a graph generated by $y(t)$ in π. By the transversality of π and $X(D^*)$ and the fact $X(\partial D) \subset C_{r_0}$, $X(\gamma)$ intersets C_{r_0}. Then y is defined at r_0 or $-r_0$. We may assume that y is defined at r_0. Then

$$|y(r_0)| \leq |(r_0, y(r_0))| \leq r_1.$$

Let $z \in A'$ and $X(z) = (r, y(r))$, $r > r_0$. By (12.53) it follows that

$$|y(r)| \leq |y(r_0)| + \left|\int_{r_0}^{r} y'(t) dt\right| \leq r_0 + r_1 + r\tan\theta.$$

Then, if $X(z) = (r, y(r))$, we have

$$\left|\frac{X(z)}{|X(z)|} \bullet N\right| = \frac{|y(r)|}{\sqrt{r^2 + y^2(r)}} \leq \frac{r_0 + r_1}{r} + \tan\theta < \frac{\sqrt{3}}{2}, \quad z \in A'. \tag{12.56}$$

Set $r_3 > \sup_{z\in(D^*-A')}\{|X(z)|\}$.

We now prove that X and $S^2(r)$ are transverse for $r \geq r_3$. If X and $S^2(r)$ are not transverse, then there exists $z \in X^{-1}(S^2(r))$ such that

$$N(z) = \frac{X(z)}{|X(z)|}.$$

Since $X(D^* - A')$ lies inside $S^2(r)$, we have that $z \in A'$ and (12.52) and (12.56) give a contradiction. Thus X is transverse to $S^2(r)$ for all $r \geq r_3$. We restrict X to A'.

Then by the *claim*, the function $h : A' \to \mathbf{R}$ defined by

$$h(z) = |X(z)|^2$$

is proper. If $z \in A'$ is a critical point of h, then $Dh(z) = (0,0)$, which means that $X_x(z)$ and $X_y(z)$ are perpendicular to $X(z)$, and so $N(z) = X(z)/|X(z)|$, contradicting to X and $S^2(r)$ are transverse. This contradiction proves that h does not have critical points. If $r > r_3$, then $h^{-1}(r^2)$ is a compact curve that does not intersect $\partial A'$. Hence $h^{-1}(r^2)$ is a finite collection of Jordan curves. If $h^{-1}(r^2)$ has more than one Jordan curve, then there is a compact domain $\Omega \subset A'$ such that $\partial\Omega$ is the union of Jordan curves of $h^{-1}(r^2)$. Then h has a maximum or minimum, hence a critical point, in the interior of Ω, which has already been proved impossible. This shows that $h^{-1}(r^2)$ is a single Jordan curve. Hence

$$\Gamma^r := X(D^*) \cap S^2(r)$$

is an immersion of S^1 and this proves item 2 in this theorem.

We observe that θ of (12.56) goes to zero as r goes to infinity. In fact θ depends on r_0, but we can let $r_0 \to \infty$ and set $r > r_0^2$. By (12.56) the curve $\gamma^r = 1/r\Gamma^r$ is contained in a strip of S^2 that converges to a great circle S as r goes to infinity. Also, by (12.52), the angle between the tangent vector of Γ^r and N goes to $\pi/2$ as r goes to infinity. Hence, Γ^r makes at least one loop around the direction N and γ^r converges C^0 to S as r goes to infinity.

Let $\alpha(\phi)$, $\phi \in \mathbf{R}$, be a parametrisation by arc length of the great circle S. Let β_r be a parametrisation of γ^r such that $\beta_r(\phi)$ lies in the great circle of S^2 that contains N and $\alpha(\phi)$. We have that

$$\left(\frac{\beta_r'}{|\beta_r'|}\bullet\alpha'\right)^2 = 1 - \left(\frac{\beta_r'}{|\beta_r'|}\bullet\alpha\right)^2 - \left(\frac{\beta_r'}{|\beta_r'|}\bullet N\right)^2.$$

As β_r' is orthogonal to $N(\beta_r)$, we have that $\beta_r'/|\beta_r'|\bullet N$ goes to zero as r goes to infinity. Since γ^r converges in the C^0 sense to α, it follows that

$$\lim_{r\to\infty} \beta_r'/|\beta_r'|\bullet\alpha = \lim_{r\to\infty} \beta_r'/|\beta_r'|\bullet \lim_{r\to\infty} \beta_r/|\beta_r| = 0.$$

Therefore γ^r converges in the C^1 sense to the great circle S, with multiplicity, and item 3 is proved.

We now prove that if X is minimal, then γ^r converges in the C^∞ sense to S. Let π be the plane orthogonal to N and containing the origin. Let Ω be the annulus $\{p \in \pi \mid 1/2 \leq |p| \leq 2\}$. Set

$$M_r := (1/rX(D^*)) \cap (\Omega \times \mathbf{R}).$$

The orthogonal projection of M_r onto Ω is a covering of Ω and locally we may write M_r as a graph of a function f_r defined over an angular sector of Ω. It follows from the C^0 convergence of M_r and convergence properties for minimal surfaces (see, e.g. Corollary (16.7) in [21]) that all derivatives of f_r of order less then $j+1$, j an integer, are uniformly bounded by a constant K_{j+1}. Since f_r converges in the C^0 sense to $f=0$ and the inclusion map of the space of C^{j+1} functions into the space of C^j functions is absolutely continuous, it follows that f_r converges in the C^j sense to $f=0$. In particular, the intersection of M_r with S^2 converges in the C^j sense to S with multiplicity for all j. This completes the proof of the theorem. $\square$

Now let $X : M = S_k - \{p_1, \cdots, p_n\} \hookrightarrow \mathbf{R}^3$ be a complete minimal surface of finite total curvature. Let $E_i = X : D_i - \{p_i\}$ be the end corresponding to p_i. Let $W_i^r = 1/rX(D_i - \{p_i\}) \cap S^2(r)$ and $\Gamma_i^r = X^{-1}(rW_i^r)$. Jorge and Meeks' theorem says that Γ_i^r is a Jordan curve in $D_i - \{p_i\}$ for r large enough and W_i^r converges to a great circle of S^2 with multiplicity I_i. We will define the multiplicity of E_i to be I_i. Clearly $I_i = 1$ if and only if E_i is embedded. An application of Jorge-Meeks' theorem is that we can get a total curvature formula via the genus k, the number of punctures n, and the multiplicities I_i.

13 Total Curvature of Branched Complete Minimal Surfaces

Let $X : M \hookrightarrow \mathbf{R}^3$ be a complete minimal surface with finite total curvature. Osserman's theorem says that conformally $M = S_k - \{p_1, \cdots, p_n\}$, $n \geq 1$, where S_k is a closed Riemann surface of genus k. Each p_i corresponds to an end E_i of M. Using Theorem 12.1, we can prove:

Theorem 13.1 *The total curvature of X is*

$$K(M) = 2\pi \left(\chi(M) - \sum_{i=1}^{n} I_i \right), \tag{13.57}$$

where $\chi(M) = 2(1-k) - n$ is the Euler characteristic of M and I_i is the multiplicity of E_i.

Proof. Let $\Gamma_i^r = X^{-1}(rW_i^r)$ be as in the proof of Theorem 12.1. Let $p_i \in D_i^r$ be the disk in S_k such that $\partial D_i^r = \Gamma_i^r$. When r is large enough the D_i^r's are disjoint from each other. Then $M_r := S_k - \bigcup_{i=1}^n \partial D_i^r$ is a Riemann surface with boundary $\bigcup_{i=1}^n \partial D_i^r$ and $\chi(M_r) = \chi(M)$. Now by the Gauss-Bonnet formula we have

$$\int_{M_r} K dA + \sum_{i=1}^{n} \int_{\Gamma_i^r} \kappa_g \, ds = 2\pi\chi(M_r) = 2\pi\chi(M),$$

where κ_g is the geodesic curvature. Since $W_i^r = \frac{1}{r} X(\Gamma_i^r)$ converges in the C^∞ sense to a great circle on S^2 with multiplicity I_i and X is an isometric immersion, we have

$$\lim_{r\to\infty} \int_{\Gamma_i^r} \kappa_g \, ds = 2\pi I_i.$$

Taking limit we have

$$K(M) = \int_M K dA = 2\pi \left(\chi(M) - \sum_{i=1}^{n} I_i \right). \tag{13.58}$$

□

In the remainder of this section, our surfaces will be branched minimal surfaces. Note that the concepts of completeness, properness, etc., can be easily generalised to branched minimal surfaces.

The Enneper-Weierstrass representation of a branched complete minimal surface of finite total curvature $X : M \to \mathbf{R}^3$ is given by

$$X(p) = \Re \int_{p_0}^{p} \left(\frac{1}{2}(1-g^2),\ \frac{i}{2}(1+g^2),\ g \right) \eta + C, \tag{13.59}$$

where $g : M = S_k - \{p_1, \cdots, p_n\} \to \mathbf{C} \cup \{\infty\}$ is a meromorphic function, η is a holomorphic 1-form on M and C is a constant vector. Both g and η can be extended to S_k as a meromorphic function and 1-form respectively. Note that we have proved this for regular minimal surfaces. But since the proof only involves the neighbourhoods of the punctures p_i, it works for branched minimal surfaces as well.

Locally, $\eta = f(z)dz$ where $z = x + iy$. The metric induced by X is given by

$$ds^2 = \Lambda^2(dx^2 + dy^2), \tag{13.60}$$

where

$$\Lambda = \frac{1}{2}|f|(1 + |g|^2). \tag{13.61}$$

From (13.61) it is clear that $q \in M$ is a branch point only if η vanishes at q. Hence all branch points are isolated and if η is a meromorphic 1-form on S_k, there is only a finite number of branch points.

Therefore, given g and η as above, we can define a metric h with isolated degenerate points on $M = S_k - \{p_1, \cdots, p_n\}$ by $h_{ij} = \Lambda^2\delta_{ij}$, where Λ is defined as in (13.61). We can study the intrinsic geometry of the branched complete Riemannian manifold (M, h) even though the mapping X in (13.59) may not be well defined. When X is well defined, it is a branched complete minimal surface.

Let U_i be a disk coordinate neighbourhood of p_i such that $z(p_i) = 0$. Let J_i be the order of Λ at p_i, i.e., J_i is an integer such that in U_i,

$$\lim_{z \to 0} |z|^{J_i}\Lambda(z) = C_i > 0,$$

for $1 \le i \le n$. Since (M, h) is complete, $J_i \ge 1$.

Suppose q_i, $1 \le i \le m$, are branch points of M. Let V_i be a disk coordinate neighbourhood of q_i such that $z(q_i) = 0$. Let K_i be the branch order of Λ, i.e.,

$$\lim_{z \to 0} |z|^{-K_i}\Lambda(z) = C_i > 0, \quad \text{in} \quad V_i.$$

There is a generalised version of (13.57) in [16] which allows X to have branch points.

Theorem 13.2 *The total curvature of (M, h) is given by*

$$\int_M K dA = 2\pi\left(\chi(M) - \sum_{i=1}^{n}(J_i - 1) + \sum_{i=1}^{m} K_i\right). \tag{13.62}$$

Proof. Let $R > 0$ be such that $D_R^i := \{|z| < R\} \subset U_i$, $1 \le i \le n$ and $D_R^i := \{|z| < R\} \subset V_{i-n}$, $n + 1 \le i \le n + m$. When R is small enough, $D_R^i \cap D_R^j = \emptyset$ for $i \ne j$.

Let $M_R = M - \bigcup_{i=1}^{n+m} D_R^i$. By the Gauss-Bonnet formula, we have

$$\int_{M_R} K dA + \sum_{i=1}^{n+m} \int_{\partial D_R^i} \kappa_g \, ds = 2\pi\chi(M_R) = 2\pi(\chi(M) - m). \tag{13.63}$$

If $g(p_i) \neq \infty$, then $\eta = z^{-J_i} f_i(z)dz$ where f_i is a holomorphic function in U_i and $f_i(0) \neq 0$. Write $z = re^{it}$. By Minding's formula, see [12], Volume I, pages 33-34, the geodesic curvature on ∂D_R^i is given by

$$\kappa_g \Lambda = -\frac{1}{R} + \frac{\partial \log \Lambda}{\partial \nu},$$

where ν is the inward unit normal (in the Euclidean metric on D_R^i) of ∂D_R^i. Now $\Lambda = \frac{1}{2}|z|^{-J_i}|f_i|(1+|g|^2)$, so

$$\frac{\partial \log \Lambda}{\partial \nu} = -\frac{\partial \log \Lambda}{\partial r} = \frac{J_i}{r} - \frac{\partial \log |f_i|}{\partial r} - \frac{\partial \log(1+|g|^2)}{\partial r},$$

and

$$\int_{\partial D_R^i} \kappa_g \, ds = \int_0^{2\pi} \kappa_g \Lambda R \, dt = \int_0^{2\pi} \left(\frac{J_i - 1}{R} - \frac{\partial \log |f_i|}{\partial r} - \frac{\partial \log(1+|g|^2)}{\partial r} \right) R \, dt.$$

Since

$$\int_0^{2\pi} \frac{\partial \log |f_i|}{\partial r} R \, dt = \int_{D_R^i} \triangle(\log |f_i|) dx \, dy = 0,$$

and $\partial \log(1+|g|^2)/\partial r$ is bounded, we have

$$\lim_{R \to 0} \int_{\partial D_R^i} \kappa_g ds = 2\pi(J_i - 1).$$

If $g(p_i) = \infty$ then $g = z^{-m_i} g_i(z)$, $m_i > 0$, and $\eta = z^{-J_i + 2m_i} f_i(z)dz$, where f_i and g_i are holomorphic functions in U_i and $f_i(0) \neq 0$, $g_i(0) \neq 0$. Then

$$\frac{\partial \log \Lambda}{\partial \nu} = -\frac{\partial \log \Lambda}{\partial r} = \frac{J_i - 2m_i}{r} - \frac{\partial \log |f_i|}{\partial r} - \frac{\partial \log(1+|g|^2)}{\partial r}.$$

Since

$$\frac{\partial \log(1+|g|^2)}{\partial r} = \frac{1}{1 + r^{-2m_i}|g_i|^2} \left(-2m_i r^{-2m_i - 1}|g_i|^2 + r^{-2m_i} \frac{\partial |g_i|^2}{\partial r} \right),$$

we have

$$\begin{aligned} -\int_0^{2\pi} \frac{\partial \log(1+|g|^2)}{\partial r} R \, dt &= \int_0^{2\pi} \frac{2m_i R^{-2m_i}|g_i|^2}{1 + R^{-2m_i}|g_i|^2} dt - \int_0^{2\pi} \frac{R^{-2m_i} \frac{\partial |g_i|^2}{\partial r}}{1 + R^{-2m_i}|g_i|^2} R \, dt \\ &\to 4m_i \pi \quad \text{as} \quad R \to 0. \end{aligned}$$

We have the same limit

$$\lim_{R \to 0} \int_{\partial D_R^i} \kappa_g ds = 2\pi(J_i - 1).$$

Similarly, for the branch ponts q_i, if $g(q_i) \neq \infty$, then $\eta = z^{K_i} f_i(z) dz$ where f_i is a holomorphic function defined in V_i and $f_i(0) \neq 0$. Similar calculation gives

$$\int_{\partial D_R^{i+n}} \kappa_g \, ds = -\int_0^{2\pi} \left(\frac{K_i + 1}{R} + \frac{\partial \log |f_i|}{\partial r} + \frac{\partial \log(1 + |g|^2)}{\partial r} \right) R \, dt.$$

Hence

$$\lim_{R \to 0} \int_{\partial D_R^{i+n}} \kappa_g ds = -2\pi(K_i + 1).$$

If $g(q_i) = \infty$, then $g(z) = z^{-m_i} g_i(z)$ and $\eta = z^{K_i + 2m_i} f_i(z)$, similar calculation still gives us the same limit.

Note that

$$\lim_{R \to 0} \int_{M_R} K dA = \int_M K dA.$$

Letting $R \to 0$ in (13.63), we get (13.62). The proof is complete. □

Remark 13.3 Suppose $X : S_k - \{p_1, \cdots, p_n\} \hookrightarrow \mathbf{R}^3$ is a regular complete minimal surface, then $h_{ij} = \Lambda^2 \delta_{ij}$ is the pull back metric of X. Comparing the proofs of Theorem 13.1 and Theorem 13.2, we see that $J_i - 1 = I_i$, thus (13.62) is a generalization of (13.57).

The calculation in the proof of Theorem 13.2 also works for boundary branch points. Let M be a compact domain of a Riemann surface with a C^2 boundary $\Gamma = \partial M$. Suppose that g and η are given meromorphic function and 1-form respectively, and h is the Riemannian metric with isolated degenerate points defined by (13.60) and (13.61). Let $q_i \in M$ $(1 \leq i \leq m)$ be the interior branch points with branch order K_i and $s_i \in M$ $(1 \leq i \leq n)$ be the boundary branch points with branch order L_i. Then:

Theorem 13.4 *The total curvature of (M, h) is given by*

$$\int_M K dA = 2\pi \left(\chi(M) + \sum_{i=1}^{m} K_i \right) + \pi \sum_{i=1}^{n} L_i - \int_\Gamma \kappa_g \, ds. \tag{13.64}$$

A sketch of the proof of (13.64) is as follows:

Define D_R^i as before and $M_R = M - \bigcup_{i=1}^{n+m} D_R^i$. By the Gauss-Bonnet formula,

$$\int_{M_R} K dA + \int_{\partial M_R} \kappa \, ds + \sum_{i=1}^{n} (\alpha_R^i + \beta_R^i) = 2\pi(\chi(M) - m),$$

where α_R^i and β_R^i are the exterior angles near the boundary branch points and

$$\lim_{R \to 0} \alpha_R^i = \frac{\pi}{2}, \qquad \lim_{R \to 0} \beta_R^i = \frac{\pi}{2}.$$

Then (13.64) follows by

$$\lim_{R \to 0} \int_{\partial D_R^i \cap \partial M_R} \kappa \, ds = \lim_{R \to 0} \int_{\epsilon_R^i}^{\delta_R^i} \left(\frac{-1}{R} - \frac{\partial \log \Lambda}{\partial r} \right) R \, dt = \lim_{R \to 0} (\epsilon_R^i - \delta_R^i)(1 + L_i) = -\pi(1 + L_i),$$

for the boundary branch points.

Remark 13.5 If X in (13.59) is well defined then X is a minimal surface and h is induced by X. In this case, (13.64) is the same as the formula in [12], Volume II, page 128.

Since if $X : S_k - \{p_1, \cdots, p_n\} \hookrightarrow \mathbf{R}^3$ is a complete minimal immersion, then $J_i \geq 2$ and $J_i = 2$ if and only if the end E_i is embedded, we get a corollary.

Corollary 13.6 *The total curvature of a regular complete minimal surface of genus k with n ends satisfies*

$$K(M) \leq 4\pi(1 - k - n) = 2\pi(\chi(M) - n). \tag{13.65}$$

Moreover,

$$K(M) = 2\pi(\chi(M) - n)$$

if and only if each end of M is embedded.

The inequality (13.65) is a result of Osserman.

14 Examples of Complete Minimal Surfaces

We have discussed complete minimal surfaces without a single example. This section is designed to give some examples and their Enneper-Weierstrass representations. But first let us sum up what we have known, in order to simplify the following discussion.

Suppose now that $X : S_k - \{p_1, \cdots, p_r\} \hookrightarrow \mathbf{R}^3$ is a complete minimal immersion of finite total curvature. Then by the Enneper-Weierstrass representation, X is given by

$$X = \Re \int \left(\frac{1}{2}(1 - g^2)\eta, \frac{i}{2}(1 + g^2)\eta, g\eta \right) + C.$$

If we want X to be embedded, then g and η should satisfy certain conditions. Here is an important necessary condition.

Theorem 14.1 *Let S be a complete minimal immersion in $\mathbf{R}^3$ of finite total curvature, defined by $X : M \to \mathbf{R}^3$, where $M \cong S_k - \{p_1, \cdots, p_r\}$. Let g and the holomorphic 1-form η be the Enneper-Weierstrass data for X. Then $\eta \neq 0$ at a point $p \in M$ unless g has a pole at p, and if g has a pole at $p \in M$ of order m, then η has a zero of order $2m$ at p.*

Suppose that E_i is an embedded end corresponding to p_i. If g has a pole of order $k \geq 1$ at p_i, then η has a zero of order $2k - 2$ at p_i. If g takes on a finite value at p_i, then η has a pole of order 2 at p_i. Furthermore, p_i is a branch point of g if and only if E_i is a flat end.

Proof. From $\Lambda^2 = \frac{1}{4}|f|^2(1 + |g|^2)^2$, we see that to make $0 < \Lambda < \infty$ on M, the zeros and poles of g and η must be as stated in the theorem.

We have already seen that an end is embedded if and only if Λ has order 2, thus g and η has to satisfy the conditions stated in this theorem.

The last statement is Remark 11.13. □

Now let us see some examples. The first one, the *catenoid*, is quite a classical one, it was discovered in 1741 by Euler, see [61], page 5. By the way, one can find a very interesting history of minimal surfaces in [61].

Example 14.2 (Catenoid) *Let $M = \mathbf{C} - \{0\}$, $g(w) = w$, $\eta = \frac{dw}{w^2}$. The Enneper-Weierstrass representation of the catenoid is given by the three 1-forms:*

$$\omega_1 = \frac{1}{2}\frac{1}{w^2}(1 - w^2)dw, \quad \omega_2 = \frac{i}{2}\frac{1}{w^2}(1 + w^2)dw, \quad \omega_3 = \frac{1}{w}dw.$$

The total curvature of the catenoid is -4π since $g(w) = w$ has degree 1. It has genus zero and two ends. At 0 and ∞ we see that ϕ_1 and ϕ_2 have poles of order 2 and ϕ_3 has a pole of order 1, so the two ends are embedded and they are catenoid ends since $g(w) = w$ has no branch points. We can prove that the catenoid is embedded and is a rotation surface.

In fact, let $X(w) = (x, y, z)(w)$ and $X(1) = (-1, 0, 0)$. We have

$$x(w) = -1 + \frac{1}{2}\Re \int_1^w \left(\frac{1}{\zeta^2} - 1\right) d\zeta = \frac{1}{2}\Re\left(-\frac{1}{w} - w\right),$$

$$y(w) = -\frac{1}{2}\Im \int_1^z \left(\frac{1}{\zeta^2} + 1\right) d\zeta = -\frac{1}{2}\Im\left(-\frac{1}{w} + w\right),$$

so

$$x(w) - i\, y(w) = -\frac{1}{2}\left(\frac{1}{w} + \overline{w}\right).$$

It is obvious that $z(w) = \log|w|$ only depends on $|w|$. Note that when w is real, $y(w) = 0$. Moreover,

$$|x - i\, y|^2(w) = \frac{1}{4}\left(\frac{1}{w} + \overline{w}\right)\left(\frac{1}{\overline{w}} + w\right) = \frac{1}{4}\left(|w|^2 + \frac{1}{|w|^2} + 2\right) \geq 1,$$

and so X sends $|w| =$ constant to a circle centred at $(0, 0, z(w))$. Hence the surface is contained in a rotation surface with height function z. Since $|w| = \exp(z(w))$, we have

$$\begin{aligned} |x - i\, y|^2(w) &= \frac{1}{4}(\exp(2z(w)) + \exp(-2z(w)) + 2) \\ &= \left(\frac{\exp(z(w)) + \exp(-z(w))}{2}\right)^2 = \cosh^2(z(w)). \end{aligned}$$

Since the two ends are embedded, when $|w|$ is sufficiently large or small, $X(\{|w| = \text{constant}\})$ must be the circle centred at $(0, 0, \log|w|)$ with radius $\cosh(\log|w|)$. Hence the catenoid coincides with the rotation surface defined by

$$x^2 + y^2 = \cosh^2(z).$$

A little calculation shows that the rotation surface is minimal, and so by the extension theorem, the catenoid must be the same rotation surface. In particular, the catenoid is embedded.

Exercise : 1. Prove that all rotation minimal surfaces are generated by functions as follows:

$$x(z) = a\cosh\left(\frac{z - z_0}{a}\right),$$

where $a > 0$ and z_0 are constants. Such a curve is called a *catenary*, the name "catenoid" comes from it. Of course we assume that the axis of rotation is parallel to the z-axis. All these rotation surfaces are homothetic to each other.

2. Use the formulas (7.34) to study the asymptotic and curvature lines of the catenoid.

Example 14.3 (Helicoid) *If we consider the conjugate surface of the catenoid, the forms will be*

$$\omega_1 = \frac{i}{2}\frac{1}{w^2}(1-w^2)dw, \quad \omega_2 = -\frac{1}{2}\frac{1}{w^2}(1+w^2)dw, \quad \omega_3 = i\frac{1}{w}dw.$$

Integrating them and taking real parts, we have a surface given by

$$(x,y,z)(w) = \left(-\Im\left(\frac{1}{w}-w\right), -\Re\left(\frac{1}{w}-w\right), -\arg w\right).$$

The third coordinate is not well defined on $\mathbf{C}-\{0\}$. If we pass to the universal covering $\exp : \mathbf{C} \to \mathbf{C}-\{0\}$, then we will get a well defined minimal surface on $\mathbf{C}$, called the *helicoid.* Moreover, being conjugate to the catenoid, the helicoid is locally isometric to the catenoid.

Let us derive the Enneper-Weierstrass representation of the helicoid. Let $w = e^\zeta$ for $\zeta \in \mathbf{C}-\{0\}$, then $dw = e^\zeta d\zeta$. Hence we have

$$g(\zeta) = e^\zeta, \quad \eta(\zeta) = e^{-\zeta}d\zeta,$$

and

$$\omega_1 = \frac{i}{2}(1-e^{2\zeta})e^{-\zeta}d\zeta = \frac{i}{2}(e^{-\zeta}-e^{\zeta})d\zeta = -i\sinh(\zeta)d\zeta,$$

$$\omega_2 = -\frac{1}{2}(1+e^{2\zeta})e^{-\zeta}d\zeta = -\cosh(\zeta)d\zeta,$$

$$\omega_3 = i\,d\zeta.$$

The helicoid is a ruled surface and is embedded. In fact, let $\zeta = u+iv$, then

$$\begin{aligned}(x,y,z)(\zeta) &= (\Im\cosh(\zeta), -\Re\sinh(\zeta), -\Im\zeta) \\ &= \left(\Im\frac{1}{2}\left(e^u e^{iv}+e^{-u}e^{-iv}\right), -\Re\frac{1}{2}\left(e^u e^{iv}-e^{-u}e^{-iv}\right), -v\right) \\ &= \left(\frac{1}{2}\sin v(e^u-e^{-u}), -\frac{1}{2}\cos v(e^u-e^{-u}), -v\right) \\ &= (\sin v\sinh(u), -\cos v\sinh(u), -v).\end{aligned}$$

Thus for fixed v, X maps the straight line $\zeta = u+i\,v$ one to one and onto the straight line generated by $(\sin v, -\cos v, 0)$ on the plane $z=-v$. Since $\mathbf{C}$ consists of all these straight lines, the helicoid is embedded and is a ruled surface.

If we change coordinates in $\mathbf{C} = \mathbf{R}^2$ by $(t,s) = (\sinh u, v)$, then we see that the helicoid is given by

$$X(t,s) = (t\sin s, -t\cos s, s) = t(\sin s, -\cos s, 0) - (0,0,s) = tY(s) + c(s).$$

When $t=1$, the curve $(\sin s, -\cos s, s)$ is a *helix.*

Since the Gauss map $g(\zeta) = e^{\zeta}$ has an essential singularity at ∞, the helicoid has infinite total curvature.

Exercise : 1. Prove that a ruled minimal surface is a piece of the helicoid in the sense of homothety.

2. Prove that there is only one straight line on the helicoid which is also an asymptotic line, and that line is not in the family discussed above.

Example 14.4 (Enneper's Surface) *Let* $M = \mathbf{C}$, $g(z) = z$, $\eta = dz$, *then*

$$\omega_1 = \frac{1}{2}(1 - z^2)dz, \quad \omega_2 = \frac{i}{2}(1 + z^2)dz, \quad \omega_3 = zdz.$$

Since $\deg g = 1$, Enneper's surface has total curvature -4π. It has genus zero and one end. Since $\chi(M) = 1$ and there is only one end, by (13.62) we have

$$-2 = (\chi(M) + 1 - J_1) = (2 - J_1),$$

so $J_1 = 4$ and thus by Lemma 11.9 Enneper's surface is not embedded.

In terms of total curvature, we have

Theorem 14.5 *The catenoid and Enneper's surface are the only complete minimal surfaces of total curvature* -4π.

Proof. Let $X : M = S_k - \{p_1, \cdots, p_n\}$ be a complete minimal surface of finite total curvature -4π. Let g be the Gauss map. Then $\deg g = 1$ means $g : S_k \to S^2$ is a conformal diffeomorphism, thus $k = 0$. We have $\chi(M) = 2 - n$. By Corollary 13.6

$$-4\pi \leq 2\pi(\chi(M) - n) = 4\pi(1 - n),$$

so we have $n = 1$ or 2.

When $n = 2$, by (13.62), $-4\pi = 2\pi(\chi(M) + 2 - J_1 - J_2) = -2\pi(J_1 + J_2 - 2)$. We know that $J_i = 2$ since $J_i \geq 2$, and hence the two ends are embedded. Since $\deg g = 1$ means that $g' \neq 0$ everywhere on M, we can assume that $g \neq \infty$ on M and take $g(z) = z$ and $M = \mathbf{C} - \{z_0\}$. Then g has a pole of order 1 at ∞. The 1-form dz has a pole of order 2 at ∞ and no zeros in $\mathbf{C}$. The ends being embedded requires that η should have neither pole nor zero at ∞ (g has a pole of order 1 at ∞), and we have that $\eta = h(z)/(z - z_0)^2 dz$, where h is a holomorphic function which is bounded at ∞. Since η should have a pole of order 2 at z_0, we know that h is also bounded near z_0; thus h is a bounded entire holomorphic function and so h is a constant function, $h \equiv c \neq 0$. Let C be a circle centred at z_0. Since X is well defined,

$$0 = 2\Re \int_C \omega_1 + i\, 2\Re \int_C \omega_2 = \overline{\int_C \frac{c}{(z - z_0)^2} dz} + c \int_C \frac{z^2}{(z - z_0)^2} dz = 4cz_0\pi i$$

forces $z_0 = 0$.

Thus we get the Enneper-Weierstrass representation of the catenoid after a homothety, i.e., $g(z) = z$, $\eta = c\,dz/z^2$.

When $n = 1$, we can take $M = \mathbf{C}$ and $g(z) = z$. By $-4\pi = 2\pi(\chi(M) + 1 - J) = 2\pi(2 - J)$ we have $J = 4$. Let $\eta = f dz$. Since $(1 - z^2)dz$ and $(1 + z^2)dz$ have a pole of order 4 at ∞ and $z\,dz$ has a pole of order 3 at ∞, f can have neither pole nor zero at ∞. Being an entire holomorphic function, f must be a constant $c = re^{i\theta} \neq 0$. Thus we achieve the Enneper-Weierstrass representation of an associated Enneper's surface after a homothety. □

Corollary 14.6 *The only embedded complete minimal surface of total curvature -4π is the catenoid.*

Proof. Enneper's surface is not embedded. □

If these notes were written eleven years ago, then the catenoid, the helicoid and the plane would comprise all the known examples of embedded complete minimal surfaces of finite topology. In 1982, Costa [7] gave a pair of Weierstrass data on a torus with three punctures. Examination by the criteria in Theorem 14.1 shows that the surface is complete, has three embedded ends and total curvature -12π. It is a good candidate for an example of a new embedded complete minimal surface with finite total curvature. The trouble is, how to prove that it is embedded. Using computer graphics, David Hoffman observed that the surface has a lot of symmetries and seems is embedded. Together with William Meeks III, he eventually proved that the surface is embedded. In their proof [30] the symmetries play an important role.

Recently, more embedded complete minimal surfaces of finite topology type, with finite or infinite total curvature, have been discovered, see [80], [26], and [27] for example.

There are also examples of embedded, periodic minimal surfaces, both old and new, such as the classical one-parameter family of Riemann's examples. For these examples and their properties, see [50], [51], [39], and [40].

Here we only mention an infinite family with finite total curvatures. Which are the earlest examples after Costa's example. They were found by Hoffman and Meeks. The proof of their embeddedness is not an easy business, so let us only list their Enneper-Weierstrass representations. The reader is recommended to read the paper [31].

Example 14.7 (Hoffman-Meeks' Surfaces) *First we introduce the special genus k Riemann surfaces, k any positive integer, given by*

$$\overline{M_k} := \{(z, w) \in (\mathbf{C} \cup \{\infty\})^2 \,|\, w^{k+1} = z^k(z^2 - 1)\}.$$

Let

$$p_0 = (0, 0), \quad p_{-1} = (-1, 0), \quad p_1 = (1, 0), \quad p_\infty = (\infty, \infty).$$

The surface we will consider is

$$M_k := \overline{M_k} - \{p_{-1}, p_1, p_\infty\}.$$

The Gauss map and the one form η will be

$$g = \frac{c_k}{w}, \quad \eta = \left(\frac{z}{w}\right)^k dz = \frac{w}{z^2 - 1} dz.$$

Here c_k is a positive constant to be determined. The determination of c_k is involved in the procedure of "killing the periods", i.e., to make (6.22) true.

Theorem 14.8 *For $k > 0$, there is a unique $c_k > 0$ such that the above Enneper-Weierstrass data g and η give an embedded, complete minimal surface $X : M_k \hookrightarrow \mathbf{R}^3$ of three ends. It has the following properties:*

1. *The total curvature of M_k is $-4\pi(k+2)$;*
2. *M_k has two catenoid ends and one flat end;*
3. *M_k intersects the x_1x_2-plane in $k+1$ straight lines, which meet at equal angles at the origin;*
4. *Removal of the $k+1$ lines disconnects M_k. What remains is, topologically, the union of two open annuli;*
5. *The intersection of M_k with any plane parallel (but not equal) to the x_1x_2-plane is a single Jordan curve;*
6. *The symmetry group of M_k is the dihedral group with $4(k+1)$ elements generated by*
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \multicolumn{2}{c}{\mathcal{R}_k} & 0 \\ & & 0 \\ 0 & 0 & -1 \end{pmatrix}$$
where $\mathcal{R}_k$ is the matrix of rotation by $\pi/(k+1)$ in the x_1x_2-plane;
7. *M_k may be decomposed into $4(k+1)$ congruent pieces, each a graph;*
8. *M_k is the unique properly embedded minimal surface of genus k with three ends, finite total curvature, and a symmetry group containing $4(k+1)$ or more elements.*

M_1 is the surface discovered by Costa, now is called the Costa-Hoffman-Meeks surface.

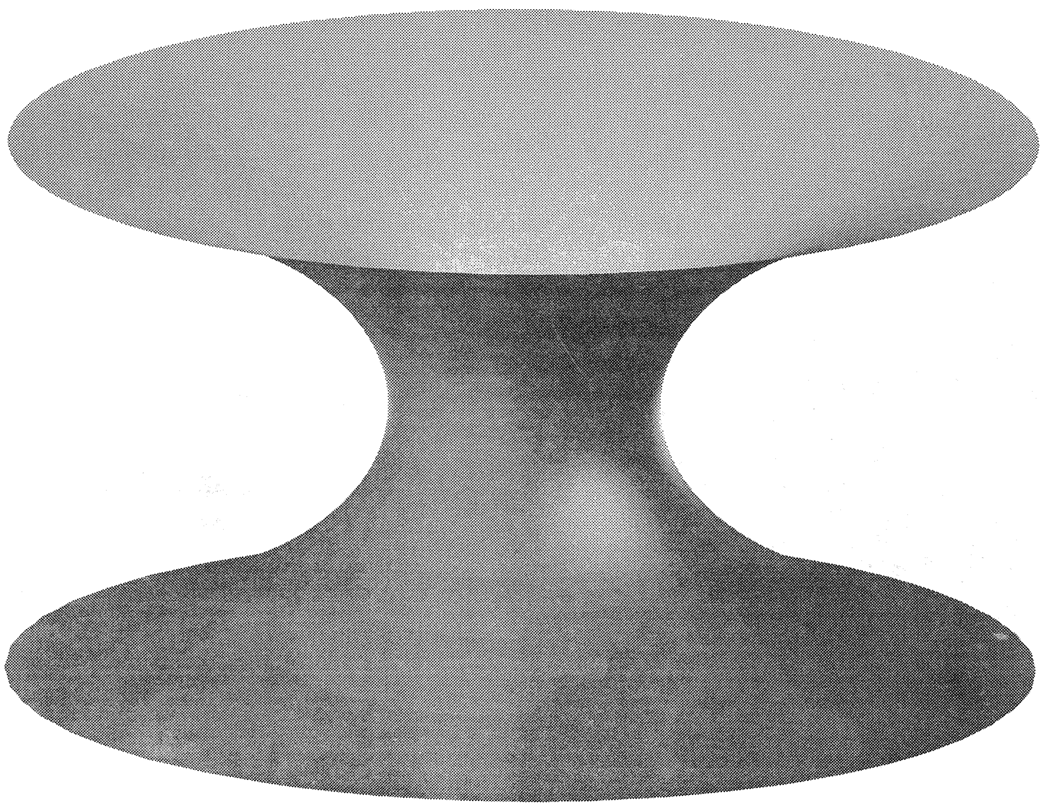

Figure 3
Catenoid, a rotation surface

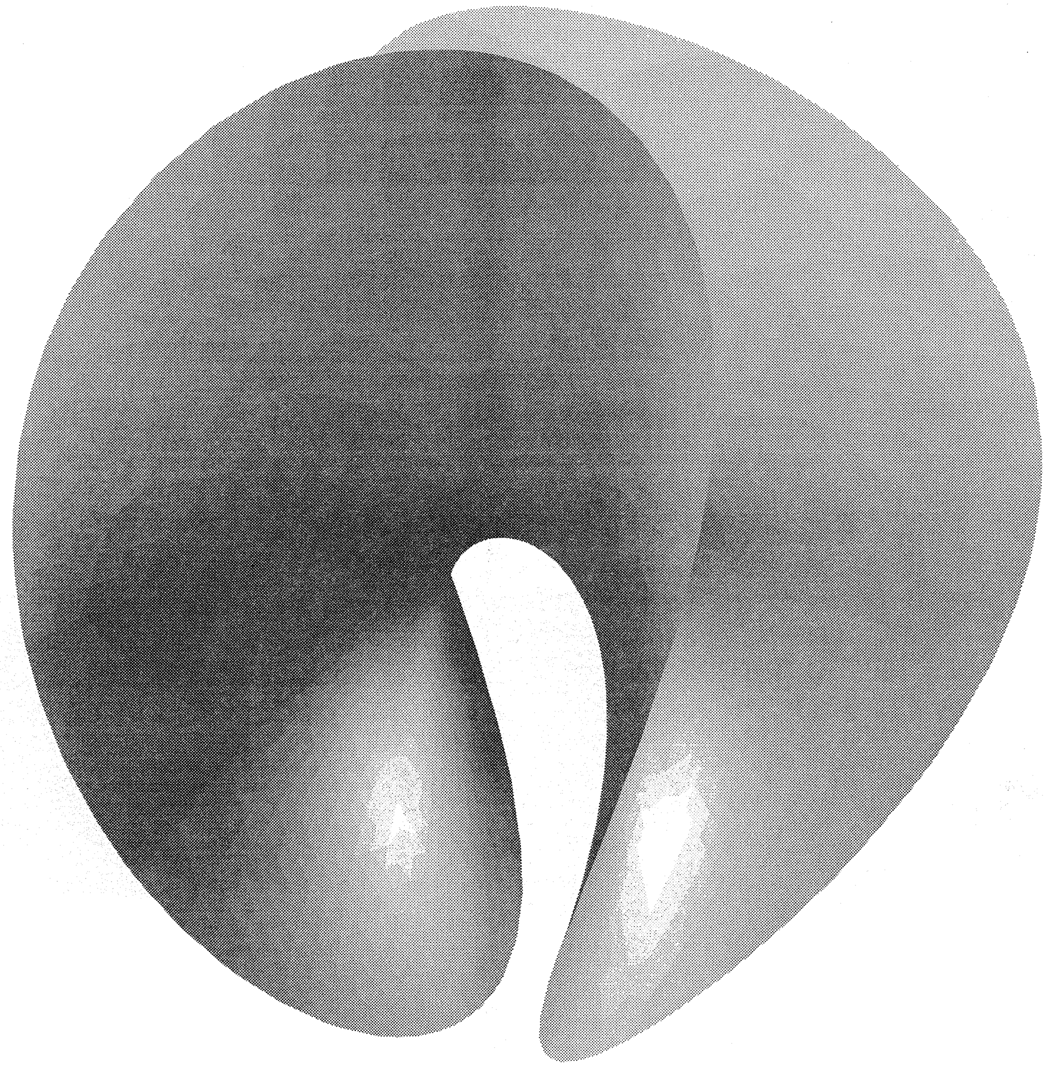

Figure 4
Enneper's Surface

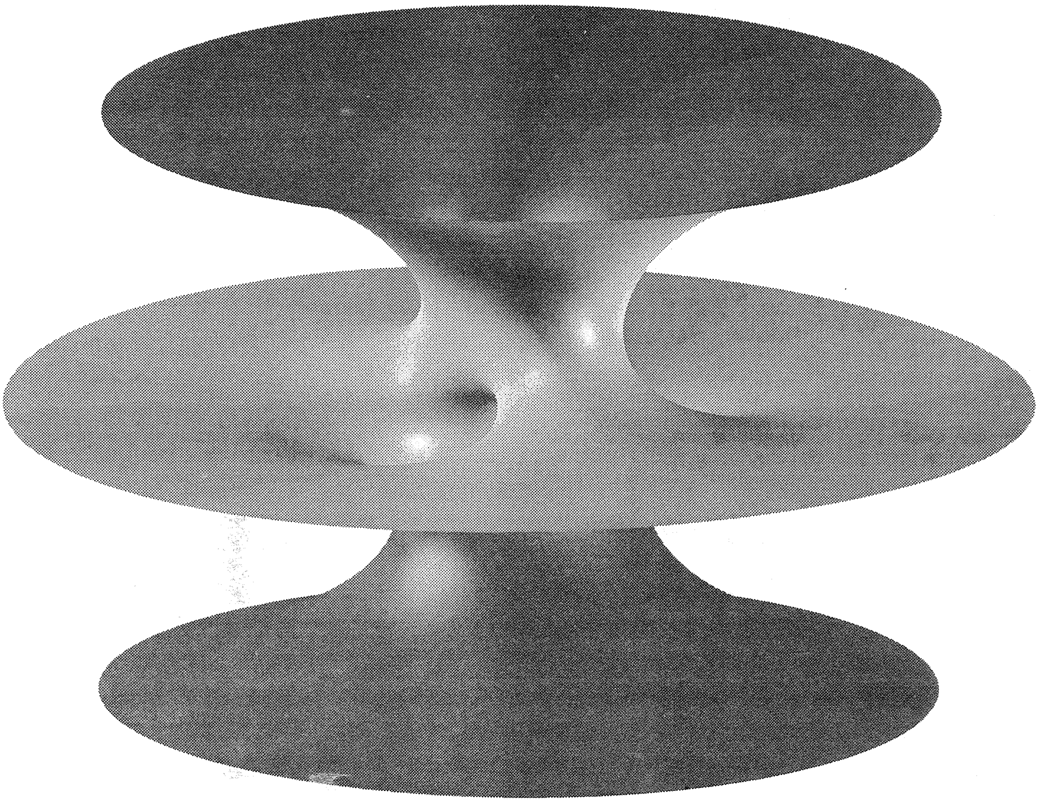

Figure 5
Costa-Hoffman-Meeks Surface

Figure 6
Genus 2 Hoffman-Meeks Surface

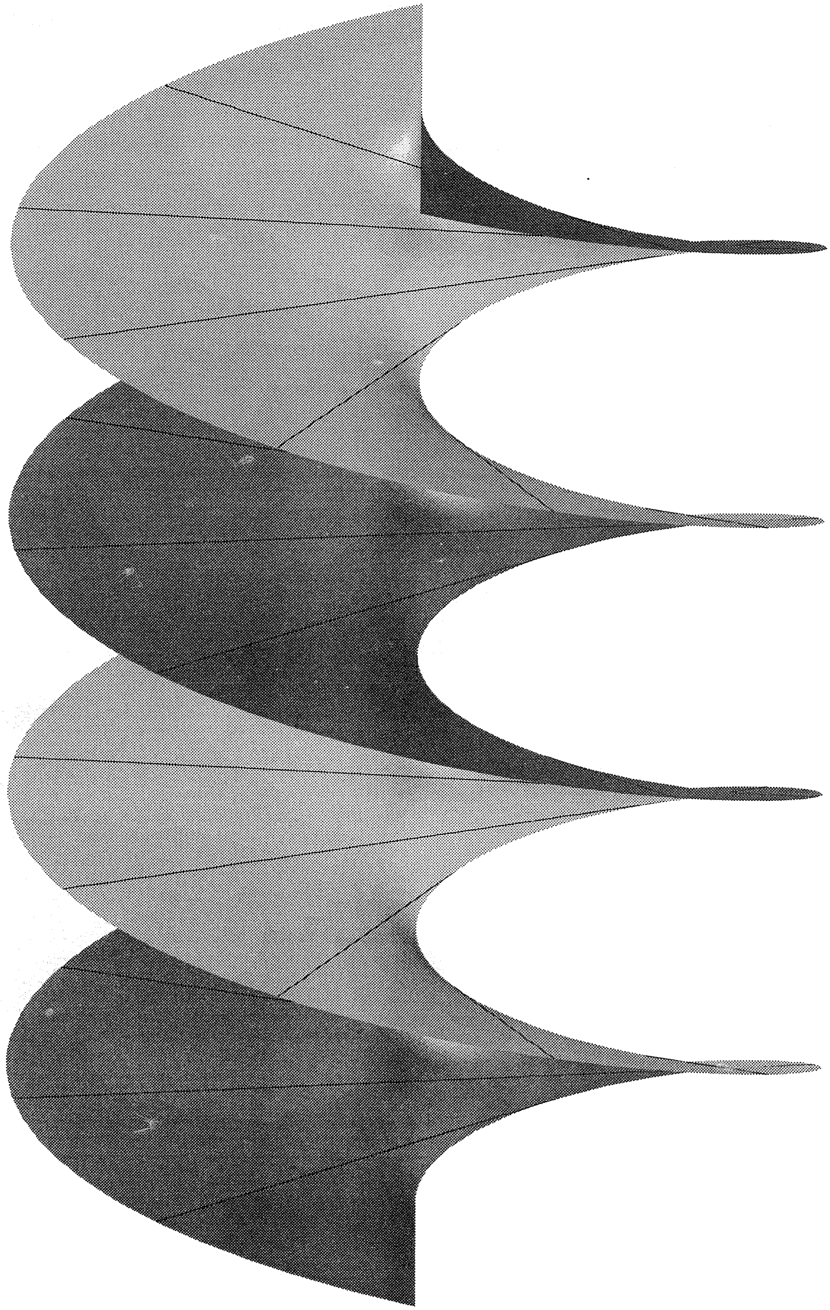

Figure 7
Helicoid, a ruled surface

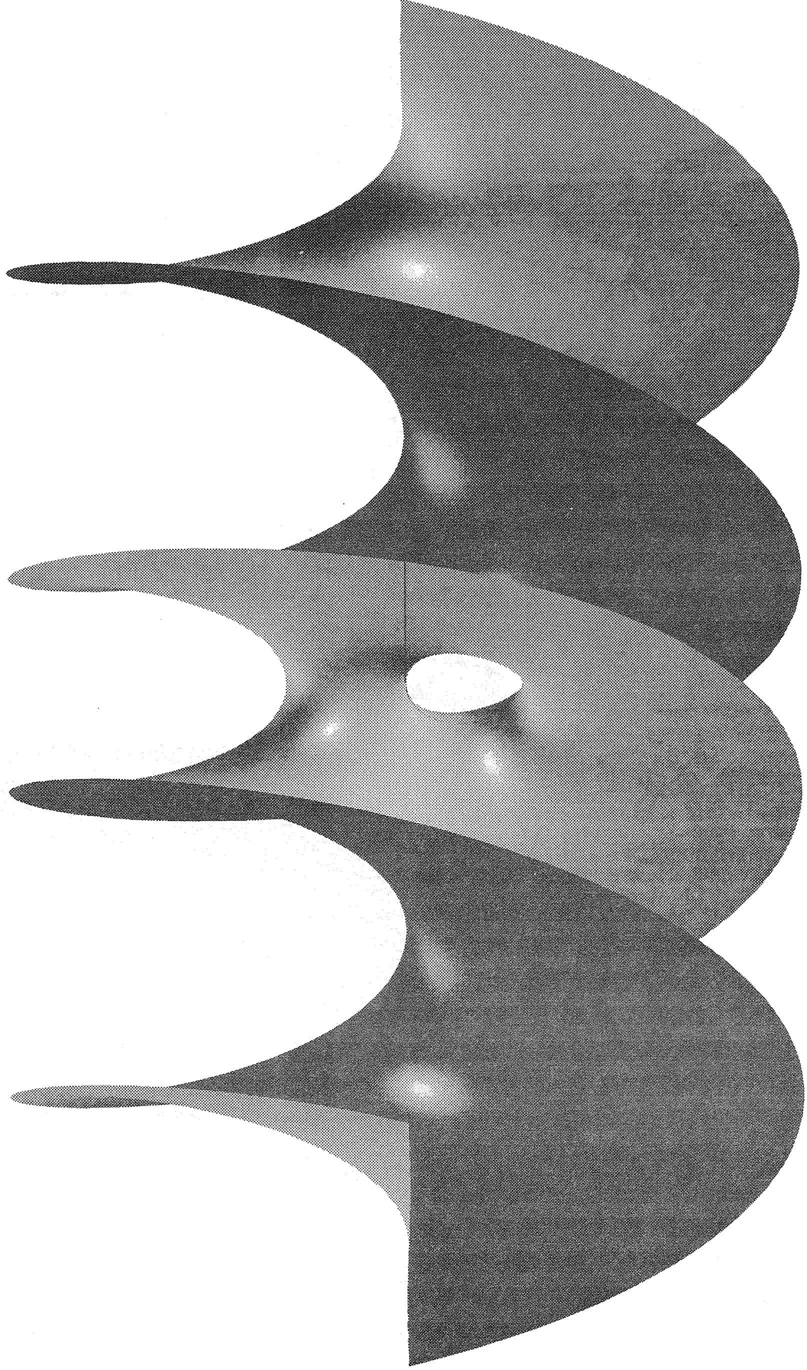

Figure 8
Hoffman-Karcher-Wei's Genus 1 Helicoid, or Helicoid with a hole

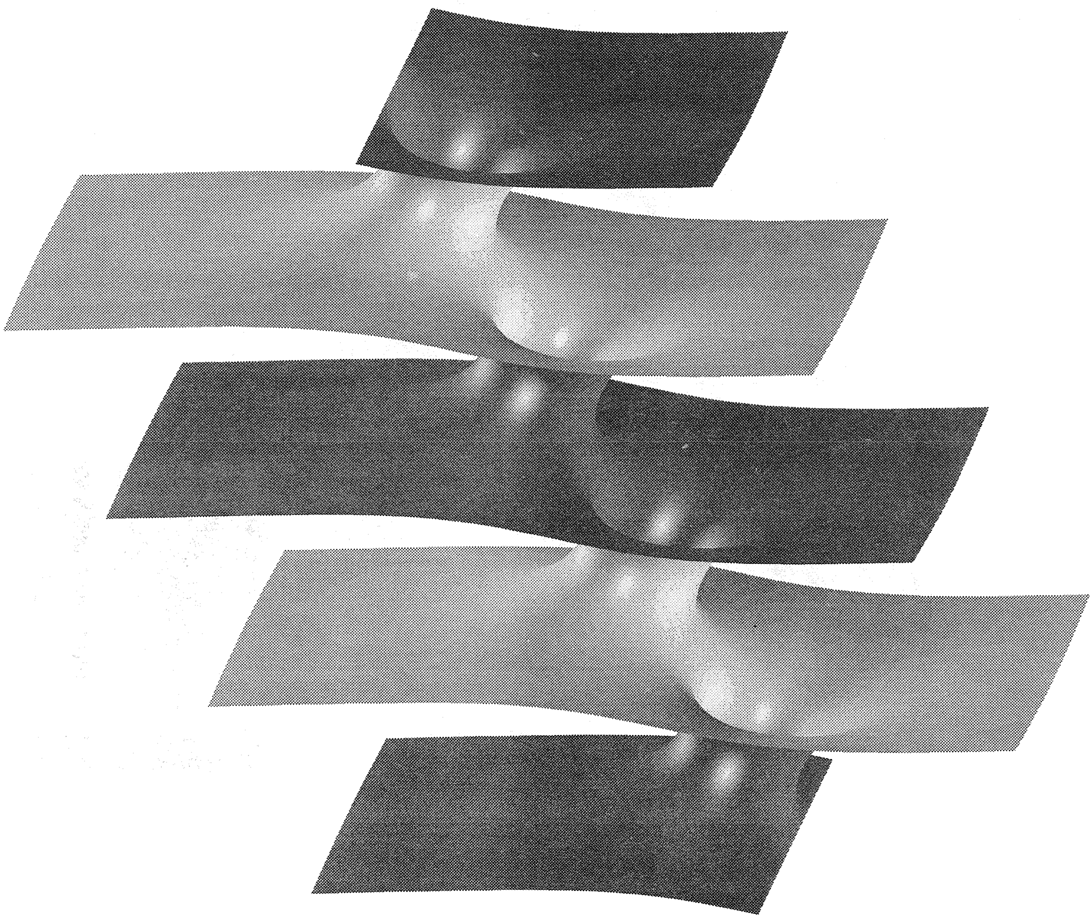

Figure 9
A Riemann's example, it is fibered by circles and straight lines

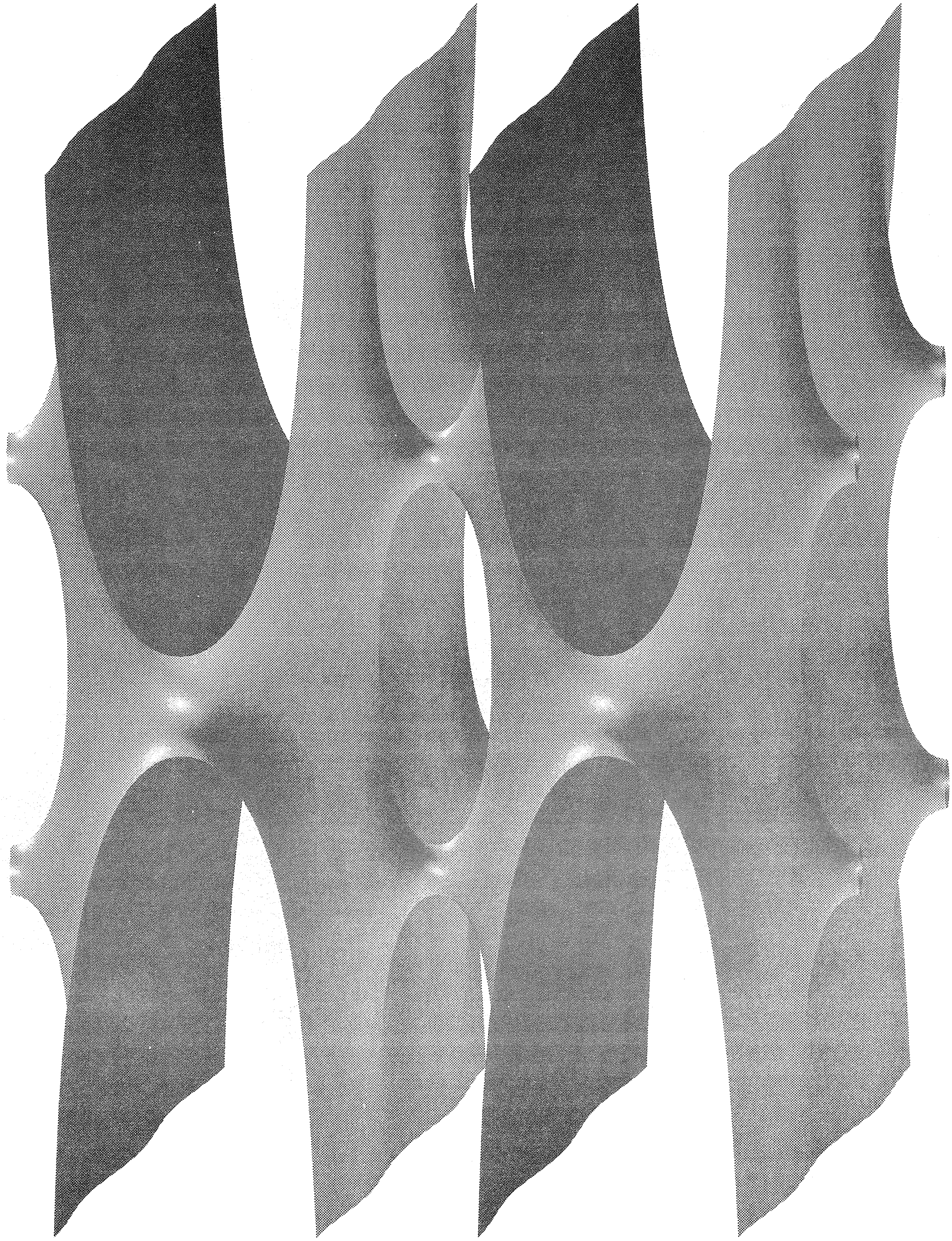

Figure 10
Wei's doubly periodic surface

15 The Halfspace Theorem and The Maximum Principle at Infinity

By Jorge and Meeks' theorem, we know that if we stand at infinity to view a complete minimal surface of finite total curvature, it looks like several planes passing through origin,

We will further discuss the image of such a surface. The basic theorem in this section is the Halfspace Theorem due to Hoffman and Meeks [32], its proof is surprisingly simple.

Theorem 15.1 (Halfspace Theorem) *A connected, proper, possibly branched, non-planar complete minimal surface M in $\mathbf{R}^3$ is not contained in a halfspace.*

Proof. Suppose the theorem is false.

Define $\mathbf{H}_t := \{(x_1, x_2, x_3) \,\big|\, x_3 \geq t\}$, $P_t = \partial \mathbf{H}_t$, $t \in \mathbf{R}$. By a translation and rotation, we may assume that $M \subset \mathbf{H}_0$. Let $T := \sup\{t \,\big|\, M \subset \mathbf{H}_t\}$. If $p \in M \cap P_T$, then P_T is the tangent plane T_pM. By Corollary 4.5, M must be on both sides of P_T, contradicting the fact that $M \subset \mathbf{H}_{T-\epsilon}$, any $\epsilon > 0$. Hence $M \cap P_T = \emptyset$. By a translation, we may assume that $T = 0$.

Let M_ϵ be the downward translation of M, then $M_\epsilon \cap P_0 \neq \emptyset$ for any $\epsilon > 0$. Let $C = C_1$ be the half-catenoid $\{(x_1, x_2, x_3) \,\big|\, x_1^2 + x_2^2 = \cosh^2(x_3),\, x_3 < 0\}$. By choosing $\epsilon > 0$ small enough, we may insure that $M_\epsilon \cap C_1 = \emptyset$ and $M_\epsilon \cap D_1 = \emptyset$, where D_1 is the unit-disk in P_0. Specifically, let $d > 0$ be the distance from M to the disk of radius $R = \cosh(1) > 1$. Outside the cylinder over D_R, C_1 lies below the plane P_{-1}. We will choose $\epsilon < \frac{1}{2}\min\{1, d\}$ small enough so that $M_\epsilon \cap C_1 = \emptyset$ and $M_\epsilon \cap P_0 \neq \emptyset$.

Denote by C_t the homothetic shrinking of C_1 by t, $0 < t \leq 1$. Observing that C_t converges smoothly, away from 0, to $P_0 - \{0\}$ we may conclude from the previous paragraph that $C_t \cap M_\epsilon \neq \emptyset$ for t sufficiently small, that $C_t \cap M_\epsilon$ lies outside the cylinder over D_1 for all t, and that $C_t \cap M_\epsilon = \emptyset$ for t close to 1.

Let $S = \{t \,|\, C_t \cap M_\epsilon \neq \emptyset\}$ and $T = \text{lubS}$. We claim that $T \in S$, i.e., $C_T \cap M_\epsilon \neq \emptyset$, thus $T < 1$.

If T is an isolated point of S, we are done. If not, we can find an increasing sequence $t_n \to T$, with $t_0 > T/2$, such that there exist points $p_n \in C_1$ with $t_n p_n \in C_{t_n} \cap M_\epsilon$. If $p_n = (x_n, y_n, z_n)$, we must have $t_n z_n \geq -\epsilon$ which implies $z_n \geq -\epsilon/t_n \geq -2\epsilon/T$. This means that p_n lies on the bounded closed subset $X_T := \{(x_1, x_2, x_3) \in C_1 \,\big|\, x_3 \geq -2\epsilon/T\}$ and must therefore possess a convergent subsequence. If $\{p_j\}$ is that subsequence and $p_j \to p_0 \in C_1$, then $t_j p_j \in C_{t_j} \cap M_\epsilon$. Since X_T is compact and M is proper, $\{t_j p_j\}$ must have a convergent subsequence in M_ϵ, still denoted by $\{t_j p_j\}$, and by continuity, $T p_0 \in C_T \cap M_\epsilon$. This proves that $C_T \cap M_\epsilon \neq \emptyset$.

Since the boundary of C_T lies inside $D_1 \subset P_0$, and that disk is disjoint from M_ϵ, $T p_0$ must be an interior point of C_T. Moreover, the fact that $T < 1$ and $C_t \cap M_\epsilon = \emptyset$ for $t > T$ means that C_T meets M_ϵ at $T p_0$, but lies locally on one side of M_ϵ near

Tp_0. We conclude that M_ϵ and C_T are tangent to each other at Tp_0 but stay on one side to each other near Tp_0. By Theorem 4.4 (which is also called *maximum principle for minimal surface*) we know that M_ϵ and C_T coincide near Tp_0. By Theorem 4.2, $M_\epsilon = C_T$. A catenoid, however, is not contained in any halfspace. This gives the desired contradiction. $\square$

Corollary 15.2 *Let $X : M \hookrightarrow \mathbf{R}^3$ be an embedded, nonplanar, complete minimal surface of finite total curvature. Then M has at least two annular ends.*

Proof. By Theorem 11.5, X is proper. If M has only one end, then by Theorem 11.8, $X(M)$ is a graph outside a large ball and is asymptotically a catenoid or flat end. Hence $X(M)$ is contained in a halfspace, which forces $X(M)$ to be a plane. $\square$

Remark 15.3 Theorem 15.1 is a kind of maximum principle at infinity. In [45], a version of maximum principle at infinity is proved, which states that if two embedded minimal surfaces with compact boundary and finite total curvature do not intersect, they are a positive distance apart. In [50] a stronger maximum principle at infinity (but it is called the *weak maximum principle at infinity*) in flat three-manifolds is proved, which says:

If two properly immersed minimal surfaces with compact boundaries in a flat three-manifold are disjoint, they stay a bounded distance apart.

The main tool in the proof of this weak maximum principle at infinity is Theorem 15.1.

The classical maximum principle (Remark 4.6) is one of the main tools in the study of minimal surfaces (and is used in an essential manner in the proof of Theorem 15.1). It is crucial in the study of regularity and in the use of barriers in the Plateau problem. Being, fundamentally, a result about elliptic equations, it is not surprising that there are applicable versions of maximum principle for surfaces with variable mean curvature. See for example Hildebrandt [24], where some of the history of this subject is discussed.

As an easy exercise we give a version of maximum principle at infinity.

Proposition 15.4 *Let $M \subset \mathbf{H}$ be a proper, complete minimal surface with compact boundary, where $\mathbf{H}$ is a halfspace. Then the distances satisfy*

$$d(M, \partial\mathbf{H}) = d(\partial M, \partial\mathbf{H}).$$

The proof is left as an exercise. Note that we only need prove that

$$d(M, \partial\mathbf{H}) \geq d(\partial M, \partial\mathbf{H}).$$

Proof. Translating M we will get a point $p \in \mathrm{int}(M_\epsilon) \cap \partial\mathbf{H}$, and by the maximum principle, we have a contradiction. $\square$

Remark 15.5 Theorem 15.1 says that two proper, complete, connected minimal surfaces must intersect each other if one of them is a plane. We call Theorem 15.1 the *Halfspace Theorem.* In fact, there is a stronger version, called the *Strong Halfspace Theorem.* It says that the conclusion of Theorem 15.1 is true without the assumption that one of the surfaces is a plane. A sketch of its proof is as follows: If the Strong Halfspace Theorem is false, then $M_1 \cap M_2 = \emptyset$. Let N be the flat three-manifold with M_1 and M_2 as boundary. The corollary of Theorem 8 in [52] says that there is a plane contained in N, thus we can apply the Halfspace Theorem. The proof of the existence of a plane in N involves the general Douglas-Plateau problem which is beyond our course.

Theorem 15.1 is essentially a three-dimensional theorem. In $\mathbf{R}^n$, $n > 3$, it is false.

16 The Convex Hull of a Minimal Surface

Recall that the *convex hull* $H(E)$ of a set $E \subset \mathbf{R}^n$ is defined as

$$H(E) = \bigcap_{E \subset \mathbf{H}} \mathbf{H}$$

where $\mathbf{H}$ is a halfspace in $\mathbf{R}^n$. Of course, if E is not contained in any halfspace, then $H(E) = \mathbf{R}^n = \bigcap_{\emptyset} \mathbf{H}$.

We want to study the convex hull of a minimal surface.

Let M be conformally a bounded plane domain and $X : M \hookrightarrow \mathbf{R}^3$ be a minimal surface such that X is continuous on $\overline{M}$. If $\partial M \neq \emptyset$, then a simple application of the maximum principle for harmonic functions shows that $X(M) \subset H(X(\partial M))$, where $H(X(\partial M))$ is the convex hull of $X(\partial M)$.

Exercise : Prove this fact.

Now using the the Halfspace Theorem, we can prove more.

Theorem 16.1 ([32]) *Suppose that $M \subset \mathbf{R}^3$ is a proper, complete, connected minimal surface in $\mathbf{R}^3$, whose boundary ∂M, which may be empty, is a compact set. Then exactly one of the following holds:*

1. *$H(M) = \mathbf{R}^3$;*
2. *$H(M)$ is a halfspace;*
3. *$H(M)$ is a closed slab between two parallel planes;*
4. *$H(M)$ is a plane;*
5. *$H(M)$ is a compact convex set. This case occurs precisely when M is compact.*

Furthermore, ∂M has nonempty intersection with each boundary component of $H(M)$.

Remark 16.2 We note that all of these cases are possible. For 1 and 2, examples are the catenoid and half-catenoid. For 3 we could take any of the examples in theorem 14.8 and consider the portion of these surfaces in the slab $|x_3| \leq 1$. This surface is bounded by two Jordan curves. For 4 we have a plane and 5 is the case for any compact example.

Proof of Theorem 16.1. Suppose now that cases 1, 4 and 5 do not occur. To prove that case 2 or case 3 must occur we need show that if H_1 and H_2 are distinct smallest halfspaces containing M, then $P_1 = \partial H_1$ and $P_2 = \partial H_2$ are parallel planes. Suppose now that P_1 and P_2 are not parallel planes. We shall derive a contradiction.

The interior of M cannot have a point in common with $P_1 \cup P_2$. (If it did then the maximum principle for minimal surface (see Theorem 4.4 and Remark 4.6) implies it would have to lie entirely on one plane or the other, contradicting the assumption that 4 does not hold. Let $C = H_1 \cap H_2$.

After a rotation, if necessary, we may assume that C lies in the halfspace $x_3 \geq 0$, that the boundary of C is a graph over the x_1x_2-plane and that $P_1 \cap P_2$ is the x_1-axis. After (if necessary) a translation of M, parallel to the x_1-axis, ∂M lies in the halfspace $x_1 \leq -1$. (This translation leaves C invariant.) In particular $0 \notin M$, and since M is closed (recall that properness implies that M is closed in $\mathbf{R}^3$), there exists an $s > 0$ such that $M \cap B_s = \emptyset$, where $B_s = \{(x_1, x_2, x_3) \,\big|\, (x_1 - s)^2 + x_2^2 + x_3^2 \leq s^2\}$. Let $\Gamma_s = \partial B_s \cap \partial C$. Since Γ_s has a 1-1 projection onto a convex plane curve (recall that ∂C is a graph over the x_1x_2-plane), by Theorem 4.1 it is the boundary of a compact minimal surface Δ_s that is the graph over a convex set in the x_1x_2-plane. By the convex hull property mentioned in the beginning of this section, $\Delta_s \subset B_s$, so Δ_s is a positive distance from M. Note that $B_s \subset \{x_1 \geq 0\}$ and $\Delta_s \subset C \cap \{x_1 \geq 0\}$.

For $t \in [1, \infty)$ consider the surfaces

$$A_t := \{tp \,|\, p \in \Delta_s\}.$$

We note: that each A_t is a nonnegative graph inside of $C \cap \{(x_1, x_2, x_3) \,|\, x_1 \geq 0\}$; that each A_t is compact; that as $t \to \infty$, A_t converges to $\{(x_1, x_2, x_3) \in C \,|\, x_1 = 0\}$; and that every point in $(C \cap \{(x_1, x_2, x_3) \,|\, x_1 > 0\}) - B_s$ lies on some A_t. Because $A = A_1$ is disjoint from M, it follows from an application of the maximum principle that none of the surfaces A_t can meet M (remember that ∂M is a distance at least 1 from any A_t, so any possible contact must occur at an interior point). However $(B_s \cup \bigcup_{t=1}^{\infty} A_t) \supset C \cap \{(x_1, x_2, x_3) \,\big|\, x_1 > 0\}$. Hence $M \subset H_3 = \{(x_1, x_2, x_3) \,\big|\, x_1 \leq 0\}$.

A similar argument will show that for some large positive integer k, $M \subset H_4 = \{(x_1, x_2, x_3) \,\big|\, x_1 \geq -k\}$. Repeating the entire procedure with H_1 and H_3 replacing H_1 and H_2 will prove that M may also be bounded in the x_3-direction and lie in some halfspace $H_5 = \{(x_1, x_2, x_3) \,\big|\, x_3 \leq N\}$ for N sufficiently large. Therefore $M \subset H_1 \cap H_2 \cap H_3 \cap H_4 \cap H_5$ which is a compact, convex set. This contradicts the assumption that 5 does not hold. This contradiction completes the proof of the main part of the theorem.

The fact that ∂M intersets each boundary component of $H(M)$ follows from Proposition 15.4. This completes the proof. □

Exercise : Prove that ∂M intersets each boundary component of $H(M)$.

Remark 16.3 All results in this section are true for minimal surfaces with branch points.

Theorem 16.1 is true for minimal submanifolds in $\mathbf{R}^n$, just replace planes by hyperplanes in the theorem.

Remark 16.4 If $X : M \hookrightarrow \mathbf{R}^3$ is a complete minimal surface of finite total curvature, then we know that X is proper. Then by the Halfspace Theorem, Theorem 15.1, $X(M)$

is not contained in any halfspace, and thus $H(X(M)) = \mathbf{R}^3$. This is a case where we know that $H(X(M)) = \mathbf{R}^3$. Here properness is necessary, as Rosenberg and Toubiana [73] have constructed complete minimal annuli which are contained in a slab.

Another example where $H(X(M)) = \mathbf{R}^3$ is a theorem of F. Xavier [85], which says that if $X : M \hookrightarrow \mathbf{R}^3$ is a complete minimal surface with bounded Gauss curvature (i.e, there is an $a > 0$ such that $K(p) > -a$ for any $p \in M$), then $H(X(M)) = \mathbf{R}^3$.

17 Flux

A simple but very useful consequence of minimal surfaces being conformal harmonic immersions is the flux formula. Let M be a compact domain on a Riemann surface. According to Stoke's Theorem, for any C^2 function $f : M \to \mathbf{R}^n$,

$$\int_M \triangle_M f dA = \int_{\partial M} df(\vec{n})ds, \tag{17.66}$$

where dA is the element of area on M, $\triangle_M$ is the Laplacian on M, ds is the line element on ∂M, $\vec{n}$ is the outward unit normal vector to M along ∂M, and $df(\vec{n})$ is the directional derivative of f in the direction $\vec{n}$. Applying (17.66) to an isometric immersion $X : M \hookrightarrow \mathbf{R}^3$, we have that $dX(\vec{n})$ is the image in $\mathbf{R}^3$ of the outward *conormal* (i.e., $dX(\vec{n})$ is tangent to $X(M)$ but normal to $\partial X(M)$); writing $n^* = dX(\vec{n})$ we have

$$\int_M \triangle_M X\, dA = \int_{\partial M} n^*\, ds.$$

If X is minimal and M is equipped with the metric induced by X, then

$$\int_{\partial M} n^*\, ds = 0. \tag{17.67}$$

In particular, if $\vec{v}$ is any fixed vector in $\mathbf{R}^3$

$$\int_{\partial M} n^* \bullet \vec{v}\, ds = 0. \tag{17.68}$$

The integral in (17.68) can be thought of as the tangential part of the flux through $X(\partial M)$ of the flow in $\mathbf{R}^3$ with constant velocity vector $\vec{v}$. While (17.67) and (17.68) are quite simple and were undoubtedly known in the 19th century, they and their modifications have only recently come into widespread use in the study of minimal and constant mean curvature surfaces [43], [44].

As a sample application of the flux formula, we consider the catenoid. It was Euler who discovered the catenoid, the first nonplanar example of a minimal surface. He did this by finding the surface of revolution that was a critical point for the area functional. Consider a surface of revolution about the z-axis with profile curve $(r(t), t)$ in the xz-plane. Let S be the compact portion of the surface that is between $z = t_1$ and $z = t_2$. S is bounded by two circles of radii $r(t_1)$ and $r(t_2)$, respectively. The conormal of S at the level set $z = t$ is

$$\frac{1}{\sqrt{1 + r'(t)^2}}(r'(t)\cos\theta,\, r'(t)\sin\theta,\, 1).$$

Then computing the flux in the z-direction ($\vec{v} = (0, 0, 1)$), we get by (17.68)

$$\int_{S\cap\{z=t_1\}} \frac{1}{\sqrt{1 + r'(t_1)^2}} ds = \int_{S\cap\{z=t_2\}} \frac{1}{\sqrt{1 + r'(t_2)^2}} ds$$

or

$$\frac{2\pi r(t_1)}{\sqrt{1+r'(t_1)^2}} = \frac{2\pi r(t_2)}{\sqrt{1+r'(t_2)^2}}.$$

But t_1 and t_2 are arbitrary, so

$$\frac{r(t)}{\sqrt{1+r'(t)^2}} = C, \tag{17.69}$$

where $C > 0$ is a constant.

The ordinary differential equation (17.69) is satisfied by the functions

$$r(t) = C\cosh(C^{-1}t + B), \tag{17.70}$$

where B is a constant. These are all possible solutions to (17.69). These curves 17.70 are catenaries, thus, nonplanar minimal surfaces of revolution are all catenoids. The definition of flux and this application are adapted from [33].

Now let us go back to the general theory of flux. Let $X : M \hookrightarrow \mathbf{R}^3$ be a minimal surface, $\Gamma \subset M$ a loop. Under the metric induced by X, we define the *flux of* X *along* Γ as

$$\mathbf{Flux}(\Gamma) = \int_\Gamma dX(\vec{n})ds, \tag{17.71}$$

where $\vec{n}$ is the unit vector orthogonal to the unit vector $\vec{s}$ tangent to Γ and $(\vec{n}, \vec{s})$ gives the orientation of M. The flux is well defined on the homology class of $[\Gamma]$. In fact, if $\gamma \in [\Gamma]$ then $\gamma \cup \Gamma$ bounds a domain Ω and we have

$$0 = \int_\Omega \Delta_M X\, dA = \int_\Gamma dX(\vec{n})ds - \int_\gamma dX(\vec{n})ds.$$

Remember that

$$X(z) = \Re \int_{z_0}^{z} (\omega_1, \omega_2, \omega_3),$$

where the ω_i's are holomorphic 1-forms. We define the (maybe multiple-valued) harmonic function

$$Y(z) = \Im \int_{z_0}^{z} (\omega_1, \omega_2, \omega_3).$$

Then $dX(\vec{n}) = dY(\vec{s})$ by the Cauchy-Riemann equations. Hence we have

$$\mathbf{Flux}(\Gamma) = \int_\Gamma dY(\vec{s})ds = Y(\Gamma(l)) - Y(\Gamma(0)) = \Im \int_\Gamma (\omega_1, \omega_2, \omega_3),$$

where l is the arc-length of Γ. Since Γ is a loop, we know that

$$\Re \int (\omega_1, \omega_2, \omega_3) = 0,$$

hence

$$\mathbf{Flux}(\Gamma) = -i \int_\Gamma (\omega_1, \omega_2, \omega_3). \tag{17.72}$$

Recall that a holomorphic 1-form ϕ is exact if and only if for any loop $\Gamma \subset M$, $\int_\Gamma \phi\, ds = 0$; if and only if $\phi = df$, where f is a holomorphic function on M. By the Enneper-Weierstrass representation

$$\omega_1 = \frac{1}{2}(1-g^2)\eta, \quad \omega_2 = \frac{i}{2}(1+g^2)\eta, \quad \omega_3 = g\eta,$$

we get the following proposition.

Proposition 17.1 ([71]) *The following are equivalent:*

1. *For each loop $\Gamma \subset M$, the flux of X along Γ vanishes;*
2. *The holomorphic 1-forms ω_i are exact;*
3. *The holomorphic 1-forms η, $g\eta$, and $g^2\eta$ are exact;*
4. *The conjugate immersion $X_{\pi/2}$ is globally well defined on M.*

Assume that $X : M \hookrightarrow \mathbf{R}^3$ is a complete minimal surafce of finite total curvature. Let $D - \{p\} \subset M$ be a punctured disk corresponding to an end. We want to calculate the flux of X along ∂D. Let g and η be the Weierstrass data for X. We have

$$\begin{aligned} 0 &= 2\Re \int_{\partial D} \omega_1 + i2\Re \int_{\partial D} \omega_2 \\ &= \overline{\int_{\partial D} \eta} - \int_{\partial D} g^2\eta = \overline{2\pi i\, \mathbf{res}(p,\eta)} - 2\pi i\, \mathbf{res}(p, g^2\eta) \\ &= -2\pi i[\overline{\mathbf{res}(p,\eta)} + \mathbf{res}(p, g^2\eta)], \end{aligned} \tag{17.73}$$

where $\mathbf{res}(p,\omega)$ is the residue of ω at p. Hence we have

$$\mathbf{res}(p,\omega_1) = \frac{1}{2}\mathbf{res}(p,\eta) - \frac{1}{2}\mathbf{res}(p,g^2\eta) = \Re\, \mathbf{res}(p,\eta),$$

$$\mathbf{res}(p,\omega_2) = \frac{i}{2}\mathbf{res}(p,\eta) + \frac{i}{2}\mathbf{res}(p,g^2\eta) = -\Im\, \mathbf{res}(p,\eta).$$

Since $\Re \int_{\partial D} \omega_3 = 0$, $c := \mathbf{res}(p, g\eta)$ is real. Thus

$$\begin{aligned} \mathbf{Flux}(\partial D) &= -i \int_{\partial D_i} (\omega_1, \omega_2, \omega_3) \\ &= 2\pi(\Re(\mathbf{res}(p,\eta)),\ -\Im(\mathbf{res}(p,\eta)),\ \mathbf{res}(p, g\eta)). \end{aligned} \tag{17.74}$$

If we write $\mathbf{R}^3 = \mathbf{C} \times \mathbf{R}$, then $X = (X^1, X^2, X^3) = (X^1 + iX^2, X^3)$, and the flux around ∂D then is

$$\mathbf{Flux}(\partial D) = 2\pi(\overline{\mathbf{res}(p,\eta)},\ \mathbf{res}(p, g\eta)). \tag{17.75}$$

Recall that for any meromorphic 1-form ω on a closed Reimann surface S_k, the summation of residues of ω is zero, i.e., for all poles $p \in S_k$ of ω,

$$\sum_p \mathbf{res}(p, \omega) = 0.$$

Let $M = S_k - \{p_1, \cdots, p_n\}$, and $p_i \in D_i$. Since η and $g\eta$ can only have poles at an end, we have

$$\sum_{i=1}^n \mathbf{Flux}(\partial D_i) = \sum_{i=1}^n \overline{(\mathbf{res}(p_i, \eta)}, \mathbf{res}(p_i, g\eta)) = 0. \tag{17.76}$$

This is consistent with 17.67 which comes from Stokes' theorem.

When an end is embedded, there is a nicer formula for the flux, i.e., the flux is a vector in the direction of the limiting normal at that end. Let us derive the formula.

After a rotation in $\mathbf{R}^3$ if necessary, we may assume that $g(p) = 0$. Let ζ be a coordinate on D such that $\zeta(p) = 0$. Then $g(\zeta) = \zeta^k \phi(\zeta)$, where $k \geq 1$, ϕ is holomorphic on D and $\phi(0) \neq 0$. Then $z = \zeta\phi^{1/k}(\zeta)$ is a coordinate on a (maybe smaller) disk $D' \subset D$. Since $\partial D'$ is homologous to ∂D, $\mathbf{Flux}(\partial D') = \mathbf{Flux}(\partial D)$. We may assume that $D' = D$. Then $g(z) = z^k$ on D. By Lemma 11.3, the 1-form $\eta = f(z)dz$ has a pole of order at least 2 at p, so

$$f(z) = \frac{1}{z^m} h(z),$$

$m \geq 2$ and h is holomorphic and $h(0) \neq 0$. The Enneper-Weierstrass representation is given by

$$\omega_1 = \frac{1}{2}\left(z^{-m} - z^{2k-m}\right) h(z)dz, \quad \omega_2 = \frac{i}{2}\left(z^{-m} + z^{2k-m}\right) h(z)dz, \quad \omega_3 = z^{k-m} h(z)dz.$$

As before, since X is well defined, we have

$$0 = 2\Re \int_{\partial D} \omega_1 + i2\Re \int_{\partial D} \omega_2 = \overline{\int_{\partial D} z^{-m} h(z)dz} - \int_{\partial D} z^{2k-m} h(z)dz. \tag{17.77}$$

If $2k \geq m$, then

$$\overline{\int_{\partial D} z^{-m} h(z)dz} = \int_{\partial D} z^{2k-m} h(z)dz = 0,$$

hence

$$\int_{\partial D} \omega_1 = \int_{\partial D} \omega_2 = 0.$$

In particular, if the end is embedded, then $m = 2$. When $k = 1$,

$$\int_{\partial D} \omega_3 = 2\pi i\, h(0).$$

Since

$$\Re \int_{\partial D} \omega_3 = 0,$$

$h(0) \neq 0$ is real. Hence we have

$$\mathbf{Flux}(\partial D) = (0, 0, 2\pi h(0)) = -2\pi h(0)(0, 0, -1). \tag{17.78}$$

When $k > 1$, we have

$$\int_{\partial D} \omega_3 = 0,$$

thus

$$\mathbf{Flux}(\partial D) = (0, 0, 0). \tag{17.79}$$

Recall that these two cases corresponding to that the end is catenoid type or flat type. We have the following lemma.

Lemma 17.2 *Let $X : S_k - \{p_1, \cdots, p_n\} \hookrightarrow \mathbf{R}^3$ be a complete minimal surface of finite total curvature. Let $E_i := X : D_i - \{p_i\}$ be an embedded end and N_i be the limiting unit normal at p_i. Then there is an $\alpha_i \in \mathbf{R}$ such that*

$$\mathbf{Flux}(\partial D_i) = \alpha_i N_i. \tag{17.80}$$

Furthermore, $\alpha_i \neq 0$ if and only if E_i is a catenoid type end.

Proof. Let $A : \mathbf{R}^3 \to \mathbf{R}^3$ be the rotation such that $AN_i = (0, 0, -1)$. Then

$$AX = \Re \int (\omega_1, \omega_2, \omega_3)$$

gives the rotated surface which has limiting normal $(0, 0, -1)$ at p_i. When E_i is a catenoid type end, by (17.78),

$$\begin{aligned}
\mathbf{Flux}(\partial D_i) &= \int_{\partial D_i} dX(\vec{n})ds = A^{-1} \int_{\partial D_i} d(AX)(\vec{n})ds \\
&= A^{-1} \int_{\partial D_i} -i(\omega_1, \omega_2, \omega_3)ds = -2\pi h(0) A^{-1}(0, 0, -1) \\
&= -2\pi h(0) N_i = \alpha_i N_i.
\end{aligned}$$

When E_i is a flat end, the proof is similar with $\alpha_i = 0$. □

Remark 17.3 Comparing the α_i here and the α (the coefficient of the logarithmic term of u) in Theorem 11.8 and its proof, we see that $\alpha_i = 2\pi\alpha$.

Theorem 17.4 *Let $X : S_k - \{p_1, \cdots, p_n\} \hookrightarrow \mathbf{R}^3$ be a complete minimal surface of finite total curvature. Suppose that all ends of M are embedded. Let p_i, $1 \leq i \leq k \leq n$, correspond to catenoid type ends and N_i be the corresponding limiting normals. Then $\{N_i\}_{1 \leq i \leq k}$ are linearly dependent.*

Proof. Let D_i be pairwise disjoint open disks in S_k such that $p_i \in D_i$, $1 \leq i \leq n$. Then $M' := M - \bigcup_i^n D_i = S_k - \bigcup_i^n D_i$ is compact. Let $\vec{n}_i$ be the inward unit normal along ∂D_i in D_i. We have

$$\begin{aligned} 0 &= \int_{M'} \triangle_X X dA = \int_{\partial M'} dX(\vec{n}) ds = \sum_{i=1}^{n} \int_{\partial D_i} dX(\vec{n_i}) ds \\ &= -\sum_{i=1}^{n} \mathbf{Flux}(\partial D_i) = -\sum_{i=1}^{k} \alpha_i N_i. \end{aligned}$$

Since $\alpha_i \neq 0$, $\{N_i\}_{1 \leq i \leq k}$ are linearly dependent. $\square$

Corollary 17.5 *Let $X : S_k - \{p_1, \cdots, p_n\} \hookrightarrow \mathbf{R}^3$ be a complete minimal surface of finite total curvature. Suppose that all ends of M are embedded. Then M has either no catenoid type ends, or has at least two catenoid ends.*

Furthermore, if X is an embedding, or has parallel embedded ends, then M has at least two catenoid type ends.

Proof. Straight forward. The last claim is a corollary of the Halfspace Theorem, Theorem 15.1. $\square$

18 Uniqueness of the Catenoid

Let $X : M = S_k - \{p_1, \cdots, p_n\} \hookrightarrow \mathbf{R}^3$ be a complete minimal surface, where S_k is a closed Riemann surface of genus k. We say that X has genus k.

The catenoid has genus zero by this definition. We have already proved that the catenoid is the only embedded complete minimal surface of total curvature -4π and is the only minimal surface which is a rotation surface. Schoen [74] proved that the catenoid is the only complete minimal surface with exactly two annular ends and finite total curvature. Thus the catenoid has many special features which describe it uniquely.

In 1989, López and Ros proved the following remarkable theorem [49].

Theorem 18.1 *The catenoid is the only embedded genus zero non-planar minimal surface of finite total curvature.*

The proof of Theorem 18.1 is a combination of the flux formula and the maximum principle at infinity. We will give a proof here adapted from [71].

Another key ingredient in the proof of Theorem 18.1 is deformation. Suppose that $X : M = S_k - \{p_1, \cdots, p_n\} \hookrightarrow \mathbf{R}^3$ is a minimal surface. If for any loop $\Gamma \subset M$, $\mathbf{Flux}(\Gamma)$ is a vertical vector, i.e., parallel to $(0, 0, 1)$, then we say that X has *vertical flux.* By Proposition 17.1, we see that X has vertical flux if and only if for any loop Γ,

$$\int_\Gamma \eta = 0, \quad \text{and} \quad \int_\Gamma g^2\eta = 0, \tag{18.81}$$

where g and η are the Weierstrass data for X.

Let $\lambda \in (0, \infty)$ and $\eta_\lambda = \lambda^{-1}\eta$, $g_\lambda = \lambda g$. Consider the corresponding Enneper-Weierstrass representation,

$$\omega_1^\lambda = \frac{1}{2}(\lambda^{-1} - \lambda g^2)\eta; \quad \omega_2^\lambda = \frac{i}{2}(\lambda^{-1} + \lambda g^2)\eta; \quad \omega_3^\lambda = g\eta = \omega_3.$$

If X has vertical flux, then we have a family of well defined minimal surfaces, deformations of the original surface, given by

$$X^\lambda = \Re \int (\omega_1^\lambda, \omega_2^\lambda, \omega_3^\lambda). \tag{18.82}$$

Note that the third coordinate function of X^λ does not depend on λ.

A point $p \in M$ such that $g(p) = 0$ or ∞ is called a *vertical point* of X. Since $g_\lambda = \lambda g$, if p is a vertical point of X then p is also a vertical point of X^λ, and vice versa. We first investigate the behaviour of X^λ when p is a vertical point.

Lemma 18.2 *Suppose that X has vertical flux and is non-planar. If p is a vertical point, then X^λ is not an embedding when λ is sufficiently large or sufficiently small.*

Proof. First we assume that $g(p) = \infty$. Let $p \in D$ be a coordinate disk in M and $z(p) = 0$. Without loss of generality, we may assume that $g(z) = z^{-k}$ on D and $k > 0$. By Theorem 14.1, η should have a zero of order $2k$ at p, so we can write $\eta = z^{2k}h(z)dz$, where h is holomorphic on D and $h(0) \neq 0$. Make a change of coordinate on D by $\zeta = \lambda^{-1/k}z$, then

$$g_\lambda(z) = \lambda z^{-k} = \zeta^{-k}, \quad \text{and} \quad \eta_\lambda = \lambda^{1+1/k}\zeta^{2k}h(\lambda^{1/k}\zeta)d\zeta.$$

Under these new coordinates,

$$\omega_1^\lambda = \frac{1}{2}\lambda^{1+1/k}(\zeta^{2k} - 1)h(\lambda^{1/k}\zeta)d\zeta, \quad \omega_2^\lambda = \frac{i}{2}\lambda^{1+1/k}(\zeta^{2k} + 1)h(\lambda^{1/k}\zeta)d\zeta,$$

$$\omega_3^\lambda = \lambda^{1+1/k}\zeta^k h(\lambda^{1/k}\zeta)d\zeta.$$

Now we dilate X^λ by a homothety of ratio $\lambda^{-(1+1/k)}$, $\tilde{X}^\lambda = \lambda^{-(1+1/k)}X^\lambda$. When $\lambda \to 0$, $\tilde{X}^\lambda$ converges uniformly on compact subsets of $\mathbf{C}$ to the minimal surface $X^0 : \mathbf{C} \hookrightarrow \mathbf{R}^3$ (note that for fixed $z \neq 0$, $\lim_{\lambda\to 0} \lambda^{-1/k}z = \infty$ and for fixed ζ, $\lim_{\lambda\to 0}\lambda^{1/k}\zeta = 0$). X^0 is determined by the Weierstrass data for X^0,

$$g_0 = \zeta^{-k}, \quad \eta_0 = h(0)\zeta^{2k}d\zeta.$$

Such data gives a complete non-embedded minimal surface. In fact, by Theorem 11.1, η_0 should have a pole of order 2 to make X^0 an embedding at $\zeta = \infty$, but our η_0 has a pole of order $2k + 2 > 2$ at $\zeta = \infty$.

Since $\tilde{X}^\lambda$ converges to X^0 uniformly on compact subset when $\lambda \to 0$, for λ small enough, $\tilde{X}^\lambda$, thus X^λ, is not an embedding.

When $g(p) = 0$, the proof is similar and when λ is large enough, X^λ is not an embedding. □

Exercise : Give a rigorous proof that X^λ is not embedded when $g(p) = 0$ and λ is large.

Note that if X has vertical flux and $X|_{D-\{p\}} : D - \{p\} \hookrightarrow \mathbf{R}^3$ is an annular end , then $X^\lambda|_{D-\{p\}} : D - \{p\} \hookrightarrow \mathbf{R}^3$ is also an annular end.

Next we will study the behaviour of X^λ at an embedded end of vertical limiting normal.

Lemma 18.3 *Suppose that $X : M \hookrightarrow \mathbf{R}^3$ is non-planar and has vertical flux. If $E = X|_{D-\{p\}} : D - \{p\} \hookrightarrow \mathbf{R}^3$ is an embedded flat annular end with vertical limiting normal, then $E^\lambda = X^\lambda|_{D-\{p\}} : D - \{p\} \hookrightarrow \mathbf{R}^3$ is not embedded for λ large or small enough.*

Proof. Let $p \in D$ be a coordinate neighbourhood with $z(p) = 0$. As before, we first assume that $g(p) = \infty$ and so $g(z) = z^{-k}$, $k > 1$ since E is a flat end. By Theorem

14.1, η has a zero of order $2k-2$, so $\eta = z^{2k-2}h(z)dz$. Again we make the change of coordinate $\zeta = \lambda^{-1/k}z$ and

$$g_\lambda = \zeta^{-k}, \qquad \eta_\lambda = \lambda^{1-1/k}\zeta^{2k-2}h(\lambda^{1/k}\zeta)d\zeta.$$

Arguing as before, we dilate X^λ by a homothety of ratio $\lambda^{-(1-1/k)}$, $\tilde{X}^\lambda = \lambda^{-(1-1/k)}X^\lambda$. When $\lambda \to 0$, $\tilde{X}^\lambda$ converges uniformly on compact subsets of $\mathbf{C} - \{0\}$ to the minimal surface $X^0 : \mathbf{C} - \{0\} \hookrightarrow \mathbf{R}^3$. X^0 is determined by the Weierstrass data for X^0,

$$g_0 = \zeta^{-k}, \qquad \eta_0 = h(0)\zeta^{2k-2}d\zeta.$$

Thus by Theorem 14.1, this complete minimal surface has an embedded end at $\zeta = 0$ and a non-embedded end at $\zeta = \infty$, since at ∞ η_0 has a pole of $2k > 2$. Hence when λ small enough, X^λ is not embedded.

When $g(p) = 0$, similar argument gives that when λ large enough, X^λ is not embedded. $\square$

Lemma 18.4 *Suppose that $X : M = S_k - \{p_1, \cdots, p_n\} \hookrightarrow \mathbf{R}^3$ is embedded and all ends have vertical normal. If X has vertical flux, then X^λ is an embedding for all $\lambda > 0$.*

Proof. First note that since X is embedded, at each puncture p_i,

$$|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 \cong \frac{1}{|z|^4},$$

where $\phi_i dz = \omega_i$. Thus for any deformation X^λ, we have

$$|\phi_1^\lambda|^2 + |\phi_2^\lambda|^2 + |\phi_3^\lambda|^2 \cong \frac{1}{|z|^4}.$$

This then tells us that each end of X^λ is embedded. By the weak maximum principle at infinity (see Remark 15.3), the distance between any two ends of X is positive. Since the third coordinate of X^λ is independent of λ, any two ends of X^λ are eventually disjoint. Thus outside of a compact set $C_\lambda \subset M$, X^λ is embedded.

Now let $B := \{\lambda \in (0,\infty) \,\big|\, X^\lambda \text{ is embedded } \}$. We want to prove that B is both open and closed; then by the connectedness of $(0,\infty)$ and $1 \in B$, we know that $B = (0,\infty)$.

Suppose $\lambda_0 \in B$. Since X^λ uniformly converges to X^{λ_0} on compact sets when $\lambda \to \lambda_0$, and each X^λ is embedded outside of a compact set, it follows for λ near λ_0 that X^λ is embedded.

Now suppose that $\{\lambda_n\} \subset B$ and $\lambda_n \to \lambda$ when $n \to \infty$. If X^λ is not embedded, then there are x and $y \in M$ such that $x \neq y$ and $X^\lambda(x) = X^\lambda(y)$. Let D_1 and D_2 be disjoint closed disk type neighbourhoods of x and y respectively, such that $X^\lambda|_{D_i}$ is embedded. Since X^{λ_n} converges uniformly on D_i and $X^{\lambda_n}(D_1) \cap X^{\lambda_n}(D_2) = \emptyset$, by shrinking D_i if necessary, $X^{\lambda_n}(D_i)$ are disjoint graphs on the same plane domain, and $\lim_{n\to\infty} X^{\lambda_n}(x) = \lim_{n\to\infty} X^{\lambda_n}(y)$. By the maximum principle (Theorem 4.4 and Remark 4.6), $X^\lambda(D_1) = X^\lambda(D_2)$. This shows that the image $X^\lambda(M)$ is an embedded minimal surface of finite total curvature and $X^\lambda : M \to X^\lambda(M)$ is a finite sheet covering. But outside a compact set, X^λ is one to one, so this covering is single sheeted, that is X^λ must be embedded. This proves that B is also closed, hence also proves this lemma. $\square$

Now we can prove Theorem 18.1.

Proof of Theorem 18.1. Without loss of generality, we may assume that all ends of X have vertical limiting normals. Let $D_i \subset S_0 = \mathbf{C} \cup \{\infty\}$ be disjoint open disks such that $p_i \in D_i$. Then ∂D_i are generators of $H_1(M)$. By (17.80), X has vertical flux on each ∂D_i, hence has vertical flux on any loop, i.e., X has vertical flux.

Since X is embedded, by Lemma 18.4 X^λ is embedded for any $\lambda \in (0, \infty)$. By Lemma 18.2 and Lemma 18.3, $g \neq 0$ or ∞ on M and X does not have flat ends. We claim that X has exactly two catenoid ends.

In fact, since $g \neq 0$ or ∞ on M, dX_3 never vanishes on M where $X = (X_1, X_2, X_3)$. Suppose X has more than two catenoid ends. Let $P_t := \{(x, y, z) \in \mathbf{R}^3 \,|\, z = t\}$; there is an $N > 0$ such that if $t < -N$ or $t > N$ then $X(M) \cap P_t$ has at least two components. By Morse theory, $X_3^{-1}(-\infty, -N)$ or $X_3^{-1}(N, \infty)$ has at least two components since X_3 has no critical points on M. Again by Morse theory, $M = X_3^{-1}(\mathbf{R})$ has at least two components, contradicting the fact that M is connected.

By Corollary 17.5, X must have at least two catenoid type ends, so X has exactly two catenoid type ends.

Now by the total curvature formula (13.57), X has total curvature -4π. By Corollary 14.6, X must be a catenoid. The proof is complete. $\square$

19 The Gauss Map of Complete Minimal Surfaces

Let $X : M \hookrightarrow \mathbf{R}^3$ be a complete minimal surface. Let g and η be the Weierstrass data for X. The question in this section is how many points does the set $\mathbf{C} \cup \{\infty\} - g(M)$ have? We will only prove a relatively easy theorem due to Osserman, which will be useful when we discuss the behavior of minimal annuli. At the end of this section we will give an up to date survey of partial results for this problem.

To prove the theorem of Osserman mentioned above we have to introduce the concept of *capacity*. Since we only need describe when a set has capacity zero, we will only define zero capacity sets.

Definition 19.1 Let $D \subset \mathbf{C}$ be a closed set. Then D has *capacity zero* if and only if the function $\log(1+|z|^2)$ has no harmonic majorant in $\mathbf{C} - D$, i.e, there is no harmonic function $h : \mathbf{C} - D \to \mathbf{R}$ such that

$$\log(1+|z|^2) \le h(z), \quad z \in \mathbf{C} - D.$$

Note that any finite set in $\mathbf{C}$ has capacity zero.

Theorem 19.2 *Let $X : M_r(:= \{1 \le |z| < r \le \infty\}) \hookrightarrow \mathbf{R}^3$ be a complete minimal surface. Then either the Gauss map g tends to a single limit as $|z| \to r$, or else in each neighbourhood of $\{|z| = r\}$ g takes on all points of $\mathbf{C} \cup \{\infty\}$ except for at most a set of capacity zero.*

Proof. Let $\eta = f(z)dz$. Suppose now that in some neighbourhood of $\{|z| = r\}$ g omits a set Z of positive capacity. This means that for some $1 \le r_1 < r$, the function $w = g(z)$ omits Z in the domain $D' := \{r_1 < |z| < r\}$. Hence there exists a harmonic function $h(w)$ defined in $\mathbf{C} - Z \supset g(D')$ such that $\log(1+|w|^2) \le h(w)$. Since the induced metric by X on M_r is $\Lambda^2 = \frac{1}{4}|f|^2(1+|g|^2)^2$, we have

$$\log \Lambda(z) \le \log \frac{|f|}{2} + h(g(z)).$$

Since g and f are holomorphic, the right hand side of the above inequality is harmonic. By Lemma 10.5 and Proposition 10.6, $r = \infty$. But then g could not have an essential singularity at infinity by Picard's theorem. Thus g tends to a limit, finite or infinite, as z tends to infinity. $\square$

Remark 19.3 Once we know that $r = \infty$ and g has a limit at infinity, we know that X has finite total curvature. The argument is as follows:

By a rotation if necessary we may assume that g has a pole at ∞. Then $g(z) = z^n h(z)$ where $h(\infty) \ne 0$ and $n > 0$. Since

$$\frac{4|g'|^2}{(1+|g|^2)^2} = O(|z|^{-2n}) \quad \text{at } \infty,$$

$$\int_{M_\infty} K dA = -\int_{M_\infty} \frac{4|g'|^2}{(1+|g|^2)^2} dx\, dy > -\infty.$$

Since the Gauss curvature is invariant under rotation, X has finite total curvature.

Combined with Corollary 10.9 we have

Corollary 19.4 *The Gauss map of a complete minimal surface of finite topology achieves every point in* $\mathbf{C}$ *except a set of zero capacity.*

In 1981, F. Xavier [84] proved that the Gauss map of a complete minimal surface cannot miss more than 6 points of $\mathbf{C} \cup \{\infty\}$.

In 1988, Fujimoto [20] proved that the Gauss map of a complete minimal surface cannot miss more than 4 points of $\mathbf{C} \cup \{\infty\}$.

In 1990, Mo and Osserman [58] proved that the Gauss map of a complete minimal surface of infinite total curvature achieves any point of $\mathbf{C} \cup \{\infty\}$, except at most 4, infinitely many times.

Scherk's first surface is an embedded complete doubly periodic minimal surface. One block of it is given by the Weierstrass data

$$g(z) = z, \quad f(z) = \frac{1}{(1+z)(1-z)(z+i)(z-i)}$$

on $\mathbf{C} - \{\pm 1, \pm i\}$. This block has four vertical straight lines as boundary. Rotating $180°$ around one of those straight lines we get a basic block S of Scherk's first surface. The whole surface is the parallel translations of S in two perpendicular directions. The Gauss map of Scherk's first surface misses 4 points, ± 1, $\pm i$, and takes any other points infinitely many times.

This example shows that Fujimoto's, and Mo and Osserman's results are the best possible results.

All known examples of surfaces whose Gauss map misses 4 points are surfaces of infinite total curvature. This is no surprise, since in 1964, Osserman [67] proved that if the surface has finite total curvature, then g can miss at most 3 points.

The catenoid has finite total curvature -4π and its Gauss map misses 2 points, 0 and ∞, say. Hence either 2 or 3 is the maximal number of points that may be omitted by the Gauss map of a complete minimal surface of finite total curvature. But there is no known example of a complete minimal surface of finite total curvature whose Gauss map misses 3 points in $\mathbf{C} \cup \{\infty\}$.

In 1987, Weitsman and Xavier [81] proved that if g misses 3 points, then the total curvature is less than or equal to -16π.

In 1993, Fang [15] proved that the total curvature must be less than or equal to -20π.

So far the problem of whether 2 or 3 is the maximal number of points that may be omitted by the Gauss map of a complete minimal surface of finite total curvature is still open.

20 The Second Variation and Stability

We now introduce the concept of *stability* of minimal surfaces which will play an important role in the proof of several theorems in the remainder of these notes.

Let Ω be a precompact domain in a Riemann surface M, $X : \Omega \to \mathbf{R}^3$ a minimal surface. From the calculus of variations definition of a minimal surface, we know that X is a minimal surface if and only if the area A of X is a stationary point of the area functional $A(t)$ for any variation $X(t)$. Note that being stationary does not mean that X has minimum area among all surfaces with the same boundary.

To study when X has locally minimum area, naturally we study the second variation, namely the second derivative $A''(0)$ of the area functional for any variation family $X(t)$. From calculus we know that if $A''(0) > 0$ then $A(0)$ is a local minimum. Note that the word *local* is significant, there are minimal surfaces such that $A''(0) > 0$ for any variation family, yet those surfaces do not have minimum area. Hence we define that X is *stable* if $A''(0) > 0$ for all possible variation families $X(t)$, otherwise X is *unstable*. Sometimes one says X is *almost stable* if $A''(0) \geq 0$.

It is important to express the formula for the second variation of X via the geometric quantities of X. Let (u^1, u^2) be the local coordinates of Ω. We use the fact that X is conformal harmonic, and write $\Lambda^2 = |X_1|^2 = |X_2|^2$, $\triangle = D_{11} + D_{22}$.

From (3.4),

$$\frac{dA(t)}{dt} = -2\int_\Omega H(t)(E(t)\bullet N(t))\, dA_t,$$

where $E(t) = \partial X(t)/\partial t$, $H(t)$ is the mean curvature of $X(t)$, and $N(t)$ is the Gauss map of $X(t)$. Let $E = \alpha X_1 + \beta X_2 + \gamma N$. Since $H(0) = 0$ we have

$$\left.\frac{d^2A(t)}{dt^2}\right|_{t=0} = -2\int_\Omega \left.\frac{dH(t)}{dt}\right|_{t=0}(E\bullet N)\, dA_0,$$

where we write $E = E(0)$, etc. Now suppose that each $X(t)$ is a C^2 surface, and the first and second fundamental forms are given on an isothermal coordinate chart U by

$$g_{ij}(t) = X_i(t)\bullet X_j(t), \quad (g^{ij}(t)) = (g_{ij}(t))^{-1}, \quad h_{ij}(t) = X_{ij}(t)\bullet N(t).$$

Then

$$H(t) = \frac{1}{2}\sum_{i,j} g^{ij}(t)h_{ij}(t),$$

hence

$$\left.\frac{dH(t)}{dt}\right|_{t=0} = \frac{1}{2}\sum_{i,j}\left.\frac{dg^{ij}(t)}{dt}\right|_{t=0} h_{ij} + \frac{1}{2}\sum_{i,j} g^{ij}\left.\frac{dh_{ij}(t)}{dt}\right|_{t=0},$$

where we write $g^{ij}(0) = g^{ij}$, etc. From

$$\sum_j g^{ij}(t)g_{jk}(t) = \delta_{ik}, \quad g^{ij} = \Lambda^{-2}\delta_{ij},$$

we see that

$$\frac{dg^{ij}(t)}{dt}\Big|_{t=0} = -\Lambda^{-4}\frac{dg_{ij}(t)}{dt}\Big|_{t=0} = -\Lambda^{-4}(E_i \bullet X_j + E_j \bullet X_i).$$

Using $h_{11} = -h_{22}$ and $X_{11}\bullet X_1 = \frac{1}{2}\Lambda_1^2$, $X_{11}\bullet X_2 = -\frac{1}{2}\Lambda_2^2$, etc., we have

$$\frac{1}{2}\sum_{i,j}\frac{dg^{ij}(t)}{dt}\Big|_{t=0} h_{ij} = \gamma\Lambda^{-4}\sum_{i,j} h_{ij}^2 - \Lambda^{-2}[\alpha_1 h_{11} + (\alpha_2+\beta_1)h_{12} + \beta_2 h_{22}].$$

One calculates that

$$\begin{aligned}
\frac{1}{2}\sum_{ij} g^{ij}\frac{dh_{ij}(t)}{dt}\Big|_{t=0} &= \frac{1}{2}\sum_{ij} g^{ij}\frac{dX_{ij}(t)}{dt}\Big|_{t=0}\bullet N + \frac{1}{2}\sum_{ij} g^{ij}X_{ij}\bullet\frac{dN(t)}{dt}\Big|_{t=0}\\
&= \frac{1}{2}\sum_{i}\Lambda^{-2}E_{ii}\bullet N + \frac{1}{2}\sum_{i}\Lambda^{-2}X_{ii}\bullet\frac{dN(t)}{dt}\Big|_{t=0}\\
&= \frac{1}{2}\Lambda^{-2}\,\triangle\, E\bullet N,
\end{aligned}$$

since $\triangle X = 0$. Using $\triangle X_i = 0$ and $N_i\bullet N = 0$, we have

$$\triangle E\bullet N = \triangle\gamma + \gamma\,\triangle N\bullet N + 2[\alpha_1 h_{11} + (\alpha_2+\beta_1)h_{12} + \beta_2 h_{22}].$$

Hence

$$\frac{dH(t)}{dt}\Big|_{t=0} = \gamma\Lambda^{-4}\sum_{ij} h_{ij}^2 + \frac{1}{2}\Lambda^{-2}(\triangle\gamma + \gamma\,\triangle\, N\bullet N).$$

Since $h_{11} = -h_{22}$, $\sum_{ij} h_{ij}^2 = -2\det(h_{ij}) = -2\Lambda^4 K$, where K is the Gauss curvature. By (8.36), $\triangle N = 2K\Lambda^2 N$, thus

$$\frac{dH(t)}{dt}\Big|_{t=0} = \frac{1}{2}\Lambda^{-2}(\triangle\gamma - 2K\Lambda^2\gamma) = \frac{1}{2}(\triangle_X\gamma - 2K\gamma).$$

Since the above formula does not depend on the local coordinates, we have the second variation formula for any variation vector field $E = \alpha X_1 + \beta X_2 + \gamma N$, that is

$$A''(0) = -\int_\Omega \gamma(\triangle_X\gamma - 2K\gamma)dA_0. \tag{20.83}$$

We see from (20.83), as in the first variation, that the second variation does not depend on the tangential part of the variation field E.

Let Ω be a plane domain, consider the Dirichlet eigenvalue problem for the second order elliptic operator $L = \triangle - 2K\Lambda^2$,

$$\begin{cases} Lu + \lambda u = 0, & \text{in} \quad \Omega \\ u = 0, & \text{on} \quad \partial\Omega \end{cases} \tag{20.84}$$

The classical theory of eigenvalues (see Appendix) says that there is a sequence

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n \leq \cdots,$$

$\lambda_n \to \infty$ as $n \to \infty$, such that (20.84) has solution if and only if $\lambda = \lambda_n$ for some $n \geq 1$. Moreover, we can select smooth ϕ_n as the solution of (20.84) when $\lambda = \lambda_n$ (ϕ_n is called the *eigenfunction* corresponding to λ_n), such that $\{\phi_n\}$ is orthonormal in $L^2(\Omega)$ and spans $W_0^{1,2}(\Omega)$. Thus if $\gamma \in W_0^{1,2}(\Omega) \subset L^2(\Omega)$ it can be decomposed as

$$\gamma = \sum_{n=1}^{\infty} a_n \phi_n.$$

and if γ is also smooth, then

$$L\gamma = \sum_{n=1}^{\infty} a_n L\phi_n = -\sum_{n=1}^{\infty} a_n \lambda_n \phi_n.$$

We have that

$$A''(0) = -\int_\Omega \gamma L\gamma \, du^1 \wedge du^2 = \int_\Omega \left(\sum_{n=1}^{\infty} a_n \phi_n\right)\left(\sum_{m=1}^{\infty} a_m \lambda_m \phi_m\right) du^1 \wedge du^2 = \sum_{n=1}^{\infty} a_n^2 \lambda_n.$$

Hence if $\lambda_n > 0$, we will have for any variation vector field $E = \alpha X_1 + \beta X_2 + \gamma N$ with smooth $\gamma \in W_0^{1,2}(\Omega)$, that $A''(0) > 0$, and hence locally X has minimum surface area.

Of course, if L has a negative eigenvalue, say $\lambda_1 < 0$, taking $\gamma = \phi_1$, we have

$$A''(0) = \lambda_1 < 0,$$

and so X cannot have minimum area.

Note that $\Delta_X = \Lambda^{-2}\Delta$ is intrinsically defined on the surface X. Based on the discussion above, we have definition equivalent to that given in the beginning of this section:

Definition 20.1 A minimal surface $X : \Omega \hookrightarrow \mathbf{R}^3$ is *stable* on a precompact domain $U \subset \Omega$ if the first eigenvalue of $L_X = \Delta_X - 2K$ in U is positive. That is, if

$$\begin{cases} L_X u + \lambda u = 0, & \text{in} \quad U \\ u = 0, & \text{on} \quad \partial U \end{cases}$$

has a non-trivial solution, then $\lambda > 0$.

In general, if $\overline{\Omega}$ is not compact, we say that X is stable on Ω if it is stable on any precompact subdomain of Ω.

For a minimal surface $X : \Omega \hookrightarrow \mathbf{R}^3$, the Gauss map $N \to S^2$ is anti-conformal. We can consider N as a surface though it may have finite branch points. The first fundamental form induced by N is

$$|N_1 \wedge N_2|\delta_{ij} = -K\Lambda^2\delta_{ij}.$$

Hence the S^2 Laplacian $\triangle_S$ induced by N on Ω is

$$\triangle_S = -K^{-1}\Lambda^{-2}\triangle = -K^{-1}\,\triangle_X\,.$$

The sphere metric induced by N then is $dS = -KdA_0$ on Ω. Suppose $K < 0$ on Ω, then since N is anti-holomorphic, by the area formula,

$$A''(0) = -\int_\Omega \gamma(\triangle_X\gamma - 2K\gamma)dA_0. = -\int_{N(\Omega)} \#(N^{-1}(x))\gamma(\triangle_S\gamma + 2\gamma)(x)dS(x).$$

Thus the corresponding operator L_S on $N(\Omega)$ is

$$L_S = -K^{-1}L_X = \triangle_S + 2.$$

If $N : U \subset \Omega \to S^2$ is one to one, then clearly $A''(0) > 0$ if and only if all eigenvalues of $\triangle_S$ on $N(\Omega)$ are larger than 2. And the eigenvalue problem becomes

$$\begin{cases} \triangle_S u + (2+\lambda)u = 0, & \text{in} \quad N(U) \\ u = 0, & \text{on} \quad \partial N(U) \end{cases}$$

It is well known that if the area of $N(U)$ is less than 2π, then the first eigenvalue of $\triangle_S$ is larger than 2, thus have proved:

Theorem 20.2 *Let $X : \Omega \hookrightarrow \mathbf{R}^3$ be a minimal surface and $U \subset \Omega$ be such that $N : U \to S^2$ is one to one and the area of $N(U)$ is less than 2π. Then $X : U \hookrightarrow \mathbf{R}^3$ is stable.*

Since N is locally one to one except at points p such that $K(p) = 0$, we see that at any point $p \in \Omega$ such that $K(p) \neq 0$, there is a neighbourhood $U \ni p$, such that $X : U \hookrightarrow \mathbf{R}^3$ is stable.

Note that if N is one to one, then

$$\mathbf{Area}(N(U)) = -\int_U KdA,$$

so if N is one to one on U and the area of $N(U)$ is less than 2π, then $-\int_U KdA < 2\pi$. Barbosa and do Carmo [2] proved:

Theorem 20.3 *If $-\int_U KdA < 2\pi$, then X is stable on U.*

In fact, Barbosa and do Carmo proved a stronger version of Theorem 20.3 in [2]:

Theorem 20.4 *If* **Area**$(N(U)) < 2\pi$, *then* X *is stable on* U.

Theorem 20.3 is stronger than Theorem 20.2 since N is not assumed to be one to one on U. Note that the converse of Theorem 20.3 is not true, there are stable minimal surfaces whose total curvature is less then -2π. See, for example, [61], page 99.

Let $X : M \hookrightarrow \mathbf{R}^3$ be a minimal surface. A *Jacobi field* is a function u defineded on M such that

$$L_X u = 0.$$

Note that each component of N is a Jacobi field. Whenever we have a Jacobi field u on M, we are interested in the *nodal set* $Z := u^{-1}(0) \subset M$ of u. The reason is that each component of $M - Z$ is a domain (*nodal domain*) $\Omega \subset M$ such that on Ω the u does not change sign and it vanishes on $\partial\Omega$. If u is continuous on $\overline{\Omega}$, then by the properties of eigenvalues (see Appendix) the first eigenvalue of L_X on Ω is zero, and any domain $\Omega' \supset \overline{\Omega}$ will have negative first eigenvalue. Thus such Ω and $\Omega' \supset \Omega$ are unstable. By Theorem 20.3, the total curvature of X on Ω is less than or equal to -2π. Similarly, any domain $\Omega' \subset \Omega$ such that $\Omega - \Omega'$ has positive area, will have positive first eigenvalue, and therefore is stable. We will apply these comments in the proof of Shiffman's theorems.

In [4], do Carmo and Peng proved that the only stable complete minimal surface in $\mathbf{R}^3$ is plane. This is a generalized version of Bernstein's theorem, which says that a complete minimal graph (which is stable by Theorem 20.4) must be a plane.

Thus all complete non-planar minimal surfaces $X : M \hookrightarrow \mathbf{R}^3$ are unstable. A measure of how unstable is a surface, is the *index*. If $\Omega \subset M$ is precompact, then index(Ω) is the number of negative eigenvalues of L_X on Ω, counting the multiplicity. Hence the index is the dimension of the subspace of $L^2(\Omega)$ spanned by the eigenfunctions corresponding to negative eigenvalues. The index of M then is defined as

$$\mathrm{index}(M) = \mathrm{lub}_{\Omega \subset \mathrm{M}} \mathrm{index}(\Omega), \tag{20.85}$$

where lub means the least upper bound and Ω is taken over all precompact domains in M.

A theorem of Fischer-Colbrie [19] says that a complete minimal surface $X : M \hookrightarrow \mathbf{R}^3$ has finite index if and only if it has finite total curvature.

Let g and η be the Weierstrass data of a complete minimal surface of finite total curvature $X : M \hookrightarrow \mathbf{R}^3$ and $k = \deg g$. A theorem of Tysk [79] says that

$$\text{index of } M \leq C \cdot k.$$

for some constant C. Tysk [79] proved that C can be taken as $C = 7.68183$. The number 7.68183 is certainly not optimal, since for a catenoid $k = 1$ and the index is also 1, see Theorem 27.8. A good problem then is what is the optimal value of C? A guess is that $C = 1$.

21 The Cone Lemma

Let X_c be the cone in $\mathbf{R}^3$ defined by the equation

$$x_1^2 + x_2^2 = (x_3/c)^2, \quad c \neq 0.$$

The complement of X_c consists of three components, two of which are convex. We label the third region W_c and note that W_c contains $P^0 - \{0\}$, where $P^t = \{x_3 = t\}$ for $t \in \mathbf{R}$. Suppose $M \subset W_c$ is a noncompact, properly immersed minimal annulus with compact boundary.

Note that as $c \to 0$, $X_c - \{0\}$ collapses to a double covering of $P^0 - \{0\}$. Note also that any horizontal plane or vertical catenoid is eventually disjoint from any X_c, hence eventually contained in W_c, no matter how small c is (by "eventually" we mean "outside of a compact set"). Since any embedded complete minimal annular end of finite total curvature is asymptotic to a plane or a catenoid (a graph with logarithmic growth), it follows that, after suitable rotation, such an end is eventually contained in any W_c. By Jorge and Meeks' theorem, Theorem 12.1, it is easy to see that a minimally immersed end of finite total curvature with a horizontal limit tangent plane is also eventually contained in every X_c. The Cone Lemma [29] shows that this property implies that the annular end must have finite total curvature if it is proper. Hence after a rotation if necessary, a proper minimal annular end has finite total curvature if and only if it is eventually contained in every X_c.

Let $A := \{z \in \mathbf{C} \,|\, 1 \leq |z| < \infty\}$.

Theorem 21.1 (The Cone Lemma) *Let $X : A \hookrightarrow \mathbf{R}^3$ be a properly immersed minimal annulus with compact boundary. If $M := X(A)$ is eventually contained in W_c for a sufficiently small c, then X has finite total curvature.*

In order to prove the Cone Lemma we need to introduce the concept of *foliation.*

Definition 21.2 Let M be a C^∞ manifold of dimension 3. A C^k, $1 \leq k \leq \infty$, foliation of M is a set of leaves $\{\mathcal{L}_\alpha\}_{\alpha \in \mathcal{A}}$ in M that satisfies the following conditions:

1. $\{\mathcal{L}_\alpha\}_{\alpha \in \mathcal{A}}$ is a collection of disjoint 2-submanifolds.

2. $\bigcup_{\alpha \in \mathcal{A}} \mathcal{L}_\alpha = M$.

3. For all points $p \in M$ there exists a neighbourhood U of M and class C^kcoordinate system (x_1, x_2, x_3) of U such that $\mathcal{L}_\alpha \cap U$ is empty or is the solution of $x_3 =$ constant in U.

Before proving Theorem 21.1, we will state a fact about the catenoid. Let C be the unit circle in P^0 centred at $(0,0)$. Let C_h be the translate of C in the plane P^h. There is an $h_2 > 0$ such that for $0 < h < h_2$ there are two catenoids bounded by C_{-h} and C_h; one is stable and the other is unstable. While the C_{-h_2} and C_{h_2} bound only one catenoid. When $h > h_2$, there is no catenoid bounded by C_{-h} and C_h. It is known that $h_2 \cong 0.6627435$. For more details, see for example [60], §515.

As we will see later, by Schffman's second theorem, Theorem 29.2, any minimal annulus bounded by C_{-h} and C_h must be a catenoid. Thus there is no minimal annulus bounded by C_{-h} and C_h when $h > h_2$.

We describe a technical result that will be used in the proof of the Cone Lemma. Let γ_t be the circle of radius t in P^0, centred at the origin. Let C_t^ϵ be the stable catenoid whose boundary circles consists of the vertical translates of γ_t by $(0, 0, \pm\epsilon)$. Let Δ_ϵ be the solid torus bounded by subsets of C_2^ϵ, C_4^ϵ, P^ϵ, and $P^{-\epsilon}$. Note that $\partial\Delta_\epsilon$ consists of two planar annuli and the two catenoids C_2^ϵ and C_4^ϵ. Let $K^0 \subset P^0$ be the annulus bounded by γ_2 and γ_4. Let δ be any smooth Jordan curve in K^0 that is homotopic to γ_2 in K^0.

Proposition 21.3 *For $\epsilon > 0$ sufficiently small, Δ_ϵ can be foliated by compact minimal annuli A_t, $2 \le t \le 4$, with the following properties:*

1. *$A_t = C_t^\epsilon$, for t sufficiently close to 2 or to 4;*
2. *Each A_t meets P^0 orthogonally;*
3. *A_3 meets P^0 in a smooth Jordan curve that converges to δ, in the C^0-norm, as $\epsilon \to 0$.*
4. *By selecting suitable δ, the foliation of Δ_ϵ satisfies the following: for any $q \in P^0 - \{0\}$, $|q| > 4$, and any line l through q, we may rotate Δ_ϵ around the x_3-axis so that $\overline{0q}$ intersects $\alpha_3^0 = A_3 \cap P^0$ in a point p, where $T_pA_3 \cap P^0$ is a line parallel to l.*

This proposition is proved in [29], its proof involves several facts about solutions to the Douglas-Plateau problem. Since we are not going to discuss this interesting problem in this lecture notes, we will skip the proof. Readers who are interested in the Douglas-Plateau problem can refer to [55], [56], [53] and [54].

Proof of Theorem 21.1. We begin by normalizing the problem.

Let $C(1)$ be the vertical catenoid with waist-circle of radius 1 and denote by C the compact component of $W_c \cap C(1)$. Choose $c > 0$ small enough so that C is a radial graph and foliate W_c by the leaves $\{t \cdot C\}$, $0 < t < \infty$. For convenience, we write C_t for $t \cdot C$.

Claim 1. After a homothetic shrinking of M (but not of the foliation) and a discarding of a compact subset of M:

1. $\partial M \subset C_1 = C$;
2. $M \subset \bigcup_{1 \le t < \infty} C_t$;

3. $M \cap C_t$ consists of a single closed immersed curve for $t \geq 1$.

Proof of Claim 1. Choose T_0 large enough so that $\partial M = X(\partial A)$ lies in the bounded component of $W_c - C_{T_0}$. Without loss of generality, we may assume that C_{T_0} intersects M transversally. Denote by Z the closure of the unbounded component of $W_c - C_{T_0}$; that is, $Z = \bigcup_{t \geq T_0} C_t$. Define $f : Z \to [T_0, \infty)$ to be the function whose level set at t is C_t. Since the C_t are minimal surfaces, the maximum principle (Theorem 4.4) implies that $f \circ X|_{X^{-1}(Z)}$ has no interior maxima or minima. Moreover, by Theorem 4.4, the intersection of two minimal surfaces in a neighbourhood of a point of tangency consists of j curves, $j \geq 2$, intersecting at that point in equal angles. This implies that $f \circ X|_{X^{-1}(Z)}$ has only index-1 critical points with multiplicity equal to $j-1$. Therefore, f may have at most $k-1$ critical points, where k is the first Betti number of $X^{-1}(Z)$, by elementary Morse theory. Consequently, outside of a compact subset of $X^{-1}(Z)$, $f \circ X|_{X^{-1}(Z)}$ is free of critical points. This means that there exists a $T_1 > 0$ such that for $t \geq T_1$, $C_t \cap M$ consists of a finite number of closed immersed curves. Since M has one end, each $C_t \cap M$, $t > T_1$, must consist of a single closed immersed curve. By a similar argument as in the proof of Lemma 11.9 and Theorem 12.1, this time using the maximum principle for minimal surfaces, each $X^{-1}(C_t)$ is a homotopically non-trivial Jordan curve in A and $A' = \bigcup_{t > T_1} X^{-1}(C_t) \subset A$ is an annulus. Conformally $A' \cong A$ since they are both equivalent to the punctured disk.

Discarding the compact subsurface $M \cap (\bigcup_{t \leq T_1} C_t)$ we get $X(A')$. Now rescaling by a factor of T_1^{-1}, we satisfy conditions 1, 2, and 3 by denoting $M = X(A')$.

We will write A' as A for convenience.

Because M is properly immersed and projection from W_c to $P^0 - \{0\}$ is also proper, the projection $\Pi \circ X : A \to P^0 - \{0\}$ is a proper map.

Claim 2. The mapping Π is a submersion outside of a compact set, provided $c > 0$ is sufficiently small.

Before proving **Claim 2**, we will show that the theorem follows from it. By Theorem 19.2, the Gauss map N of X either takes on all points of $\mathbf{C} \cup \{\infty\}$, except for at most a set of capacity zero, or it has a unique limiting value. In the latter case, X must have finite total curvature as remarked in Remark 19.3. But **Claim 2** implies that, outside of some compact set B, the Gauss map of $X : A - X^{-1}(B) \hookrightarrow \mathbf{R}^3$ will not take values in the great circle $S^1 \subset S^2$. Since X is proper, $X^{-1}(B)$ is compact. Taking the connected component W in $A - X^{-1}(B)$ which is connected to ∞, we infer that the image $N(W)$ is contained in a hemisphere, so the first case of Theorem 19.2 is precluded. Hence, X has finite total curvature.

Proof of Claim 2. In this proof, we will need at several points to restrict the size of $c > 0$. At each point, we will continue to assume that $M \subset W_c$. Let $\Delta := \Delta_\epsilon$ be the foliated annulus from Proposition 21.3. Reduce the size of c so that the Δ has its top and bottom boundaries disjoint from W_c. Let K be the intersection of W_c with the vertical cylinder over the disk of radius 4 in P^0. Note $\Delta \cap W_c \subset K$. Shrink $c > 0$ even more if necessary, so that the following is true. If the distance from $q \in K$ to P^0

is τ, then the vertical translation of Δ by τ has the property that its top and bottom boundaries are disjoint from W_c.

Suppose now that $\Pi : M \to P^0$ is not a submersion outside of any compact set. This is equivalent to the statement that the points on M with vertical tangent plane form an unbounded set. In particular, there is a point $p \in M - K$ whose tangent plane is vertical.

If $\hat{p}$ is the projection of p onto P^0, we can assume that $|\hat{p}| > 4$. According to Proposition 21.3, we may rotate M about the vertical axis so that the following holds: the line $\overline{0\hat{p}}$ intersects α_3 at a point where the tangent line to α_3 is parallel to $T_pM \cap P^0$. We perform this rotation of M and shrink M so that $\hat{p}$ actually lies on α_3. Since the original M satisfied the conditions of **Calim 1**, and the foliation $\{C_t \,|\, 0 < t < \infty\}$ is rotationally symmetric, it follows easily that the modified M also satisfies condition 3 of **Claim 1**. We also discard $M \cap \bigcup_{t<1} C_t$. We will refer to this modified surface as M. It is clear that to prove the claim, it is sufficient to prove it for this modified surface.

Vertically translate Δ so that $\hat{p}$ coincides with p, and label this translated torus $\hat{\Delta}$. Also translate the foliation A_t of Δ to be a foliation $\hat{A}_t$ of $\hat{\Delta}$. Recall that we have chosen $c > 0$ small enough so that the top and bottom boundaries of $\hat{\Delta}$ are disjoint from W_c. Also recall that for t near 2 and 4, the leaves of the foliation of $\hat{\Delta}$ are catenoids. Make c smaller, if necessary, to insure that these catenoids are radial graphs.

We will now extend $\hat{A}_t$ to be a smooth foliation of a region that contains $\bigcup_{4 \le t < \infty} C_t$. Let $\hat{A}_t = \frac{t}{4}\hat{A}_4$, $t \geq 4$, be the homothetic expansion of $\hat{A}_t$. The boundary $\partial\hat{A}_2$ consists of two concentric circles. By making $c > 0$ smaller if necessary, we may insure that C is a subset of a stable catenoid $\hat{C}$, whose boundaries are concentric circles exterior to W_c on the parallel planes that contain $\partial\hat{A}_2$. We interpolate between $\partial\hat{A}_2$ and $\partial\hat{C}$ with a smooth family, each member of which is a pair of circles centred on the vertical axis. The vertical distance between circles in each pair is an increasing function of t, $1 \leq t \leq 2$. Note that each pair of circles bounds a unique stable catenoid. Label that catenoid $\hat{A}_t$, $1 \leq t \leq 2$. It is evident that this family may be chosen to insure that the resulting foliation $\{\hat{A}_t \,|\, 1 \leq t < \infty\}$ is smooth. By construction, $M \subset \bigcup_{1 \le t < \infty} \hat{A}_t$, $\partial M \subset \hat{A}_1$, and $\hat{A}_3$ is tangent to M at p.

Let h be the smooth function, defined on the union of the leaves $\hat{A}_t$, whose level set at t is $\hat{A}_t$. Restriction of h to M yields a proper function $h \circ X$ on A that satisfies $h \circ X \geq 1$ and is equal to 1 precisely on ∂A. Repeating the argument in the proof of **Claim 1** will show that *all* the critical points of h have index 1, possibly with multiplicity. However, A is an annulus and $(h \circ X)^{-1}(1) = \partial A$, so by elementary Morse theory, it follows that $h \circ X$ can have *no* critical points. But $\hat{A}_3$ is tangent to M at $p = X(q) \in M$, which shows that q is a critical point of $h \circ X$. This contradiction completes the proof of **Claim 2** and also of the theorem. □

Remark 21.4 Let X_c and W_c be as in the Cone Lemma, and M be a proper, connected, complete minimal surface with compact boundary. Then by Theorem 16.1, M is eventually disjoint from X_c is equivalent to the fact that M is eventually contained in W_c. Thus suppose that M is a proper, connected, complete minimal surface with compact boundary and finite topology. If after a rotation if necessary, M is eventually disjoint from X_c, for some $c > 0$ sufficiently small, then M has finite total curvature.

22 Standard Barriers and The Annular End Theorem

The study of ends of complete minimal surfaces leads to the Annular End Theorem of Hoffman and Meeks [29], and its corollaries.

Theorem 22.1 (The Annular End Theorem) *If M is a properly embedded minimal surface in $\mathbf{R}^3$, then at most two distinct annular ends of M can have infinite total curvature.*

To prove the Annular End Theorem, we need some preparation. First we introduce the notion of a *standard barrier.*

Definition 22.2 A *standard barrier* in $\mathbf{R}^3$ is one of the following two minimal surfaces with boundary: the complement of a disk in a plane in $\mathbf{R}^3$; a component of the complement of a simple, closed, homotopically nontrivial curve on a catenoid.

We will say that a surface $M \subset \mathbf{R}^3$ *admits a standard barrier* if it is disjoint from some standard barrier. We will use the word "eventually" to mean "outside of some sufficiently large compact set of $\mathbf{R}^3$". Thus, two surfaces $M \subset \mathbf{R}^3$ and $N \subset \mathbf{R}^3$ are "eventually disjoint" if they have compact intersection. It is straightforward to see that M admits a standard barrier if and only if it is eventually disjoint from some standard barrier.

Given a standard barrier S and a ball B large enough to contain ∂S, it is clear that $S - B$ divides $\mathbf{R}^3 - B$ into two components. Two surfaces M, $N \subset \mathbf{R}^3$ will be said to be *separated by a standard barrier* if such an S and B can be found so that M and N eventually lie in different components of $\mathbf{R}^3 - (B \cup S)$.

Two disjoint standard barriers divide the complement of a sufficiently large ball $B \subset \mathbf{R}^3$ into three components, only one of which contains portions of both barriers on its boundary. A surface $M \subset \mathbf{R}^3$ that eventually lies in such a component will be said to *lie between two standard barriers.* After a rotation of $\mathbf{R}^3$, if necessary, the region of $\mathbf{R}^3$ between two standard barriers eventually lies in the complement of any $X_c = \{x_1^2 + x_2^2 = (x_3/c)^2\}$ (in the component that contains $P^0 - \{0\}$) for any $c > 0$, no matter how small. It follows from Theorem 21.1 and Remark 21.4 that:

Proposition 22.3 *If $X : M \hookrightarrow \mathbf{R}^3$ is a properly immersed complete minimal surface of finite topology, with compact boundary ∂M, and eventually lies between two standard barriers, then M must have finite total curvature.*

Our strategy in proving the Annular End Theorem is to trap ends between standard barriers. The next lemma contains the critical technical construction.

Before proving the lemma, we introduce the notion of *linking number.*

Definition 22.4 Let γ be an embedded curve in $\mathbf{R}^3$ such that $\mathbf{R}^3 - \gamma$ is homotopic to $\mathbf{R}^2 - \{0\}$. The first homology group of $\mathbf{R}^3 - \gamma$ is $H_1(\mathbf{R}^3 - \gamma) \cong \mathbf{Z}$. Let $\beta \subset \mathbf{R}^3 - \gamma$ be a closed curve. Then the *linking number* of β with γ is the homology class of $[\beta]$ in $H_1(\mathbf{R}^3 - \gamma)$. This is an integer, denoted by $l(\beta, \gamma)$.

If we use the homology group $H_1(\mathbf{R}^3 - \gamma; \mathbf{Z_2})$, then $l_2(\beta, \gamma) = 0$ or 1.

Intuitively, if β is a Jordan curve, then $l(\beta, \gamma) \neq 0$ means that any disk $D \subset \mathbf{R}^3$, such that $\partial D = \beta$, intersects γ. In the homology group $H_1(\mathbf{R}^3 - \gamma; \mathbf{Z_2})$, if γ is a proper curve $\gamma : \mathbf{R} \to \mathbf{R}^3$, then $l_2(\beta, \gamma) \neq 0$ if and only if there is a disk $D \subset \mathbf{R}^3$ such that $\partial D = \beta$ and D intersects γ an odd number of times.

Lemma 22.5 *Suppose M is a properly embedded, piecewise-smooth surface that is a smooth minimal surface outside of some ball and that has at least two ends. Let $\gamma : \mathbf{R} \to M$ be a proper curve that diverges into two distinct ends of M, depending on whether $t \to +\infty$ or $t \to -\infty$. Then M admits a standard barrier whose boundary has linking number 1 with γ.*

Proof. Let $B \subset \mathbf{R}^3$ be a ball large enough to contain the nonsmooth, nonminimal portion of M, and expand it, if necessary, so that the ends of M in question correspond to distinct components of $M - B$. If one has such a ball, any larger one will have the same property. We may also choose B so that ∂B intersects M transversally. Suppose that M_1 and M_2 are the two components of $M - B$ that contain the unbounded components of $\gamma - B$. Since the proper arc γ intersects ∂M_1 and ∂M_2 an odd number of times, we can choose exactly one component of $\mathbf{R}^3 - M$ whose closure, $\mathcal{N}$, has the following property: the arc γ has odd linking number with any 1-cycle in $\mathrm{Int}(\mathcal{N})$ homologous to ∂M_1 in $\mathcal{N}$. Note that ∂M_1 is not homologous to zero in $\mathcal{N}$.

Let $\Sigma_1 \subset \cdots \subset \Sigma_n \subset \cdots$ be an exhaustion of M_1 by smooth compact subdomains, with $\partial M_1 \subset \partial \Sigma_1$. Let $\tilde{\Sigma}_i$ denote a least-area integral current (roughly speaking, piecewise C^1 minimal surface) in $\mathcal{N}$ with boundary $\partial \Sigma_i$, which is $\mathbf{Z_2}$-homologous to Σ_i (i.e., $[\Sigma_i] = [\tilde{\Sigma}_i]$ in $H_1(\mathcal{N}; \mathbf{Z_2})$). Since $\Sigma_i \cup \tilde{\Sigma}_i$ is a boundary in $\mathcal{N}$, $\tilde{\Sigma}_i$ is orientable. Interior regularity of least-area currents (see, for example, [75]) shows that $\tilde{\Sigma}_i \cap \mathrm{Int}(\mathcal{N})$ is a regular embedded minimal surface. Since $\partial \mathcal{N} - \partial B$ has zero mean curvature, the maximum principle and the extension theorem for minimal surfaces imply that either $\tilde{\Sigma}_i \cap (\mathcal{N} - \partial B)$ is regular and $\tilde{\Sigma}_i \cap M_1 = \partial \Sigma_i$ or $\tilde{\Sigma}_i \subset M_1$. Standard compactness theorems imply that a subsequence of the surfaces $\{\tilde{\Sigma}_i\}$ converges to a least-area orientable surface $\Sigma \subset \mathcal{N}$ with $\partial \Sigma = \partial M_1$. Suppose, for the moment, $\Sigma \cap M = \partial \Sigma$.

The surface $\Sigma - \partial B$ is a stable, properly embedded, orientable minimal surface in $\mathbf{R}^3$ with compact boundary and hence has finite total curvature (see, [57], Theorem 1 as well as [19]). Hence, Σ has a finite number, say n, of ends, and each end is asymptotic to a plane or to a catenoid. Let S_R be the sphere of radius R centred at the origin. For R sufficiently large, by Theorem 12.1, $\Sigma \cap S_R$ consists of n parallel almost-great-circles, each of which is the boundary of one of the annular ends of Σ.

By our choice of $\mathcal{N}$, γ has odd linking number with one of the curves in $\Sigma \cap S_R$, and hence has linking number 1 with one of the annular ends of $\Sigma - B_R$. Call this annular end F.

By the weak maximum principle at infinity (Remark 15.3), since $F \cap M = \emptyset$, $\mathbf{dis}(F, M) > 0$. On the other hand, F is asymptotic to an end, C', of a plane or a catenoid. Hence, C' contains a subend, C, whose boundary is a circle that has linking number 1 with γ. Moreover, C is contained in the interior of $\mathcal{N}$. This proves the lemma in the case $\Sigma \cap M = \partial\Sigma$.

In case $\Sigma \subset M$, the extension theorem implies that $\Sigma = M_1$, which means that $\gamma \cap M_1$ is eventually contained in a catenoid-type end or a flat end, say $C'' \subset \Sigma$. This is, in fact, the easier case and can be treated directly, but we prefer to reduce it to the previous case. We may choose C'' so that it is as close as desired to a standard barrier. Moving C'' a small amount in the direction of its limiting normal produces a minimal surface that is disjoint from M_1 and its boundary has linking number with γ equal to either 0 or 1. Move in the direction that makes the linking number equal to 1. The maximum principle at infinity shows that if C'' is moved a small amount, then it is also disjoint from M. We can now apply the argument in the previous case to complete the proof. □

Corollary 22.6 *Suppose M_1, M_2, and M_3 are three pairwise-disjoint, properly embedded minimal surfaces in $\mathbf{R}^3$, each of which has compact boundary and one end. Then at least one of the surfaces lies between two standard barriers.*

Proof. Choose a ball $B \subset \mathbf{R}^3$ that is big enough to contain $\bigcup_i \partial M_i$. The ball can be chosen to intersect $\bigcup M_i$ transversally. After removal of $B \cap M_i$ from each M_i, we may assume that $\partial M_i \subset \partial B$. The curves $\bigcup_{i=1}^3 \partial M_i$ bound a region S on ∂B with the property that the boundary of at least one component of S touches the boundary of more than one of M_i. We will refer to this component as $\mathcal{S}$ and relabel the M_i, if necessary, so that both $\partial\mathcal{S} \cap \partial M_1$ and $\partial\mathcal{S} \cap \partial M_2$ are nonempty.

Let $M = \mathcal{S} \bigcup_i M_i$. We intend to apply Lemma 22.5 to M. Toward that end, choose a proper curve $\gamma : \mathbf{R} \to M$, with $\gamma(\mathbf{R}) \cap \mathcal{S}$ consisting of a single connected arc in $\mathcal{S}$ from ∂M_1 to ∂M_2. We may assume that γ diverges in M_1 (resp. M_2) as $t \to +\infty$ (resp. $t \to -\infty$). By Lemma 22.5, there exists a standard barrier disjoint from M whose boundary has linking number 1 with γ.

We now expand the ball B to be large enough to contain the boundary of this barrier and discard from each M_i the subset $M_i \cap B$. Similarly, let C_1 be the component of the barrier exterior to B. C_1 is still a barrier for M, and γ has linking number 1 with ∂C_1. Moreover, C_1 divides $\mathbf{R}^3 - B$ into two components. Clearly, M_1 and M_2 are in different components. Without loss of generality, we may assume that M_3 is in the same component as M_1.

The curve ∂C_1 divides ∂B into two disks. Let D be the disk containing $\partial M_1 \cup \partial M_3$. We now repeat the construction in the previous paragraph. This time, let S' be a region of D bounded by $\partial C_1 \cup \partial M_1 \cup \partial M_3$. Since ∂C_1 is almost a great circle on ∂B and M_1 and M_3 are in the same component of $\mathbf{R}^3 - C_1$, ∂M_1 and ∂M_3 are in the same half-sphere bounded by ∂C_1, therefore, S' has a component, say $\mathcal{S}'$, with boundary points on both ∂M_1 and ∂M_3. Let $M' = M_1 \cup M_2 \cup M_3 \cup C_1 \cup \mathcal{S}'$. Choose a proper arc $\gamma' \subset M'$ whose intersection with S' lies in $\mathcal{S}'$ and consists of connected arc from ∂M_1 to ∂M_3. (Note that , since $(\partial B - D) \cap \gamma' = \emptyset$, $\partial D = \partial C_1$, γ' has linking number 0 with ∂C_1.) Lemma 22.5 implies that there exists another standard barrier, C_2, that is disjoint from $M_1 \cup M_2 \cup M_3 \cup C_1 \cup \mathcal{S}'$ and whose boundary has linking number 1 with γ'.

Expand B again so that $\partial C_2 \subset B$. It is possible to do this so that ∂B meets M transversally. Note that $C_2 \cap \partial B$ is a single closed curve. Again, we discard from M_1, M_2, M_3, C_1 and C_2 the intersection of those surfaces with B. Therefore, all of these surfaces have their boundaries on ∂B.

The barrier C_2 divides $\mathbf{R}^3 - B$ into two regions, as does the barrier C_1. Since they are disjoint, $C_1 \cup C_2$ divides $\mathbf{R}^3 - B$ into three components. Let T_1 (resp. T_2) be the component of $\mathbf{R}^3 - (C_1 \cup C_2)$ whose boundary contains C_1 but is disjoint from C_2 (resp. contains C_2 but is disjoint from C_1). Let F be the third component, whose boundary contains $C_1 \cup C_2$. Since γ' has linking number 1 with ∂C_2, C_2 must separate M_1 from M_3. But clearly, $M_1 \cup M_3 \subset T_1 \cup F$. Hence, either M_1 or M_3 lies in F. That is, either M_1 or M_3 lies between two standard barriers. □

Remark 22.7 Lemma 22.5 and Corollary 22.6 hold even when the minimal surfaces in question are properly immersed rather than properly embedded. The proofs are essentially the same as the proof of the embedded case. See [50] for these types of arguments.

Proof of Theorem 22.1. If M has two or fewer annular ends, there is nothing to prove. If M has three or more annular ends, we apply Corollary 22.6 to any choice of three annular ends of M. It implies that one of them lies between two standard barriers. But by Proposition 22.3, this end must have finite total curvature. Thus, M can have at most two annular ends of infinite total curvature. □

Corollary 22.8 *Suppose M is a properly embedded complete minimal surface in $\mathbf{R}^3$. Then M can have at most two annular ends that are not conformally diffeomorphic to a punctured disk. In particular, if M has finite topology, then M is conformally equivalent to a closed Riemann surface from which a finite number of points, and zero, one, or two pairwise-disjoint closed disks, have been removed.*

Proof. Since any complete annular end of finite total curvature must be conformally a punctured disk, the conclusion is obvious. □

23 Annular Ends Lying above Catenoid Ends

The Cone Lemma (Theorem 21.1) gives a criterion for a minimal surface to have finite total curvature by looking at the picture of its image. In this section we will give another such criterion due to Fang and Meeks [18].

Consider the family of catenoids

$$C_t = \{(x, y, z) \in \mathbf{R}^3 \mid t^2x^2 + t^2y^2 = \cosh^2(tz)\},$$

for $t > 0$. We will show that a properly immersed, complete minimal annulus with one compact boundary that lies above some C_t must have finite total curvature. More precisely:

Theorem 23.1 *Let*

$$W_t = \{(x, y, z) \in \mathbf{R}^3 \mid t^2x^2 + t^2y^2 \leq \cosh^2(tz),\ z \geq 0\}.$$

Suppose $X\colon M \to \mathbf{R}^3$ is a complete, proper minimal immersion of an annulus with smooth compact boundary such that the image is contained in W_t for some $t > 0$. Then M has finite total curvature.

We will break the proof of Theorem 23.1 into several lemmas. First let us fix the notation.

Let C be a catenoid in $\mathbf{R}^3$ with the z-axis as symmetry axis. Let W be the closure of the component of $\mathbf{R}^3 - C$ that contains the z-axis. Let $\mathbf{H} = \{(x, y, z) \in \mathbf{R}^3 \mid z > 0\}$ and $\overline{\mathbf{H}}$ be its closure.

Conformally we can write $M = \{\zeta \in \mathbf{C} \,|\, 0 < r_1 \leq |\zeta| < r_2\}$. The smooth compact boundary of X corresponding to $|\zeta| = r_1$. Complete means that $X \circ \gamma$ has infinite arc length as γ diverges to $|\zeta| = r_2$. Let $A = X(M)$.

After homothetically shrinking or expanding C and A, we can assume that C is the standard catenoid, i.e., C has the conformal structure of $\mathbf{C} - \{0\}$ and is embedded in $\mathbf{R}^3$ as follows:

$$F : \mathbf{C} - \{0\} \hookrightarrow \mathbf{R}^3$$

$$F(\zeta) = \Re\left(\int_1^\zeta \omega_1, \int_1^\zeta \omega_2, \int_1^\zeta \omega_3\right) + (-1, 0, 0),$$

where

$$\omega_1 = \frac{1}{2}\frac{(1-\zeta^2)}{\zeta^2}\,d\zeta, \quad \omega_2 = \frac{i}{2}\frac{(1+\zeta^2)}{\zeta^2}\,d\zeta, \quad \omega_3 = \frac{d\zeta}{\zeta}.$$

The Gauss map of C is

$$N^C(\zeta) = \frac{1}{1+|\zeta|^2}(2\Re\zeta, 2\Im\zeta, |\zeta|^2 - 1).$$

All lemmas in the following having the same assumptions as for Theorem 23.1.

The first lemma is the key point of the proof of Theorem 23.1.

Lemma 23.2 *Let $p \in \mathrm{Int}(M)$ and P be the tangent plane of A through $X(p)$ and suppose $P \cap \partial A = \emptyset$. Then the component of $P \cap A$ that contains $X(p)$ is noncompact.*

Proof. Since A is noncompact, we may assume that A is not part of a plane. If $\vec{n}$ is the normal vector of P, then $h = (X - X(p)) \bullet \vec{n}$ is a harmonic function on M and $X^{-1}(A \cap P) = h^{-1}(0)$. Since h is harmonic and $h^{-1}(0) \subset \mathrm{Int}(M)$, the maximum principle implies that every component of $h^{-1}(0)$ is a one-dimension analytic subvariety of M. Suppose that the component of $P \cap A$ containing $X(p)$ is compact. Let Δ denote the preimage of this component on M. Note that Δ is compact since X is proper. Furthermore, by Corollary 4.6, p is a critical point of the harmonic function h, thus Δ is a *singular* compact analytic one-dimensional variety in M. But the complement of any such singular variety in the annulus M disconnects M into at least three components. One of the components of $M - \Delta$ has $\{|\zeta| = r_2\}$ as a component of its boundary, another contains $\{|\zeta| = r_1\}$ and at least one, say Σ, has compact closure $\overline{\Sigma}$ and $h|\partial\overline{\Sigma} = 0$. By the maximum principle, $X(\Sigma) \subset P$, which forces A to be contained in the plane P. This contradiction proves the lemma. □

The second lemma clarifies the conformal type of M and gives a specific representation of the third coordinate function X_3.

Lemma 23.3 *If $A \subset W \cap \overline{\mathbf{H}}$ then A contains a proper subannulus A' that is conformally parametrized by $E = \{\zeta \in \mathbf{C} \mid |\zeta| \geq 1\}$. Moreover, in this parametrization $G : E \hookrightarrow \mathbf{R}^3$ of A', the third component of G is*

$$G_3(\zeta) = a \log |\zeta| + b$$

for some a, $b \in \mathbf{R}$, $a > 0$, $b \geq 0$.

Proof. Since $X = (X_1, X_2, X_3) : M \hookrightarrow \mathbf{R}^3$ is a proper minimal immersion and $A = X(M) \subset W \cap \overline{\mathbf{H}}$, $X_3 : M \to \mathbf{R}$ is a proper harmonic function.

We claim that X_3 is unbounded. In fact, if X_3 is bounded, then $A = X(M)$ is contained in a compact set, contradicting the fact that X is proper.

Then by properness and $A \subset W \cap \mathbf{H}$, $X_3(\zeta) \to \infty$ as $|\zeta| \to r_2$. If $r_2 < \infty$, letting $g_{ij} = e^{X_3}\delta_{ij}$, we get a complete flat metric on M. By Proposition 10.6 this is impossible. Thus $r_2 = \infty$.

We claim that if $X_3(\zeta) > c := \max_{\zeta \in \partial M}\{X_3(\zeta)\}$, then $DX_3(\zeta) \neq (0,0)$. In fact, if $DX_3(\zeta) = (0,0)$, then the tangent plane P of A at $X(\zeta)$ is horizontal, hence by Lemma 23.2 $A \cap P$ should have an uncompact component, which contradicts that $A \subset W$ and X is proper.

Now let $t > c_1 > c$. Then $\gamma = X_3^{-1}(c_1)$ and $\gamma_t = X_3^{-1}(t)$ are compact one-dimensional submanifolds of M and thus are Jordan curves. The annulus A_t bounded by γ and γ_t is conformally $M_{R(t)} := \{1 \leq |\zeta| \leq R(t)\}$ for some $R(t) > 1$. Let $f_t : A_t \to M_{R(t)}$ be the conformal diffeomorphism.

Solving a Dirichlet problem on $M_{R(t)}$ we have

$$X_3 \circ f_t^{-1}(\zeta) = c_1 + \frac{t - c_1}{\log R(t)} \log |\zeta|.$$

This shows that for any $t > s > c_1$, $f_t(\gamma_s)$ is the circle

$$|\zeta| = R(t)^{(s-c_1)/(t-c_1)},$$

hence f_t sends A_s to $M_{R(s)}$, where

$$R(s) = R(t)^{(s-c_1)/(t-c_1)}.$$

In particular,

$$\frac{t - c_1}{\log R(t)} = \frac{s - c_1}{\log R(s)}.$$

Since the modulus of A_s must be $R(s)$, we know that $f_t|_{A_s} = f_s$. Thus we can define a conformal diffeomorphism

$$f : \bigcup_{t \geq c_1} A_t \to E := \{\zeta \in \mathbf{C} \,|\, |\zeta| \geq 1\},$$

such that

$$X_3 \circ f^{-1}(\zeta) = c_1 + a \log |\zeta|, \quad a = \frac{t - c_1}{\log R(t)}, \quad \text{for any} \quad t > c_1.$$

Taking $b = c_1$ and $G = X \circ f^{-1}$, we have proved the lemma. □

Suppose A' is the subannulus of A described in Lemma 23.3. Since A and A' both have finite total curvature or both have infinite total curvature, we will assume, without loss of generality, that $A = A'$.

Suppose now that A has infinite total curvature. We will exhibit a family of tangent planes P_n of A at $G(p_n)$ such that the component of $P_n \cap A$ containing $G(p_n)$ is compact. Furthermore, for n large enough, $P_n \cap \partial A = \emptyset$. The existence of such tangent planes contradicts Lemma 23.2.

For the part of C in $\overline{\mathbf{H}}$ we have the following non-parametric expression: $x^2 + y^2 = \cosh^2 z$, $z \geq 0$. Hence, at any point $p = (x, y, z) \in C \cap \overline{\mathbf{H}}$, the normal vector is

$$N^C(p) = \frac{1}{\sqrt{1 + z_x^2 + z_y^2}}(-z_x, -z_y, 1),$$

where $z_x = 2x/\sinh 2z$, $z_y = 2y/\sinh 2z$, and

$$1 + z_x^2 + z_y^2 = (\sinh^2 2z + 4\cosh^2 z)/\sinh^2 2z = [4\cosh^2 z(\sinh^2 z + 1)]/\sinh^2 2z$$

$$= 4\cosh^4 z/(4\cosh^2 z \sinh^2 z) = \cosh^2 z/\sinh^2 z.$$

Suppose $p = (x, y, z) \in C \cap \overline{\mathbf{H}}$. Let $\theta(p)$ be the angle such that

$$\cos\theta(p) = N^C(p)\bullet(0,0,1) = \frac{1}{\sqrt{1 + z_x^2 + z_y^2}} = \frac{\sinh z}{\cosh z}.$$

Then

$$\sin\theta(p) = \sqrt{1 - \cos^2\theta(p)} = \frac{1}{\cosh z}.$$

Thus $\sin\theta(p)$ is independent of x and y. We denote it by $\sin\theta(z)$. For $p_0 = (x_0, y_0, z_0) \in A \cap W \cap \overline{\mathbf{H}}$, $z_0 \geq 1$, consider the solid cylinders

$$L^{z_0} = \{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 \leq \cosh^2(z_0 + 1)\},$$

$$L_1^{z_0} = \{(x, y, z) \in L^{z_0} \mid z_0 - 1 \leq z \leq z_0 + 1\}.$$

If P is a plane passing through $p_0 = (x_0, y_0, z_0)$ and ν_P is the normal vector of P, define $-\pi/2 \leq \Psi_P \leq \pi/2$ by the formula $\cos\Psi_P = |\nu_P\bullet(0,0,1)|$.

Lemma 23.4 *If z_0 is large and*

$$|\Psi_P| < \frac{1}{16\cosh z_0} = \frac{\sin\theta(z_0)}{16},$$

then the component of $P \cap A$ that contains p_0 is compact and $P \cap \partial A = \emptyset$.

Proof. Since $p_0 = (x_0, y_0, z_0) \in L_1^{z_0}$, for any $(x, y, z) \in P \cap \partial L^{z_0}$ we have

$$|z - z_0| \leq 2\cosh(z_0 + 1)\tan|\Psi_P| = 2\cosh(z_0 + 1)\frac{\sin|\Psi_P|}{\cos|\Psi_P|}.$$

Since $\cos|\Psi_P| > \frac{1}{2}$ and $|\Psi_P| < \frac{1}{16\cosh z_0}$,

$$|z - z_0| < 4\frac{\cosh(z_0 + 1)}{16\cosh z_0}.$$

Note that $\cosh(z_0+1) = \cosh z_0 \cosh 1 + \sinh z_0 \sinh 1$, $\sinh 1 < \cosh 1 < 2$, and $\sinh z_0 < \cosh z_0$. Hence, $\cosh(z_0+1) < 4\cosh z_0$, and so $|z - z_0| < 1$. Hence, $P \cap \partial L^{z_0} = P \cap \partial L_1^{z_0}$ and $P \cap L^{z_0} = P \cap L_1^{z_0}$. This implies that the component γ of $A \cap P$ that contains p_0 must be compact (since $\gamma \subset P \cap L_1^{z_0}$ and $L_1^{z_0}$ is compact).

Let $z_0 - 1 > \max_{x \in \partial A}\{|x|\}$, then clearly $P \cap \partial A = \emptyset$. $\square$

Now we prove Theorem 23.1.

Proof of Theorem 23.1. Assume A has infinite total curvature. Let $g : E \to \mathbf{C} \cup \{\infty\}$ be the Gauss map of A composed with stereographic projection. Similarly define $\tilde{g} : \mathbf{C} - \{0\} \to \mathbf{C} \cup \{\infty\}$ to be the Gauss map of C composed with stereographic projection. Recall, in fact, that in our original parametrization F of C, $\tilde{g}(\zeta) = \zeta$ for $\zeta \in \mathbf{C} - \{0\}$.

Since A has infinite total curvature, g has an essential singularity at ∞. Recall that the Gauss map of C is

$$N^C(\zeta) = \frac{1}{1+|\zeta|^2}(2\Re\zeta, 2\Im\zeta, |\zeta|^2 - 1)$$

for $\zeta \in E$, and the Gauss map of A is

$$N^A(\zeta) = \frac{1}{1+|g(\zeta)|^2}(2\Re g(\zeta), 2\Im g(\zeta), |g(\zeta)|^2 - 1).$$

Also, recall that $\sin\theta(x,y,z) = \frac{1}{\cosh z}$. For any $(x,y,z) = F(\zeta)$, $\cos\theta(z) = N^C \bullet (0,0,1) = \frac{|\zeta|^2-1}{1+|\zeta|^2}$, so

$$\sin\theta(z) = \sqrt{1-\cos^2\theta(z)} = \frac{2|\zeta|}{1+|\zeta|^2}. \tag{23.86}$$

Similarly define the angle $-\pi/2 \le \phi(\zeta) \le \pi/2$ such that $\cos\phi(\zeta) = N^A \bullet (0,0,1) = \frac{|g(\zeta)|^2-1}{1+|g(\zeta)|^2}$. Then

$$\sin\phi(\zeta) = \sqrt{1-\cos^2\phi(\zeta)} = \frac{2|g(\zeta)|}{1+|g(\zeta)|^2}. \tag{23.87}$$

Since $z = G_3(\zeta) = a\log|\zeta| + b = F_3(\zeta^a \cdot \exp b)$, for some $a > 0$, $b \ge 0$,

$$\frac{\sin\phi(\zeta)}{\sin\theta(z)} = \frac{|\zeta^a \cdot \exp b|}{|g(\zeta)|}\left(\frac{1+1/|\zeta^a \cdot \exp b|^2}{1+1/|g(\zeta)|^2}\right). \tag{23.88}$$

Choose a positive integer $m > a$. Since $(\zeta^m \cdot \exp b)/g(\zeta)$ has an essential singularity at ∞, there is a divergent sequence $\{\zeta_n\}$ such that $|\zeta_n^m \cdot \exp b|/|g(\zeta_n)| \to 0$ as $n \to \infty$.

Delete a ray l in $\mathbf{C}$ such that l does not contain any ζ_n. Then on $\mathbf{C} - l$, ζ^a is well-defined and

$$\frac{|\zeta_n^a \cdot \exp b|}{|g(\zeta_n)|} < \frac{|\zeta_n^m \cdot \exp b|}{|g(\zeta_n)|} \to 0 \tag{23.89}$$

as $n \to \infty$. In particular, $g(\zeta_n) \to \infty$ as $n \to 0$. So $\theta(F_3(\zeta_n^a \cdot \exp b)) \to 0$, $\phi(\zeta_n) \to 0$ as $n \to \infty$. We see by (23.88) and (23.89) that

$$\frac{\phi(\zeta_n)}{\sin\theta(F(\zeta_n^a \cdot \exp b))} = \frac{\phi(\zeta_n)}{\sin\phi(\zeta_n)} \bullet \left(\frac{\sin\phi(\zeta_n)}{\sin\theta(F(\zeta_n^a \cdot \exp b))}\right) \to 0, \tag{23.90}$$

as $n \to \infty$. Here $\sin\theta(F_3(\zeta_n^a \exp b)) = \sin\theta(z_n) = 1/\cosh z_n$, and $z_n = F_3(\zeta_n^a \exp b) = G_3(\zeta_n) \to \infty$ as $n \to \infty$.

By Lemma 23.4, we can choose n so large that the tangent plane of A at $G(\zeta_n)$ does not intersect ∂A. By (23.90), we can also choose n so that

$$\frac{\phi(\xi_n)}{\sin\theta(F(\zeta n^a \cdot \exp b))} < 1/16.$$

It follows from Lemma 23.4 that the tangent plane of A at $G(\zeta_n)$ will have a compact component that contains $G(\zeta_n)$. The existence of such a tangent plane contradicts Lemma 23.2. This contradiction proves the theorem. $\square$

Remark 23.5 Rosenberg and Toubiana [73] have shown that there exist minimally immersed annuli in $\overline{\mathbf{H}}$ with proper third coordinate function which have infinite total curvature. Theorem 23.1 shows that such annuli do not lie above any catenoid.

24 Complete Minimal Surfaces of Finite Topology

Based on Corollary 24.5, Hoffman and Meeks made the following conjecture in [31]:

Conjecture 24.1 *Let $X : M \hookrightarrow \mathbf{R}^3$ be a properly embedded complete minimal surface of finite topology with more than one end. Then X has finite total curvature.*

With the help of Theorem 23.1, we can give a clearer picture of properly embedded complete minimal surfaces with more than one end.

Theorem 24.2 *Suppose M is a properly embedded minimal surface in $\mathbf{R}^3$ that has two annular ends, each having infinite total curvature. Then these two ends have representatives E_1 , E_2 satisfying the following:*

1. *There exist disjoint closed halfspaces $\mathbf{H}_1$, $\mathbf{H}_2$ such that $E_1 \subset \mathbf{H}_1$ and $E_2 \subset \mathbf{H}_2$.*

2. *All other annular ends of M are asymptotic to flat planes parallel to $\partial\mathbf{H}_1$.*

3. *M has only a finite number of normal vectors parallel to the normal vector of $\partial\mathbf{H}_1$.*

Proof. Given two properly embedded minimal annuli A_1, A_2 each with compact boundary curve, if $A_1 \cap A_2 = \emptyset$ then there exists a standard barrier between them. This means that there exists a half-catenoid or a plane C such that outside of a sufficiently large ball B the barrier C is disjoint from $A_1 \cup A_2$ and also $C \cup B$ separates $A_1 - B$ from $A_2 - B$. Now consider the two annular ends E_1 and E_2 of M with infinite total curvature; Theorem 23.1 implies that C must be a plane. Since C is disjoint from $E_1 \cup E_2$ outside of some ball, $C \cap (E_1 \cup E_2)$ is compact. Hence, after removing compact subannuli of E_1 and E_2, we may choose E_1 and E_2 to lie in the disjoint halfspaces determined by C. The weak maximum principle at infinity (Remark 15.3) implies that E_i and C stay a bounded distance apart for $i = 1, 2$. Therefore, the distance from C to $E_1 \cup E_2$ is greater than some $\epsilon > 0$. It follows that we can choose closed disjoint halfspaces $\mathbf{H}_1$, $\mathbf{H}_2$ with $E_1 \subset \mathbf{H}_1$ and $E_2 \subset \mathbf{H}_2$. This proves the first statement of the theorem.

Suppose now that E_3 is another annular end of M that is disjoint from E_1 and E_2. Corollary 22.6 says that at least one of E_1, E_2 and E_3 lying between two standard barriers. By Proposition 22.3, an end lies between two standard barriers must have finite total curvature. Hence it is evident that E_3 has finite total curvature and lies between two standard barriers, and hence between E_1 and E_2. If E_3 is a catenoid end, then either E_1 or E_2 lies above a catenoid. By Theorem 23.1, E_1 or E_2 has finite total curvature, contradicting our hypotheses. Hence E_3 is asymptotic to a flat plane P. By the weak maximum principle at infinity the end of this plane P stays a positive distance from both E_1 and E_2. This implies that P intersects both E_1 and E_2 in a compact set and hence E_1 and E_2 have proper subends that are a positive distance from P. Hence we may assume that $E_i \cap P = \emptyset$ for $i = 1, 2$. By Theorem 16.1, the convex hulls of

E_1 and E_2 are either a halfspace or a slab since E_1 and E_2 are not compact. Since $E_i \cap P = \emptyset$ for $i = 1, 2$, P must be parallel to $\partial \mathbf{H}_1$. Since E_3 is an arbitrary annular end different from E_1 and E_2, the second part of the theorem is proved.

The proof of the third part of the theorem is quite long. Since we are not interested in the problem of image of Gauss map, we skip it here. The interested reader can read the article [18]. $\square$

We have some direct corollaries of Theorem 23.1.

Corollary 24.3 *Suppose $X : M \hookrightarrow \mathbf{R}^3$ is a smooth properly immersed minimal surface with smooth compact boundary and having finite topology. A sufficient condition for M to have finite total curvature is that $X(M)$ intersects some catenoid in a compact set. If M is embedded, this is also a necessary condition.*

Proof. If M is embedded, has finite total curvature and compact boundary, then the ends of M have a well-defined tangent plane parallel to a fixed plane P, which we take to be the xy-plane. Furthermore, annular end representatives of M can be chosen to be graphs over P, each of some fixed logarithmic growth in terms of $r = \sqrt{x^2 + y^2}$. Any catenoid C with waist circle $P \cap C$, and whose ends are graphs over P with logarithmic growth greater than the logarithmic growths of all the ends of M, must intersect M in a compact set. This proves the necessary part of the theorem.

Now suppose that C is a catenoid such that $B = C \cap X(M)$ is compact. After removing a regular neighbourhood of $X^{-1}(B)$ from M, we may assume that each component of $X(M)$ is disjoint from C. Since M has finite topology, we may assume that, without loss of generality, M is connected and $X(M) \cap C = \emptyset$. Let W and Y be the closures of the components of $\mathbf{R}^3 - C$ and assume W is the component that contains the symmetry axis of C. Thus either $X(M) \subset W$ or $X(M) \subset Y$. For the first case we apply Theorem 23.1 (in fact every annular end has a representative contained in the intersection of W with a halfspace). For the second case we can use Theorem 21.1. $\square$

Corollary 24.4 *Let $X : M \hookrightarrow \mathbf{R}^3$ be a smooth properly embedded minimal surface with smooth compact boundary and having finite topology. Suppose M has two catenoid ends, each a graph over the xy-plane of opposite signed logarithmic growth. Then M has finite total curvature.*

Proof. In this case we may assume that M has a catenoid end E_+ with positive z-coordinate and an end E_- with negative z-coordinate. Since M is proper, every end of M eventually is contained in the region above E_+, below E_-, or in the region between E_+ and E_-. As in the proof of Corollary 24.3, all of the ends of M must have finite total curvature. Thus M has finite total curvature since M has only a finite number of ends. $\square$

Corollary 24.5 *Suppose M is a properly embedded complete minimal surface in $\mathbf{R}^3$ with at least one catenoid type annular end. Then M can have at most one annular ends that is not conformally diffeomorphic to a punctured disk. In particular, if M has finite topology, then M is conformally equivalent to a closed Riemann surface from which a finite number of points, and zero or one closed disks, have been removed.*

Proof. One of the two possible infinite total curvature ends of M must lie above a catenoid, hence by Theorem 23.1 it has finite total curvature. This shows that there is at most one end which has infinite total curvature. $\square$

Remark 24.6 Recently Meeks and Rosenberg [51] proved that if a properly embedded minimal annulus A with smooth compact boundary is contained in a halfspace $\mathbf{H} \subset \mathbf{R}^3$, say $\mathbf{H} = \{(x, y, z), \mid z > 0\}$, then:

1. $A \cap \{(x, y, z) \mid z = c\}$ is a Jordan curve for $c > 0$.
2. The conformal structure of A is a punctured disk.

Combining the above result of Rosenberg and Meeks and Theorem 24.2, we have:

Theorem 24.7 *If $X : M \hookrightarrow \mathbf{R}^3$ is a proper complete minimal embedding with more than one annular end and M has finite topology, then there is a closed Riemann surface S_k of genus k such that*

$$M \cong S_k - \{p_1, \cdots, p_n\}.$$

25 Minimal Annuli

The catenoid is topologically an annulus, and is the only embedded complete minimal annulus of finite total curvature by Theorem 18.1. Since any complete minimal surface has annular end, we want to study minimal surfaces of annular type, with or without boundary.

All the results in this section are due to Osserman and Schiffer [70].

First we fix $A := \{r_1 < |z| < r_2\} \subset \mathbf{C}$, $0 < r_1 \le 1 \le r_2 \le \infty$ (by Lemma 9.1 and Proposition 9.2 we can always select such a representation of A). Let $X : A \to \mathbf{R}^3$ be a minimal annulus. Let g and $\eta = f(z)dz$ be the Weierstrass data for X and ϕ_i be as (6.15), $i = 1, 2, 3$. Let $\psi_i = z\phi_i$. Write $X = (X_1, X_2, X_3)$ and let

$$t = \log r = \log |z|.$$

We define

$$\overline{X}(r) := \frac{1}{2\pi}\int_0^{2\pi} X(re^{i\theta})d\theta.$$

Lemma 25.1

$$\frac{d^2\overline{X}(r)}{dt^2} = 0.$$

Proof.

$$\begin{aligned}
\frac{d^2\overline{X}(r)}{dt^2} &= r\frac{d}{dr}\left(r\frac{d\overline{X}(r)}{dr}\right) = \frac{1}{2\pi}\int_0^{2\pi} r\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}X(r^{i\theta})\right]d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} r^2 \triangle X\, d\theta = 0,
\end{aligned}$$

since

$$0 = \triangle X = r^{-1}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}X(re^{i\theta})\right] + r^{-2}\frac{\partial^2 X(re^{i\theta})}{\partial\theta^2}$$

and

$$\int_0^{2\pi}\frac{\partial^2 X(re^{i\theta})}{\partial\theta^2}d\theta = 0.$$

□

Let $C = \{|z| = 1\} \subset A$. Since C is the generator of the first homology group of A, by (17.72), we can define

$$\begin{aligned}
\mathbf{Flux}(X) &= \mathbf{Flux}(C) = \Im\int_C \phi(z)dz = \Im\int_{|z|=r}\phi(z)dz = -i\int_{|z|=r}\phi(z)dz \\
&= \int_0^{2\pi}\phi(re^{i\theta})re^{i\theta}d\theta = \int_0^{2\pi}\psi(re^{i\theta})d\theta \qquad (25.91)
\end{aligned}$$

where $\psi = z\phi$, for any $r_1 \le r \le r_2$. We have used the fact that since X is well defined,

$$\Re \int_{|z|=r} \phi\, dz = (0,0,0).$$

Lemma 25.2

$$\frac{d\overline{X}(r)}{dt} = r\frac{d\overline{X}(r)}{dr} = \frac{1}{2\pi}\mathbf{Flux}(X). \tag{25.92}$$

Proof.

$$\begin{aligned}
\frac{d\overline{X}(r)}{dt} &= r\frac{d\overline{X}(r)}{dr} = \frac{1}{2\pi}\int_0^{2\pi} r\frac{\partial X(re^{i\theta})}{\partial r}\, d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial X(re^{i\theta})}{\partial x}\cos\theta + \frac{\partial X(re^{i\theta})}{\partial y}\sin\theta\right) r\, d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \Re\left[re^{i\theta}\left(\frac{\partial X}{\partial x} - i\frac{\partial X}{\partial y}\right)(re^{i\theta})\right] d\theta \\
&= \Re\frac{1}{2\pi}\int_0^{2\pi}\left[re^{i\theta}\left(\frac{\partial X}{\partial x} - i\frac{\partial X}{\partial y}\right)(re^{i\theta})\right] d\theta \\
&= \Re\frac{1}{2\pi}\int_{|z|=r} -i\left(\frac{\partial X}{\partial x} - i\frac{\partial X}{\partial y}\right)(z)dz \\
&= \Im\frac{1}{2\pi}\int_{|z|=r} \phi(z)\, dz = \frac{1}{2\pi}\mathbf{Flux}(X).
\end{aligned}$$

□

Corollary 25.3 *Either*

$$\mathbf{Flux}(X) = (0,0,0),$$

or by a homothety, if necessary, we may assume that

$$\int_0^{2\pi} \psi(re^{i\theta})d\theta = 2\pi\frac{d\overline{X}(r)}{dt} = \mathbf{Flux}(X) = (0,0,2\pi). \tag{25.93}$$

Proof. Assume that $\mathbf{Flux}(X) \neq 0$. By Lemma 25.1, $\overline{X}(r) = \log r(c_1, c_2, c_3) + (d_1 + d_2 + d_3)$, where c_i and d_i are constants, $i = 1, 2, 3$. Thus the points $\overline{X}(r)$ lie on the straght line $t(c_1, c_2, c_3) + (d_1, d_2, d_3)$. After a linear homothety $H : \mathbf{R}^3 \to \mathbf{R}^3$, we may assume that $H[(c_1, c_2, c_3)] = (0,0,1)$. Thus by Lemma 25.2,

$$\mathbf{Flux}(H\circ X) = 2\pi\frac{d\overline{H\circ X}(r)}{dt} = 2\pi H\left[\frac{d\overline{X}(r)}{dt}\right] = 2\pi H[(c_1, c_2, c_3)] = (0,0,2\pi).$$

□

Remark 25.4 Thus we can always assume that X has vertical flux and that if $\mathbf{Flux}(X) \neq 0$ then

$$\mathbf{Flux}(X) = (0, 0, 2\pi)$$

after a suitable homothety. We will say that a minimal annulus with the above flux is *normalised.*

We are interested in the arclength of the closed curve $X|_{|z|=r}$. By (7.28)

$$\Lambda = \frac{1}{2}|f|(1+|g|^2) = \frac{1}{2}|\phi_3|\left(\frac{1}{|g|} + |g|\right).$$

The arclength $L(r)$ of the closed curve $X|_{|z|=r}$ is

$$L(r) = \int_{|z|=r} ds = \int_0^{2\pi} r\Lambda\, d\theta = \int_0^{2\pi} \frac{1}{2}\left(\left|\frac{\psi_3}{g}\right| + |\psi_3 g|\right) d\theta. \tag{25.94}$$

Theorem 25.5 *For any minimal annulus,*

$$\frac{d^2L}{dt^2} \geq L. \tag{25.95}$$

Equality holds if and only if the surface is the portion of a catenoid bounded by parallel coaxial circles, or an annulus in the plane.

Proof. The same calculation as in the proof of Lemma 25.1 leads to:

$$\frac{d^2L}{dt^2} = \int_{|z|=r} \frac{r^2}{2} \triangle \left(\left|\frac{\psi_3}{g}\right| + |\psi_3 g|\right) d\theta. \tag{25.96}$$

Now we have

$$\psi_1 = \frac{1}{2}\psi_3\left(\frac{1}{g} - g\right), \quad \psi_2 = \frac{i}{2}\psi_3\left(\frac{1}{g} + g\right)$$

and

$$\frac{\psi_3}{g} = \psi_1 - i\psi_2, \quad \psi_3 g = -\psi_1 - i\psi_2.$$

Since both d^2L/dt^2 and L are continuous functions of r, it follows from (25.93), (25.94), (25.96) that in order to prove (25.95) it suffices to prove the following lemma. □

Lemma 25.6 *Let $F(z)$ be holomorphic in A, and satisfy*

$$\int_0^{2\pi} F(re^{i\theta})d\theta = 0. \tag{25.97}$$

Then on every circle $|z| = r$ where F has no zeros, the inequality

$$\int_0^{2\pi} r^2 \triangle |F| d\theta \geq \int_0^{2\pi} |F| d\theta \tag{25.98}$$

is valid. Equality holds if and if F is a constant multiple of z or $1/z$.

Proof. Let $G(z)$ be holomorphic in an annulus, and have the Laurent expansion

$$G(z) = \sum_{-\infty}^{\infty} a_n \, z^n. \tag{25.99}$$

Then

$$\int_0^{2\pi} |G(re^{i\theta})|^2 d\theta = 2\pi \sum_{-\infty}^{\infty} |a_n| \, r^{2n} \tag{25.100}$$

and

$$\int_0^{2\pi} |G'(re^{i\theta})|^2 d\theta = 2\pi \sum_{-\infty}^{\infty} n^2 |a_n| \, r^{2n-2}. \tag{25.101}$$

Thus

$$\int_0^{2\pi} r^2 |G'(re^{i\theta})|^2 d\theta \geq \int_0^{2\pi} |G(re^{i\theta})|^2 d\theta - 2\pi |a_0|^2 \tag{25.102}$$

and since $\triangle |G|^2 = 4(\partial^2/\partial z \partial \overline{z})(G\overline{G}) = 4|G'|^2$,

$$\int_0^{2\pi} r^2 \triangle |G(re^{i\theta})|^2 d\theta \geq 4 \int_0^{2\pi} |G(re^{i\theta})|^2 d\theta - 8\pi |a_0|^2. \tag{25.103}$$

Since $F \neq 0$ on $|z| = r$, we may choose an annulus (by "thickening" this circle) in which $F \neq 0$. Since F is holomorphic in this annulus,

$$\int_0^{2\pi} \frac{\partial}{\partial \theta} \arg F(re^{i\theta}) d\theta = 2\pi k$$

for some integer k. Corresponding to k is even or odd, there are two possibilities:

either **Case** 1. $F = G^2$ or **Case** 2. $F = zG^2$,

where G is holomorphic in the annulus.

Case 1. If G has the expansion (25.99), then the constant term in the expansion of $F = G^2$ is

$$a_0^2 + 2 \sum_{n=1}^{\infty} a_n \, a_{-n}.$$

But condition (25.97) is equivalent to the vanishing of the constant term in the Laurent expansion of F. Thus

$$a_0^2 = -2 \sum_{n=1}^{\infty} a_n \, a_{-n}$$

and

$$|a_0^2| = 2 \Big| \sum_{n=1}^{\infty} a_n r^n \, a_{-n} r^{-n} \Big| \leq \sum_{n=1}^{\infty} (|a_n|^2 r^{2n} + |a_{-n}|^2 r^{-2n}) = \frac{1}{2\pi} \int_0^{2\pi} |G(re^{i\theta})|^2 \, d\theta - |a_0|^2,$$

or

$$4\pi |a_0^2| \leq \int_0^{2\pi} |G(re^{i\theta})|^2 \, d\theta.$$

Substituting this in (25.103) and using $|F| = |G|^2$, yields

$$\int_0^{2\pi} r^2 \triangle |F| d\theta \geq 2 \int_0^{2\pi} |F| d\theta. \tag{25.104}$$

Thus in Case 1, not only does (25.98) hold but in fact a stronger form is valid, implying in particular that the inequality in (25.98) is strict.

Case 2. Here $F = zG^2$ and $|F| = r|G|^2$. We have

$$\begin{aligned} \triangle|F| &= |G|^2 \triangle r + r \triangle |G|^2 + 2D\, r \bullet D|G|^2 \\ &= r^{-1}|G|^2 + 4r|G'|^2 + 4r^{-1}(x,y)\bullet\left(\Re(\overline{G}G'), -\Im(\overline{G}G')\right) \\ &= r^{-1}|G|^2 + 4r|G'|^2 + 4r^{-1}\Re(z\overline{G}G'). \end{aligned}$$

Note that

$$\int_0^{2\pi} re^{i\theta} G'(re^{i\theta}) d\theta = 0$$

and

$$\begin{aligned} 4\Re \int_0^{2\pi} [re^{i\theta}\overline{(G-a_0)}G'] d\theta &\geq -4\int_0^{2\pi} r|(G-a_0)G'|\, d\theta && (25.105) \\ &\geq -2\int_0^{2\pi} |G-a_0|^2\, d\theta - 2\int_0^{2\pi} r^2|G'|^2\, d\theta. && (25.106) \end{aligned}$$

We have

$$\begin{aligned} \int_0^{2\pi} r^2 \triangle |F|\, d\theta &= \int_0^{2\pi} r|G|^2\, d\theta + 4\int_0^{2\pi} r^3|G'|^2\, d\theta + 4\int_0^{2\pi} r\Re(z\overline{G}G') d\theta \\ &= \int_0^{2\pi} |F|\, d\theta + 4\int_0^{2\pi} r^3|G'|^2\, d\theta + 4\Re\int_0^{2\pi} r[re^{i\theta}\overline{(G-a_0)}G'] d\theta \\ &\geq \int_0^{2\pi} |F|\, d\theta + 4\int_0^{2\pi} r^3|G'|^2\, d\theta \\ &\quad -2\int_0^{2\pi} r|(G-a_0)|^2 d\theta - 2\int_0^{2\pi} r^3|G'|^2 d\theta \\ &= \int_0^{2\pi} |F|\, d\theta + 2\int_0^{2\pi} r^3|(G-a_0)'|^2\, d\theta - 2\int_0^{2\pi} r|(G-a_0)|^2 d\theta \\ &\geq \int_0^{2\pi} |F|\, d\theta. \end{aligned}$$

The equality holds if and only if (25.105) and (25.106) are both equalities, and

$$\int_0^{2\pi} r^2|(G-a_0)'|^2\, d\theta = \int_0^{2\pi} |(G-a_0)|^2 d\theta. \tag{25.107}$$

In particular, by (25.100) and (25.101), (25.107) holds if and only if

$$a_n = 0 \quad \text{for} \quad |n| \neq 1 \text{ or } 0.$$

But if

$$G - a_0 = \frac{a_{-1}}{z} + a_1 z,$$

then

$$4\Re \int_0^{2\pi} re^{i\theta}\overline{(G-a_0)}G'\,d\theta = 8\pi(|a_1|^2r^2 - |a_{-1}|^2r^{-2}),$$

$$-2\int_0^{2\pi} |G-a_0|^2\,d\theta - 2\int_0^{2\pi} r^2|(G-a_0)'|^2\,d\theta = -8\pi(|a_1|^2r^2 + |a_{-1}|^2r^{-2}).$$

Comparing these two we have $a_1 = 0$.

Thus we have

$$G(z) = \frac{a_{-1}}{z} + a_0, \quad G^2(z) = \frac{a_{-1}^2}{z^2} + a_0^2 + 2\frac{a_0a_{-1}}{z},$$

and

$$F(z) = zG^2(z) = 2a_0a_{-1} + a_0^2 z + \frac{a_{-1}^2}{z}.$$

Condition (25.97) then implies that $a_0a_{-1} = 0$, so that F is of the form stated. $\square$

Remark 25.7 Note that in Case 2 the assumption (25.97) is not needed to deduce the inequality (25.98). Only in Case 1 did we use it, and there it is clearly necessary since, for example, (25.104) is false if F is a non-zero constant.

We now complete the proof of Theorem 25.5 by analysing when equality can hold in (25.95). Returning to (25.96) we see that for equality to hold in (25.95) we must have

$$\frac{\psi_3}{g} = c_1 z \text{ or } \frac{c_2}{z}, \quad \psi_3 g = b_1 z \text{ or } \frac{b_2}{z}.$$

We therefore have four cases.

Case 1.

$$\frac{\psi_3}{g} = c_1 z, \quad \psi_3 g = b_1 z.$$

Then g is a constant, and so is ϕ_3. It follows from (6.15), (6.18) and (6.26) that

$$\phi_1 = \frac{1}{2}c\left(\frac{1}{d} - d\right), \quad \phi_2 = \frac{i}{2}c\left(\frac{1}{d} + d\right), \quad \phi_3 = c.$$

The image surface is in a plane, and the map $X : A \hookrightarrow \mathbf{R}^3$ is a complex linear map into the plane.

Case 2.

$$\frac{\psi_3}{g} = \frac{c_2}{z}, \quad \psi_3 g = \frac{b_2}{z}.$$

Again g is a constant, but this time

$$\phi_1 = \frac{1}{2}\frac{c}{z^2}\left(\frac{1}{d} - d\right), \quad \phi_2 = \frac{i}{2}\frac{c}{z^2}\left(\frac{1}{d} + d\right), \quad \phi_3 = \frac{c}{z^2}.$$

The image is again a plane, but the map is this time the composition of $1/z$ with a complex linear map.

Case 3.

$$\frac{\psi_3}{g} = c_1 z, \qquad \psi_3 g = \frac{b_2}{z}.$$

Then we have $\psi_3 = c$, $g = d/z$, and $\phi_3 = c/z$. Thus the Weierstrass data are $g = d/z$, $\eta = (\phi_3/g)dz = (c/d)dz$. Making change $z \to d/\zeta$, we see that $g(\zeta) = \zeta$ and $(c/d)dz = -c\, d\zeta/\zeta^2$. Thus c is real and the surface is part of a catenoid.

Case 4.

$$\frac{\psi_3}{g} = \frac{c_2}{z}, \qquad \psi_3 g = b_1 z.$$

Again these give $g = cz$ and $\eta = b\, dz/z^2$, and the surface is part of a catenoid.

To prove the isoperimetric inequality for minimal annuli in the next section, we need further study the function $L(r)$ for normalised surfaces.

Lemma 25.8 *For a non-zero flux normalised surface,*

$$L(r) \geq 2\pi \quad \text{for all } r. \tag{25.108}$$

Equality can hold for at most one value of r. Moreover $L(r_0) = 2\pi$ if and only if the circle $|z| = r_0$ maps onto a horizontal plane $x_3 = c$ and each radial direction along the circle maps into a vertical direction in $\mathbf{R}^3$.

Proof. Since $\overline{X_3}(r) = \log r$,

$$0 = \frac{1}{2\pi}\int_0^{2\pi} [X_3(re^{i\theta}) - \log r]d\theta.$$

Thus there is a well defined harmonic function v conjugate to $X_3 - \log r$ in A such that $f := X_3 - \log r + iv$ is holomorphic in A. Then by the Cauchy-Riemann equations and $r^2 = z\bar{z}$, we have

$$f' = (X_3 - \log r)_x + iv_x = (X_3)_x - (\log r)_x - i(X_3)_y + i(\log r)_y = \phi_3 - \frac{x - iy}{r^2} = \phi_3 - \frac{1}{z}$$

and $\psi_3(z) = 1 + zf'(z)$. Since $\phi_1^2 + \phi_2^2 + \phi_3^2 = 0$, we have

$$\Lambda^2 = \frac{1}{2}\sum_{i=1}^{3} |\phi_i|^2 \geq \frac{1}{2}(|\phi_1^2 + \phi_2^2| + |\phi_3|^3) = |\phi_3|^2.$$

Thus

$$\begin{aligned} L(r) &= \int_0^{2\pi} \Lambda(re^{i\theta})r\,d\theta \geq \int_0^{2\pi} |\psi_3|d\theta \geq \left|\int_0^{2\pi} \psi_3(re^{i\theta})\,d\theta\right| \\ &= \left|\int_0^{2\pi} (1 + re^{i\theta} f'(re^{i\theta}))d\theta\right| \qquad (25.109) \\ &= \left|2\pi - i\int_{|z|=r} f'(z)dz\right| = 2\pi. \end{aligned}$$

The fact that $L(r)$ can attain the minimum value 2π for at most one value of r is an immediate consequence of Theorem 25.5, which says L is a strictly convex function of $\log r$.

$L(r_0) = 2\pi$ if and only both of the inequalities in (25.109) become equalities. This means

$$r_0\Lambda(r_0e^{i\theta}) = |\psi_3(r_0e^{i\theta})|, \quad 0 \leq \theta \leq 2\pi, \qquad (25.110)$$

$$\int_0^{2\pi} |\psi_3(r_0e^{i\theta})|d\theta = \left|\int_0^{2\pi} \psi_3(r_0e^{i\theta})d\theta\right|. \qquad (25.111)$$

Using the relation $\psi_3(z) = 1 + zf'(z)$ gives

$$2\pi = \int_0^{2\pi} \psi_3(r_0e^{i\theta})d\theta = \int_0^{2\pi} \Re[\psi_3(r_0e^{i\theta})]d\theta + i\int_0^{2\pi} \Im[\psi_3(r_0e^{i\theta})]d\theta.$$

Hence

$$\int_0^{2\pi} \Im[\psi_3(r_0e^{i\theta})]d\theta = 0$$

and

$$\begin{aligned} \left|\int_0^{2\pi} \psi_3(r_0e^{i\theta})d\theta\right| &= \left|\int_0^{2\pi} \Re[\psi_3(r_0e^{i\theta})]d\theta\right| \leq \int_0^{2\pi} |\Re[\psi_3(r_0e^{i\theta})]|d\theta \qquad (25.112) \\ &\leq \int_0^{2\pi} |\psi_3(r_0e^{\theta})|d\theta. \end{aligned}$$

For (25.111) to hold, we must have

$$\Im[\psi_3(r_0e^{\theta})] = 0, \quad 0 \leq \theta \leq 2\pi. \qquad (25.113)$$

But

$$\begin{aligned} \psi_3 &= (x+iy)[(X_3)_x - i(X_3)_y] = x(X_3)_x + y(X_3)_y + i[y(X_3)_x - x(X_3)_y] \\ &= r(X_3)_r - i(X_3)_\theta. \qquad (25.114) \end{aligned}$$

Thus (25.113) holds if and only $X_3(r_0e^{i\theta})$ is constant, and DX_3 is orthogonal to the circle $|z| = r_0$. From (25.110), at each point of $|z| = r_0$,

$$\Lambda = |DX_3| = |(X_3)_r|. \qquad (25.115)$$

But $\Lambda = |X_r|$, and so (25.115) holds if and only if $(X_i)_r = 0$ for $i = 1,\ 2$.

Conversely, X_r is vertical means that (25.115) holds, and this implies $(X_3)_r \neq 0$. Thus by (25.114), $\Re\psi_3$ cannot change sign on the circle $|z| = r_0$. The condition that X_3 is constant on this circle implies (25.113), again using (25.114). These two facts yield equality in (25.112), hence in (25.111), while (25.115) gives equality in (25.110). This completes the proof of the lemma. □

To prove the next theorem we need a lemma.

Lemma 25.9 *Let $f(t)$ satisfy $f''(t) \geq f(t)$ in some interval I. Then for all t_0, t in I,*

$$f(t) \geq f(t_0)\cosh(t - t_0) + f'(t_0)\sinh(t - t_0). \tag{25.116}$$

Equality holds for some $t_1 \neq t_0$ if and only if it holds for all t between t_0 and t_1 if and only if $f''(t) = f(t)$ for all t between t_0 and t_1.

Proof. We have

$$\frac{d}{dt}\left[f'(t)\cosh t - f(t)\sinh t)\right] = (f'' - f)\cosh t \geq 0. \tag{25.117}$$

Hence $t > t_0$ implies

$$f'(t)\cosh t - f(t)\sinh t \geq f'(t_0)\cosh t_0 - f(t_0)\sinh t_0.$$

Thus

$$\frac{d}{dt}\left(\frac{f(t)}{\cosh t}\right) \geq [f'(t_0)\cosh t_0 - f(t_0)\sinh t_0]\frac{1}{\cosh^2 t}$$

and

$$\frac{f(t)}{\cosh t} - \frac{f(t_0)}{\cosh t_0} \geq [f'(t_0)\cosh t_0 - f(t_0)\sinh t_0](\tanh t - \tanh t_0).$$

Multiplying out and simplifying, we obtain (25.116).

An analogous argument holds for $t < t_0$.

For equality to hold, it must hold in (25.117), so that $f'' = f$. □

Theorem 25.10 *Let $X : A \hookrightarrow \mathbf{R}^3$ be a minimal annulus. Further assume (by a reparametrisation of the form $z = \zeta/c$ if necessary) that $L(r)$ attains a minimum L_0 for $r = 1$. Then the lengths of the boundary curves are greater than or equal to $L_0/2\pi$ times the lengths of the corresponding boundary circles of the standard catenoid (the Weierstrass data are $g = z$, $\eta = dz/z^2$) based on the same annulus. Equality can hold only if X is itself the standard catenoid.*

Proof. There are three cases, depending on whether $L(r)$ is increasing throughout, decreasing throughout, or has an interior minimum. Again using the notation $t = \log r$, the minimum occurs at $r = 1$ or $t = 0$. If we use primes to denote derivatives with respect to t, then by Theorem 25.5, $L''(t) \geq L(t)$, and by Lemma 25.9,

$$L(t) \geq L(0) \cosh t + L'(0) \sinh t.$$

In the case of an interior minimum, then $L'(0) = 0$, and

$$L(t_1) \geq L(0) \cosh t_1, \quad L(t_2) \geq L(0) \cosh t_2$$

for the values t_1, t_2 corresponding to the boundary curves. But we have seen that for the catenoid, the length function is $L(t) = 2\pi \cosh t$.

If L is decreasing, then the boundary values are $t = 0$ and $t = t_1 < 0$. Using $L'(0) \leq 0$ we obtain $L(t_1) \geq L(0) \cosh t_1$, and the result is again true. A similar procedure applies if L is increasing.

For equality to hold in any of these cases, it follows from Lemma 25.9 that $L''(t) = L(t)$. According to Theorem 25.5, X must be a standard catenoid or else a plane annulus. However, a direct computation shows that for the plane annulus one has either $L(t) = L(0)e^t$, $t \geq 0$, or $L(t) = L(0)e^{-t}$, $t \leq 0$, and which is strictly greater than $L(0) \cosh t$. □

26 Isoperimetric Inequalities for Minimal Surfaces

It is well known that for a plane Jordan curve with length L, the area $\mathbf{A}$ enclosed by the curve is less than or equal to $L^2/4\pi$, with equality holding if and only if the curve is a circle. In this section we give such *isoperimetric inequalities* for simply or doubly connected minimal surfaces. For more general discussions and applications of the isoperimetric inequalities the reader can see [69].

The proof of the next theorem is from [68].

Theorem 26.1 *Let $M \subset \mathbf{R}^3$ be an immersed simply connected minimal surface with $C = \partial M$ a closed curve. Let L be the arclength of C, $\mathbf{A}$ the area of M, then*

$$L^2 - 4\pi\mathbf{A} \geq 0. \tag{26.118}$$

Proof. From (3.6) we have

$$2\mathbf{A} = \int_C (X-a)\bullet\vec{n}\,ds$$

for any $a \in \mathbf{R}^3$. Here X is the coordinate function of M, $\vec{n}$ is the outward unit conormal to C and ds is the line element of C. Select $a \in C$. We need prove that

$$2\pi\int_C (X-a)\bullet\vec{n}\,ds \leq L^2.$$

Let $x(s)$ be the parametrisation of C by arclength and $x(0) = x(L) = a$. We want to select suitable frames in each $T_{x(s)}\mathbf{R}^3$. For this purpose, let $B(s) : T_{x(s)}M \to T_{x(s)}M$ be the linear mapping which rotates $\vec{n}$ by $\pi/2$ and is zero in $T^{\perp}_{x(s)}$. If we let $(\vec{n}, B\vec{n}, N)$ be the orthonormal basis of $T_{x(s)}\mathbf{R}^3$, then B has the matrix form

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

From this it is clear that

1. $|Bv| \leq |v|$ for any $v \in \mathbf{R}^3$.

2. $u\bullet Bv = -v\bullet Bu$.

Let $(e_1, e_2, e_3)(s)$ be vector fields along C such that

$$e_i'(s) = \frac{\pi}{L}Be_i(s), \quad i = 1,\ 2,\ 3, \tag{26.119}$$

and $(e_1, e_2, e_3)(0)$ is an orthonormal basis of $\mathbf{R}^3$. Then property 2 guarantees that $(e_i \bullet e_j)(s)$ is a constant, thus $(e_1, e_2, e_3)(s)$ is an orthonormal basis of $\mathbf{R}^3$ for any $s \in [0, L]$. We can write

$$x(s) - a = \sum_{i=1}^{3} c_i(s)e_i(s).$$

Then

$$x'(s) = \sum_{i=1}^{3} c_i'(s)e_i(s) + \frac{\pi}{L}\sum_{i=1}^{3} c_i(s)Be_i(s) = \sum_{i=1}^{3} c_i'(s)e_i(s) + \frac{\pi}{L}B[x(s) - a].$$

Thus

$$\begin{aligned}
|x'(s)|^2 &= \frac{\pi}{L}x'(s)\bullet B[x(s) - a] + x'(s)\bullet\sum_{i=1}^{3} c_i'(s)e_i(s) \\
&= \frac{\pi}{L}x'(s)\bullet B[x(s) - a] + \left[\sum_{i=1}^{3} c_i'(s)e_i(s)\right]\bullet\left[\sum_{i=1}^{3} c_i'(s)e_i(s)\right] \\
&\quad + \frac{\pi}{L}B[x(s) - a]\bullet\left[\sum_{i=1}^{3} c_i'(s)e_i(s)\right] \\
&= \frac{\pi}{L}x'(s)\bullet B[x(s) - a] + \sum_{i=1}^{3} c_i'(s)^2 + \frac{\pi}{L}B[x(s) - a]\bullet\left\{x'(s) - \frac{\pi}{L}B[x(s) - a]\right\} \\
&= \frac{2\pi}{L}x'(s)\bullet B[x(s) - a] + \sum_{i=1}^{3} c_i'(s)^2 - \frac{\pi^2}{L^2}B[x(s) - a]\bullet B[x(s) - a] \\
&= \frac{2\pi}{L}x'(s)\bullet B[x(s) - a] + \sum_{i=1}^{3} c_i'(s)^2 - \frac{\pi^2}{L^2}|x(s) - a|^2 \\
&\quad + \frac{\pi^2}{L^2}\left(|x(s) - a|^2 - |B[x(s) - a]|^2\right) \\
&= \frac{2\pi}{L}x'(s)\bullet B[x(s) - a] + \sum_{i=1}^{3}\left[c_i'(s)^2 - \frac{\pi^2}{L^2}c_i^2(s)\right] \\
&\quad + \frac{\pi^2}{L^2}\left(|x(s) - a|^2 - |B[x(s) - a]|^2\right).
\end{aligned}$$

Thus we have

$$\frac{2\pi}{L}x'(s)\bullet B[x(s) - a] = |x'(s)|^2 - \sum_{i=1}^{3}\left[c_i'(s)^2 - \frac{\pi^2}{L^2}c_i^2(s)\right] - \frac{\pi^2}{L^2}\left(|x(s) - a|^2 - |B(x(s) - a)|^2\right).$$

Since $Bx'(s) = -\vec{n}$,

$$[x(s) - a]\bullet\vec{n} = -[x(s) - a]\bullet Bx'(s) = x'(s)\bullet B[x(s) - a],$$

we find that

$$2\pi\int_C (X - a)\bullet\vec{n}\,ds = 2\pi\int_0^L x'(s)\bullet B[x(s) - a]\,ds$$

$$= L^2 - L\int_0^L \sum_{i=1}^{3}\left[c_i'(s)^2 - \frac{\pi^2}{L^2}c_i^2(s)\right] ds - \frac{\pi^2}{L}\int_0^L \left(|x(s)-a|^2 - |B[x(s)-a]|^2\right) ds.$$

The fact $x(0) = a$ and $x'(0)$ exists give that $c_i(0) = 0$, $c_i'(0) \in \mathbf{R}$, $i = 1, 2, 3$, thus the functions

$$b_i(s) = \frac{c_i(s)}{\sin\left(\frac{\pi s}{L}\right)}$$

are well defined for $i = 1, 2, 3$. Using the identities

$$\begin{aligned} c_i'(s)^2 - \frac{\pi^2}{L^2}c_i^2(s) &= b_i'(s)^2 \sin^2\frac{\pi}{L} + \frac{\pi}{2L}\frac{d}{ds}\left(b_i^2(s)\sin\frac{2\pi s}{L}\right) \\ &= b_i'(s)^2 \sin^2\frac{\pi}{L} + \frac{\pi}{L}\frac{d}{ds}\left(c_i^2(s)\cot\frac{\pi s}{L}\right), \end{aligned}$$

and $|B[x(s)-a]| \leq |x(s)-a|$, we obtain

$$L^2 - 2\pi\int_C [x(s)-a]\bullet\vec{n}\, ds \geq L\sum_{i=1}^{3}\int_0^L b_i'(s)^2\sin^2\frac{\pi s}{L}\, ds \geq 0.$$

□

Remark 26.2 This isoperimetric inequality is also true for simply connected minimal surfaces in $\mathbf{R}^n$, $n \geq 3$. The proof is the same as above. See [68].

Next we study the doubly connected case, the proof is from [70]. We will use the notation in the last section.

Theorem 26.3 *Let $\mathbf{A}$ be the area of a minimal annulus $X : A \hookrightarrow \mathbf{R}^3$, L_1 and L_2 the length of its boundary curves C_1 and C_2, and let $L = L_1 + L_2$. If $\mathbf{Flux}(X) = 0$ or there are no planes separating the two boundary curves, then*

$$L_1^2 + L_2^2 \geq 4\pi\mathbf{A} \tag{26.120}$$

or, equivalently,

$$L^2 - 4\pi\mathbf{A} \geq 2L_1L_2. \tag{26.121}$$

For arbitrary minimal annulus, we have

$$L^2 - 4\pi\mathbf{A} \geq 2L_1L_2(1 - \log 2). \tag{26.122}$$

Proof. From the area formula (3.6) we have

$$2\mathbf{A} = \int_{C_1} X\bullet\vec{n}\, ds + \int_{C_2} X\bullet\vec{n}\, ds.$$

In the proof of Theorem 26.1, we have

$$M_1 := L_1^2 - 2\pi \int_{C_1} (X - p_1)\bullet\vec{n}\, ds \geq 0, \quad M_2 := L_2^2 - 2\pi \int_{C_2} (X - p_2)\bullet\vec{n}\, ds \geq 0,$$

where $p_i \in C_i$. (Note that we did not use that C_i encloses a simply connected minimal surface in the proof of the above inequalities.) Now remember that

$$-\int_{C_1} \vec{n}\, ds = \int_{C_2} \vec{n}\, ds = \mathbf{Flux}(X).$$

We have

$$L_1^2 + L_2^2 - 4\pi\mathbf{A} = M_1 + M_2 - 2\pi(p_2 - p_1)\bullet\mathbf{Flux}(X).$$

So if $\mathbf{Flux}(X) = 0$, then we have (26.120). If $\mathbf{Flux}(X) \neq 0$, then take a plane P_d defined by $x\bullet\mathbf{Flux}(X) = d$. All $d \in \mathbf{R}$ such that $P_d \cap C_i \neq \emptyset$ form two closed intervals in $\mathbf{R}$. If no planes separate C_1 and C_2, then these two intervals have common points, and thus we can find $p_i \in C_i$ such that $p_1\bullet\mathbf{Flux}(X) = p_2\bullet\mathbf{Flux}(X)$; again we get (26.120).

Now we consider the case that $\mathbf{Flux}(X) \neq 0$ and there is a plane separating C_1 and C_2. Note that after a homothety, both sides of (26.122) are multiplied by a positive constant, thus by Corollary 25.3 we can assume that

$$\mathbf{Flux}(X) = (0, 0, 2\pi).$$

So we have $\overline{X}_3(r) = \log r$. This implies that the planes $P_i := \{x_3 = \log r_i\}$ intersect C_i respectively. Thus selecting $p_i \in P_i \cap C_i$, we have

$$2\pi(p_2 - p_1)\bullet\mathbf{Flux}(X) = 4\pi^2(\log r_2 - \log r_1) = 4\pi^2 \log\frac{r_2}{r_1},$$

and

$$L_1^2 + L_2^2 - 4\pi\mathbf{A} = M_1 + M_2 - 4\pi^2 \log\frac{r_2}{r_1}. \tag{26.123}$$

We now apply Theorem 25.10. Recall that $r_1 \leq 1 \leq r_2$ and that $L(r)$ is a minimum for $r = 1$. We let

$$K_i := \pi\left(r_i + \frac{1}{r_i}\right), \quad i = 1,\ 2,$$

be the lengths of the corresponding boundary circles on the standard catenoid. Then

$$\pi^2\frac{r_2}{r_1} < K_1K_2 < 4\pi^2\frac{r_2}{r_1}.$$

By Theorem 25.10 and Lemma 25.8, $K_1K_2 \leq L_1L_2$. Finally, if we let $k_i = L_i/\pi$, we have

$$\begin{aligned}
2L_1L_2 - 4\pi^2\log\frac{r_2}{r_1} &= 2\pi^2\left(k_1k_2 - 2\log\frac{r_2}{r_1}\right)\\
&> 2\pi^2\left(k_1k_2 - 2\log\frac{K_1K_2}{\pi^2}\right)\\
&\geq 2\pi^2(k_1k_2 - 2\log k_1k_2)\\
&\geq 2\pi^2k_1k_2(1 - \log 2).
\end{aligned}$$

The last inequality follows from the elementary fact that

$$2 \log x < x \log 2 \quad \text{for} \quad x > 4,$$

combined with $K_i \geq 2\pi$, $k_1 k_2 = L_1 L_2/\pi^2 \geq K_1 K_2/\pi^2 \geq 4$. Substituting in (26.123) gives (26.122), and the theorem is proved. $\square$

Remark 26.4 The inequalities (26.120) and (26.121) are also true for minimal annuli in $\mathbf{R}^n$, $n \geq 3$, satisfying the corresponding conditions. The proof is similar, see [70]. The inequality (26.122) is true in $\mathbf{R}^3$ since we have Theorem 25.10, thus if Theorem 25.10 is true in $\mathbf{R}^n$ then (26.122) is also true in $\mathbf{R}^n$.

27 Minimal Annuli in a Slab

Recall that a catenoid is a rotation surface, hence is foliated by circles in parallel planes. A good question to ask is whether there are other minimal annuli which can be foliated by circles. It was B. Riemann [72] and Enneper [14] who solved this problem very satisfactorily. The answer is that there is only one one-parameter family of such surfaces up to a homothety. Each minimal annuli in this one-parameter family is contained in a slab and foliated by circles, and its boundary is a pair of parallel straight lines. Rotating repeatly about these boundary straight lines gives a one-parameter family of singly periodic minimal surface; these surfaces are called *Riemann's examples.*

For the details of the proof of existence and other properties of Riemann's examples, see [61], section 5.4, Cyclic minimal surfaces. For constructions of Riemann's examples using the Weierstrass functions please see [25]. It is also known that a pair of parallel straight lines can only bound a piece of Riemann's example, if they bound any minimal annulus at all, see for example, [17].

Now we are going to study minimal annuli in a slab. Let $P_t = \{(x, y, z) \in \mathbf{R}^3 | z = t\}$ and $S(t_1, t_2) = \{(x, y, z) \in \mathbf{R}^3 | t_1 \le z \le t_2,\ t_1 < t_2\}$. Consider a minimal annulus $X : A_R \hookrightarrow S(t_1, t_2)$ such that $X(\{|z| = 1/R\}) \subset P_{t_1}$, $X(\{|z| = R\}) \subset P_{t_2}$ and X is continuous on A_R. We will call such a minimal annulus a *minimal annulus in a slab*. By a homothety we can normalize t_1 and t_2 such that $t_1 = -1$ and $t_2 = 1$. We will denote the image $X(A_R) \subset S(-1, 1)$ by A and let $A(t) = A \cap P_t$ for $-1 \le t \le 1$. When discussing a minimal annulus in a slab, we often just refer to it by the image $A = X(A_R)$.

We want to derive the Enneper-Weierstrass representation of a minimal annulus in a slab. Let A be a minimal annulus in a slab. The third coordinate function X^3 is harmonic, $X^3|_{\{|z|=1/R\}} = -1$, and $X^3|_{\{|z|=R\}} = 1$. By uniqueness of solutions to the Dirichlet problem

$$\begin{cases} \triangle u = 0 & \text{in } \mathrm{Int}(A_R) \\ \\ u|_{\{|z|=1/R\}} = -1, & u_{|\{|z|=R\}} = 1, \end{cases}$$

where $\mathrm{Int}(A_R)$ is the interior of A_R, we have $X^3 = \frac{1}{\log R} \log |z|$, and

$$\omega_3 = f(z)g(z)dz = 2\frac{\partial}{\partial z}X^3\, dz = \frac{d}{dz}\left(\frac{1}{\log R}\log z\right) dz = \frac{1}{\log R}\frac{1}{z}\, dz.$$

Hence $f(z) = \frac{1}{\log R}\frac{1}{zg(z)}$. Here of course g is the Gauss map in the Enneper-Weierstrass representation and $f(z)dz = \eta$. Thus by (6.26) we have

$$\begin{cases} \omega_1 &= \frac{1}{\log R}\frac{1}{2z}(\frac{1}{g} - g)dz \\ \\ \omega_2 &= \frac{1}{\log R}\frac{i}{2z}(\frac{1}{g} + g)dz \\ \\ \omega_3 &= \frac{1}{\log R}\frac{1}{z}dz. \end{cases} \tag{27.124}$$

The immersion is given by

$$X(p) = \frac{1}{\log R} \Re \int_1^p \left(\frac{1}{2z}(\frac{1}{g} - g),\ \frac{i}{2z}(\frac{1}{g} + g),\ \frac{1}{z} \right) dz + C, \tag{27.125}$$

where $C = (a, b, 0) \in \mathbf{R}^3$. Since X is well defined, for $\gamma = \{|z| = 1\} \subset A_R$,

$$\Re \int_\gamma \left(\frac{1}{2z}(\frac{1}{g} - g),\ \frac{i}{2z}(\frac{1}{g} + g),\ \frac{1}{z} \right) dz = \vec{0}. \tag{27.126}$$

On the other hand, if g and f are meromorphic and holomorphic functions in A_R, such that (27.124) defines three holomorphic 1-forms and (27.126) is satisfied, then (27.125) defines a minimal annulus in the slab $S(-1, 1)$.

The conformal factor of a minimal annulus in a slab is

$$\Lambda^2 = \frac{1}{4(\log R)^2 |z|^2} \left(\frac{1}{|g|} + |g| \right)^2, \tag{27.127}$$

and the Gauss curvature is

$$K = -\left[\frac{4 \log R |z| |g| |g'|}{(1 + |g|^2)^2} \right]^2. \tag{27.128}$$

One observation about the Gauss map of a minimal annulus in a slab is:

Proposition 27.1 *Let A be a minimal annulus in a slab such that X is smooth up to the boundary (in fact, C^2 will be enough), then the Gauss map g of A has no zeros or poles on A_R. Furthermore, $|g|$ and $|g|^{-1}$ are both bounded.*

Proof. From (27.125) we see that for any $-1 \le t \le 1, A(t) = A \cap P_t$ is the image $X(\{|z| = R^t\})$. From Corollary 4.5 we get immediately that g has no zeros or poles in $\mathrm{Int}(A_R)$, because otherwise the preimage of $A(t)$ will have an equiangular system of at least order 4 at the pole or zero points.

Since X is continuous on A_R, A is compact. It remains only to prove that on the boundary of A, the Gauss map N is not perpendicular to the xy-plane. Since our boundary is smooth, the projection of the boundary into the xy plane satisfies the sphere condition, inner or outer. By boundary regularity theory, X is $C^{1,\alpha}, \alpha \in (0, 1)$, up to the boundary (see [12], Vol. 2, Chapter 7), hence at every boundary point there is a well defined normal direction.

Near any boundary point p that has a vertical normal, the surface is a graph over a small open disk $D \subset P_1$ with p on ∂D, assuming that $p \in A(1)$. Then we can use the minimal surface equation (4.8). We write $(x, y, z) \in A$, where $z = z(x, y)$ satisfies

$$(1 + z_y^2) z_{xx} - 2 z_x z_y z_{xy} + (1 + z_x^2) z_{yy} = 0.$$

Since X^3, the third coordinate function of A, is harmonic, by the maximum principle we have for any $(x, y) \in D$ that $z(x, y) < 1 = z(p)$. Define a uniformly elliptic operator on a smaller domain if necessary,

$$Lu = (1 + z_y^2)u_{xx} - 2z_x z_y u_{xy} + (1 + z_x^2)u_{yy}.$$

Then z satisfies $Lz = 0$. It is well known that

$$\frac{\partial z}{\partial \nu}(p) > 0,$$

where ν is the outward normal to ∂D at p (see [21], Lemma 4, page 34). But this means that the normal is not vertical. This contradiction proves that N is never vertical on the boundary of A. Since X is smooth, g and g^{-1} are continuous up to boundary (we can see this by $g = \tau \circ N$); hence by the maximum principle both $|g|$ and $|g|^{-1}$ are bounded. $\square$

If a minimal annulus A in a slab satisfies that $A(-1)$ and $A(1)$ are continuous convex Jordan curves, we will call A a *convex boundary minimal annulus* or CBA.

Theorem 27.2 *If A is a CBA and $\Gamma = \partial A$ is smooth, then $A \cap P_t$ is a strictly convex Jordan curve for every $-1 < t < 1$. In particular, $X : A_R \hookrightarrow S(-1, 1)$ is an embedding.*

Proof. By Proposition 9.2 we have $A = X(A_R)$ for some $R > 1$. And by regularity theory, X is smooth up to the boundary. At any point of $A(t) = A \cap P_t$, $-1 \leq t \leq 1$, draw a tangent vector to the curve $A(t)$, and let ψ be the angle made by this tangent vector with the positive x-axis. The ψ may be a multivalued function, but we will see that it is harmonic. To see this, consider the unit normal vector $\vec{n}$ of the curve $A(t)$, and its angle with the positive x-axis ϕ. If we orient the surface such that the normal is inward to the unbounded component of $S(-1, 1) - A$, then we have $\psi = \phi + \pi/2$. By Proposition 27.1, $g \neq 0$ or ∞ on A_R, hence the unit normal vector $\vec{n}$ must be $\frac{g}{|g|} \in \mathbf{C} \cong \mathbf{R}^2$ in complex form. Because $\phi = \arg g = \Im \log g$, ϕ is harmonic and so is ψ.

Now suppose that s is the arc length parameter of the curve $A(t)$ and notice that by (27.125) $X^{-1}(A(t)) = \{z : |z| = r = R^t\}$. Writing $z = re^{i\theta}$, we can calculate the curvature of A_t as follows:

$$\begin{aligned}
\kappa &= \psi_s = \phi_s = \frac{d}{ds}(\Im \log g) = \Im\left(\frac{d}{ds}\log g\right) = \Im\left(\frac{d}{dz}\log g \frac{dz}{ds}\right) \\
&= \Im\left(\frac{g'}{g}\frac{dz}{d\theta}\frac{d\theta}{ds}\right) = \Im\left(\frac{g'}{g} izr^{-1}\Lambda^{-1}\right) = r^{-1}\Lambda^{-1}\Re\left(z\frac{g'}{g}\right).
\end{aligned}$$

Here we have used the facts that on the curve $\{|z| = r = R^t\}$,

$$\frac{dz}{d\theta} = ire^{i\theta} = iz, \quad \text{and} \quad ds = \Lambda|dz| = \Lambda r d\theta.$$

Since $h = \Re\left(z\frac{g'}{g}\right) = r\Lambda\kappa$ is harmonic and $r\Lambda > 0$, we see that if Γ is smooth (in fact $C^{2,\alpha}$ is enough) convex then $h \geq 0$ on ∂A_R, and hence by the maximum principle, $h > 0$ in $\mathrm{Int}(A_R)$ and so $\kappa = r^{-1}\Lambda^{-1}h$ is also positive. Thus $A(t)$ is locally strictly convex. Since $\Gamma = A(1) \cup A(-1)$ consists of two Jordan curves, we have

$$\int_{|z|=R^{\pm 1}} \kappa ds = 2\pi.$$

By continuity it must be that

$$\int_{|z|=R^t} \kappa ds = 2\pi \quad \text{for} \quad -1 \leq t \leq 1.$$

This proves that $A(t)$ must be simple. Since $\kappa > 0$ on $A(t)$, we conclude that $A(t)$ is a strictly convex Jordan curve for $-1 < t < 1$. □

Remark 27.3 We have used the non-trivial regularity theorem which says that if ∂A is $C^{2,\alpha}$ then $X : A_R \hookrightarrow S(-1,1)$ is also $C^{2,\alpha}$. See [12] II, Theorem 1, page 33.

Theorem 27.4 *Let A be a CBA and ∂A be smooth. Then there is a $\rho > 1$ such that the Gauss map $g : A_R \to \mathbf{C}$ is a conformal diffeomorphism to $\overline{\Omega} \subset A_\rho = \{z \in \mathbf{C} : 1/\rho \leq |z| \leq \rho\}$.*

Proof. By Proposition 27.1, $|g|$ and $|g|^{-1}$ are both bounded, and so we need only prove that g is a diffeomorphism. Indeed, by Theorem 27.2,

$$r^{-1}\Lambda^{-1}\Re(zg'/g) = \kappa > 0,$$

and so $g' \neq 0$ in $\mathrm{Int}(A_R)$ and hence g is a local diffeomorphism.

Consider the set $\gamma = \{z : \phi(z) = \text{const}\}$. Since $\arg g = \phi = \psi - \pi/2$ is strictly increasing on each $\{|z| = r\} \subset \mathrm{Int}(A_R)$ (remember that $\kappa = \phi_s > 0$, in fact ϕ takes every value between 0 and 2π on $\{|z| = r\}$ exactly once), we see that γ is a smooth Jordan arc connecting $\{|z| = 1/R\}$ and $\{|z| = R\}$. Let $\vec{t}$ be the unit tangent vector of γ and $\vec{n}$ its unit normal vector, such that $(\vec{t}, \vec{n})$ has positive orientation. Then since $\log g = \log|g| + i \arg g$ is holomorphic, we have $-\vec{n}\log|g| = \vec{t}\phi = 0$ and so $\vec{t}\log|g| \neq 0$ on γ, as otherwise we would have $g' = 0$. Thus whenever $\arg g(z_1) = \arg g(z_2)$ and $z_1 \neq z_2$, then $\log|g(z_1)| \neq \log|g(z_2)|$, so $g(z_1) \neq g(z_2)$. The holomorphic function g is a one-to-one local diffeomorphism, hence is a conformal diffeomorphism. □

Corollary 27.5 *The total Gauss curvature of a CBA is larger than -4π.*

One interesting corollary of Theorem 27.4 is that

Corollary 27.6 *If A is a CBA with smooth boundary then the second eigenvalue of L_A is positive.*

Proof. By Theorem 27.4, N is an anti-conformal diffeomorphism. By Corollary 32.7 of Appendix, the second eigenvalue of Δ_S on $(N(A_R))$ is larger than 2, thus $\lambda_2(A) > 0$. $\square$

Remember that the index of A is

$$\mathbf{Index}(A) = \sum_{\lambda<0} \dim V_\lambda(A),$$

where V_λ is the eigenspace corresponding to the eigenvalue λ.

Corollary 27.7 *Let A be a CBA, then*

$$\text{Index}(A) = \begin{cases} 0, & \text{if } A \text{ is stable or almost stable;} \\ \\ 1, & \text{if } A \text{ is unstable.} \end{cases} \tag{27.129}$$

Proof. We need only prove the unstable case. First assume that ∂A is smooth. By Corollary 32.9 of Appendix and Corollary 27.6, $\dim V_{\lambda_1} = 1$ and $\lambda_2(A) > 0$, hence $\text{Index}(A) \leq 1$. But if A is unstable, $\text{Index}(A) \geq 1$, thus $\text{Index}(A) = 1$.

If ∂A is only continuous, we define a family of diffeomorphisms of A_R into itself by

$$f_t(z) = f_t(re^{i\theta}) = r^{1-t}e^{i\theta}, \quad 0 \leq t < 1.$$

Then $f_0 = \text{Id}_{A_R}$, $f_t(A_R) \subset f_s(A_R)$ for $0 \leq s < t < 1$, and $\lim_{t\to 1} f_t(A_R) = \{z : |z| = 1\}$; thus $\lim_{t\to 1}\text{Vol}(f_t(A_R)) = 0$.

Using the embedding X, we get a family of diffeomorphisms of A into A, $c_t = X \circ f_t \circ X^{-1}$, $0 \leq t < 1$, satisfying $c_t(A) = A \cap S(t-1, 1-t)$. Note that by Theorem 29.1 of Section 29, each $c_t(A)$, $0 < t < 1$ is a CBA and has smooth boundary, we can use Theorem 27.4 and Corollary 27.6. Moreover, we have

1. $c_0 =$ identity;
2. $c_t(A) \subset c_s(A)$, for $0 \leq s < t < 1$;
3. $\lim_{t\to 1}\text{Vol}(c_t(A)) = 0$.

Recall that $\text{nullity}(c_t(A)) = \dim V_0(c_t(A))$. By a theorem of Morse, Simons, and Smale (see [46], p 52) we have that

$$\text{Index}(A) = \sum_{t>0} \text{nullity}(c_t(A)).$$

If $c_t(A)$ is almost stable then 0 is the first eigenvalue of $c_t(A)$, so by Corollary 32.9 of Appendix, $\text{nullity}(c_t(A)) = \dim V_0(c_t(A)) = 1$. For any $s > t$, $c_s(A) \subset c_t(A)$ is a proper subdomain, so $\lambda_1(c_s(A)) > \lambda_1(c_t(A)) = 0$ and $\text{nullity}(c_s(A)) = 0$. If $c_t(A)$ is unstable and $\text{nullity}(c_t(A)) > 0$, then 0 is at least the second eigenvalue of $c_t(A)$, contradicts Corollary 27.6. Hence we have proved that at most one $t \in (0,1)$ can be such that $\text{nullity}(c_t(A)) = 1$ and for the other t we must have $\text{nullity}(c_t(A)) = 0$. We conclude that $\text{Index}(A) \leq 1$. But if A is unstable, $\text{Index}(A) \geq 1$, thus $\text{Index}(A) = 1$. $\square$

Theorem 27.8 *The index of the catenoid is 1.*

Proof. Let C be the catenoid given by Example 14.2. $C(t) := C \cap S(-t, t)$ is a CBA for $t > 0$. Thus $\text{index}(C(t)) \leq 1$. Since any precompact domain B in $\mathbf{C} - \{0\}$ is contained in some A_R, it follows $X(B) \subset X(A_R) = C(\log R)$. By the definition of index of C, see (20.85), we have $\text{index}(C) \leq 1$.

Since $g(z) = z$ is one-to-one we know by Section 20 that any precompact domain $\Omega \subset S^2 - \{(0,0,1), (0,0,-1)\}$ such that the first eigenvalue of $\triangle_S$, $\lambda_1(\Omega) < 2$, corresponding to an unstable precompact domain on C. Since the first eigenvalue of $\triangle_S$ on S^2 is 0, there are plenty precompact domains in $S^2 - \{(0,0,1), (0,0,-1)\}$ with the first eigenvalue less than 2, a consequence of the fact that λ_1 is continuously dependent on domains. Thus C is not stable and $\text{index}(C) \geq 1$. $\square$

28 The Existence of Minimal Annuli in a Slab

Given two Jordan curves Γ_1, Γ_2 in $\mathbf{R}^3$, does $\Gamma := \Gamma_1 \bigcup \Gamma_2$ bound a minimal annulus? This is called the Douglas-Plateau problem which is a generalisation of the original Plateau problem. If the answer to the Douglas-Plateau problem for a given Γ is yes, then we can ask that how many such minimal annuli are there?

These are very hard and interesting problems. Generally, they are attacked with concepts and techniques, such as those from the geometric measure theory which are quite different from the classical setting as in our notes,

One classical result due to Douglas says that if A_1 and A_2 are the areas of least area minimal disks bounded by Γ_1 and Γ_2 respectively, and

$$\inf\{\mathbf{Area}(S)\} < A_1 + A_2,$$

then there is a minimal annulus bounded by Γ. Here the infimum is taken over all surfaces of annular type bounded by Γ. See [13], or [9].

In many cases the answers to the Douglas-Plateau problem are no. One example is that of two coaxial unit circles C_1 and C_2. If the distance d between their centres is large then $C_1 \cup C_2$ cannot bound a catenoid, and therefore as Shiffman's second theorem (Theorem 29.2) shows, $C_1 \cup C_2$ cannot bound a minimal annulus.

When Γ_1 and Γ_2 are smooth convex planar Jordan curves lying in parallel (but different) planes, the Douglas-Plateau problem has a very satisfactory answer. The combined result of Hoffman and Meeks [28], and Meeks and White [53], says,

Let $\Gamma = \Gamma_1 \bigcup \Gamma_2$. Then there are exactly three cases:

1. *There are exactly two minimal annuli bounded by Γ, one is stable and one is unstable.*

2. *There is a unique minimal annulus A bounded by Γ; it is almost stable in the sense that the first eigenvalue of L_A is zero. This case is not generic.*

3. *There are no minimal annuli bounded by Γ.*

4. *Moreover, if A is a minimal annulus bounded by Γ, then the symmetry group of A is the same as the symmetry group of Γ.*

We are not going to discuss the Douglas-Plateau problem in these notes. Rather, we would like to point out some necessary conditions on Γ if it bounds a minimal annulus.

The next theorem is due to Osserman and Schiffer [70], we follow their proof.

Theorem 28.1 *Let δ_1, δ_2, c, d be positive numbers satisfying*

$$\left(\frac{c^2}{2} + d^2\right)^{1/2} \geq \delta_1 + \delta_2. \tag{28.130}$$

Let Γ_1 and Γ_2 be closed curves in $\mathbf{R}^3$. Let

$$D_1 := \{x \in \mathbf{R}^3 \,|\, x_1^2 + x_2^2 < \delta_1^2, x_3 = 0\},$$

$$D_2 := \{x \in \mathbf{R}^3 \,|\, (x_1 - c)^2 + x_2^2 < \delta_2^2, x_3 = d\}.$$

Then if $\Gamma_1 \subset D_1$ and $\Gamma_2 \subset D_2$, there does not exist any minimal annulus spanning Γ_1 and Γ_2. More generally, the same conclusion holds if we replace D_i by D_i', $i = 1, 2$, where

$$D_1' := \left\{x \in \mathbf{R}^3 \,\Big|\, \left(x_1 - \frac{c}{d}x_3\right)^2 + x_2^2 \leq \delta_1^2, x_3 \leq 0\right\},$$

$$D_2' := \left\{x \in \mathbf{R}^3 \,\Big|\, \left(x_1 - \frac{c}{d}x_3\right)^2 + x_2^2 \leq \delta_2^2, x_3 \geq d\right\}.$$

Remark 28.2 Note that Γ_1 or Γ_2 need not be Jordan curves. Moreover, the theorem is true for minimal annuli in $\mathbf{R}^n$ where $n \geq 3$, with the same proof, see [70].

Suppose $\Gamma_1 \subset P_0$ and $\Gamma_2 \subset P_d$. Let C_1 and C_2 in P_0 and P_d be the smallest circles which enclose Γ_1 and Γ_2 respectively. Let their radii be δ_1 and δ_2. The vertical distance between the centres of C_1 and C_2 is of course d. Let c be the horizontal distance between the centres of C_1 and C_2. Since we can alway adopt coordinates such that C_1 and C_2 are the boundaries of D_1 and D_2 in Theorem 28.1, we conclude that if Γ_1 and Γ_2 span a minimal annulus then

$$\left(\frac{c^2}{2} + d^2\right)^{1/2} \leq \delta_1 + \delta_2. \tag{28.131}$$

In case Γ_1 and Γ_2 are Jordan curves, this is a result of Nitsche, see [63].

To prove Theorem 28.1 we need a lemma.

Lemma 28.3 *Let u be harmonic in an annulus $A := \{r_1 \leq |z| \leq r_2\}$. Suppose $b \geq a$, and*

$$\liminf_{r \to r_1} u(re^{i\theta}) \leq a, \quad \limsup_{r \to r_2} u(re^{i\theta}) \geq b.$$

Then for $r_1 < r < r_2$,

$$\int_0^{2\pi} r\frac{\partial u}{\partial r}(re^{i\theta})d\theta \geq 2\pi\frac{b-a}{\log(r_2/r_1)}.$$

Proof. Given $\epsilon > 0$, let

$$v := u - a - \frac{b - a - \epsilon}{\log(r_2/r_1)}\log\frac{r}{r_1}.$$

Then v is harmonic in A, and

$$\liminf_{r \to r_1} v(re^{i\theta}) \leq 0, \qquad \limsup_{r \to r_2} v(re^{i\theta}) \geq \epsilon. \tag{28.132}$$

Choose ϵ', $0 < \epsilon' < \epsilon$, such that $Dv \neq 0$ on the level curve $C := \{z \in A \,|\, v(z) = \epsilon'\}$. Then C must consist of one or more analytic Jordan curves. But if any subset C' of C bounds a domain $\Omega \subset A$, the function v would be constant on Ω, hence in the whole A, which contradicts (28.132). Thus C consists of a single curve not homologous to zero. Choose δ such that

$$r_1 < \delta < \min_{z \in C} |z|.$$

Then C is homologous to the circle $|z| = \delta$, and hence

$$\int_C \frac{\partial v}{\partial n}\, ds = \int_{|z|=\delta} \frac{\partial v}{\partial n}\, ds.$$

But $v \geq \epsilon'$ outside C and $v = \epsilon'$ on C. Therefore $\partial v/\partial n \geq 0$ on C, where $\partial/\partial n$ is the exterior normal derivative. Thus

$$\int_0^{2\pi} \frac{\partial v}{\partial r}(\delta e^{i\theta})\delta\, d\theta = \int_C \frac{\partial v}{\partial n}\, ds \geq 0.$$

Using the explicit expression for v, we obtain

$$\int_0^{2\pi} \frac{\partial u}{\partial r}(\delta e^{i\theta})\delta\, d\theta \geq 2\pi \frac{b - a - \epsilon}{\log(r_2/r_1)}.$$

Since u is harmonic, the expression on the left side is independent of δ, hence this inequality holds on every circle $|z| = r$. Since ϵ was arbitrary, the lemma is proved. $\square$

Proof of Theorem 28.1. Suppose $X : A = \{r_1 \leq |z| \leq r_2\} \hookrightarrow \mathbf{R}^3$ is a minimal annulus such that $X\Big|_{|z|=r_i}$ is a parametrisation of Γ_i, $i = 1, 2$. We shall show that (28.130) cannot hold.

We define a function $u(z)$ in A by

$$u(z) = \left(X_1(z) - \frac{c}{d}X_3(z)\right)^2 + X_2^2(z). \tag{28.133}$$

Using the fact that X_i's are harmonic, one can calculate that

$$\Delta u = 2\left(\left|\phi_1 - \frac{c}{d}\phi_3\right|^2 + |\phi_2|^2\right) = 2\left(\left|\phi_1 - \frac{c}{d}\phi_3\right|^2 + |\phi_1 + \phi_3|^2\right)$$

by (6.19).

We assert next that if b is an arbitrary real number then

$$\min\{|w - b|^2 + |w^2 + 1|\} = \frac{b^2}{2} + 1, \tag{28.134}$$

where the minimum is taken over all complex numbers w. Namely, setting $w = b + re^{i\theta}$ gives

$$\begin{aligned} |w-b|^2 + |w^2+1| &= r^2 + |b^2 + 2bre^{i\theta} + r^2e^{2i\theta} + 1| \\ &\geq r^2 + b^2 + 1 + 2br\cos\theta + r^2\cos 2\theta \\ &= b^2 + 1 + 2br\cos\theta + 2r^2\cos^2\theta \qquad (28.135) \\ &= b^2 + 1 + 2r^2\left(\cos\theta + \frac{b}{2r}\right)^2 - \frac{b^2}{2} \geq \frac{b^2}{2} + 1. \end{aligned}$$

This gives a lower bound which is actually attained when $w = b/2$. This proves (28.134).

Returning to Δu, we therefore have

$$\Delta u = 2|\phi_3|^2\left[\left|\frac{\phi_1}{\phi_3} - \frac{c}{d}\right|^2 + \left|\left(\frac{\phi_1}{\phi_3}\right)^2 + 1\right|\right] \geq \left[\left(\frac{c}{d}\right)^2 + 2\right]|\phi_3|^2.$$

Using the notation

$$t = \log r, \quad U(t) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})d\theta,$$

we find, as in the proof of Lemma 25.1, that

$$\frac{d^2U}{dt^2} = \frac{1}{2\pi}\int_0^{2\pi} r^2 \Delta u(re^{i\theta})d\theta \geq \frac{c^2+2d^2}{2\pi d^2}\int_0^{2\pi}|\psi_3(re^{i\theta})|^2 d\theta. \qquad (28.136)$$

But

$$\begin{aligned} 2\pi\int_0^{2\pi}|\psi_3(re^{i\theta})|^2d\theta &\geq \left(\int_0^{2\pi}|\psi_3(re^{i\theta})|d\theta\right)^2 \geq \left(\int_0^{2\pi}\Re[\psi_3(re^{i\theta})]d\theta\right)^2 \\ &= \left(\int_0^{2\pi} r\frac{\partial X_3(re^{i\theta})}{\partial r}d\theta\right)^2 \qquad (28.137) \end{aligned}$$

by virtue of (25.114). Now the assumption that $\Gamma_1 \subset D_1'$, $\Gamma_2 \subset D_2'$ implies that $X_3(r_1e^{i\theta}) \leq 0$ and $X_3(r_2e^{i\theta}) \geq d$. By Lemma 28.3, we have

$$\int_0^{2\pi} r\frac{\partial X_3(re^{i\theta})}{\partial r}d\theta \geq 2\pi\frac{d}{T}, \qquad (28.138)$$

where

$$T = \log\frac{r_2}{r_1}. \qquad (28.139)$$

Combining (28.136), (28.137), (28.138) gives

$$\frac{d^2U}{dt^2} \geq \frac{c^2+2d^2}{T^2}. \qquad (28.140)$$

By the definition of D_i', the statement $\Gamma_i \subset D_i'$ implies $u(re^{i\theta}) \le \delta_i^2$, and hence

$$U(t_i) \le \delta_i^2, \quad i = 1,\ 2. \tag{28.141}$$

We may assume that $t_1 = \log r_1 = 0$ and $t_2 = \log r_2 = T$. Set

$$B = \frac{c^2}{2} + d^2 \tag{28.142}$$

so that (28.140) becomes

$$\frac{d^2U}{dt^2} \ge \frac{2B}{T^2}, \quad 0 < t < T. \tag{28.143}$$

Define $V(t)$ to be the parabola

$$V(t) = at^2 + bt + \delta_1^2,$$

satisfying

$$\frac{d^2V}{dt^2} = \frac{2B}{T^2}, \quad V(0) = \delta_1^2, \quad V(T) = \delta_2^2. \tag{28.144}$$

It follows from (28.141), (28.143), (28.144) that

$$U(t) \le V(t), \quad 0 < t < T. \tag{28.145}$$

The conditions (28.144) determine the coefficients a, b of V:

$$a = \frac{B}{T^2}, \quad b = \frac{1}{T}(\delta_2^2 - \delta_1^2 - B). \tag{28.146}$$

Since $a > 0$, $V(t)$ has a minimum at $t = t_0$, where

$$t_0 = -\frac{b}{2a} = T\left(\frac{1}{2} - \frac{\delta_2^2 - \delta_1^2}{2B}\right). \tag{28.147}$$

It follows that

$$t_0 > 0 \Leftrightarrow \delta_2^2 - \delta_1^2 < B,$$
$$t_0 < T \Leftrightarrow \delta_2^2 - \delta_1^2 > -B.$$

Thus

$$0 < t_0 < T \Leftrightarrow |\delta_2^2 - \delta_1^2| < B. \tag{28.148}$$

We consider two cases, according to whether (28.148) does or does not hold. If it does not hold, then

$$B \le |\delta_2^2 - \delta_1^2| = |\delta_2 - \delta_1||\delta_2 + \delta_1| < |\delta_2 + \delta_1|^2. \tag{28.149}$$

On the other hand, if (28.148) does hold, then, by virtue of (28.145) and the fact that $U(t) > 0$ for all t,

$$V(t_0) \ge U(t_0) > 0.$$

But by (28.146) and (28.147),

$$\begin{aligned} V(t_0) &= -\frac{b^2}{4a} + \delta_1^2 > 0 \\ &\Leftrightarrow\ b^2 < 4a\delta_1^2 \Leftrightarrow (\delta_2^2 - \delta_1^2) - 2B(\delta_2^2 + \delta_1^2) + B^2 < 0 \\ &\Rightarrow\ B < (\delta_2^2 + \delta_1^2) + \sqrt{(\delta_2^2 + \delta_1^2)^2 - (\delta_2^2 - \delta_1^2)^2} = (\delta_2 + \delta_1)^2. \end{aligned}$$

Comparing with (28.149), we see in both cases we must have $B < (\delta_1 + \delta_2)^2$. But going back to the definition (28.142) of B, we see that under the assumption that a spanning surface exists, inequality (28.130) must be violated. This proves the theorem. □

29 Shiffman's Theorems

Recall that we defined a CBA as a minimal annulus $A \in S(-1,1)$ such that $A(1) = A \cap P_1$ and $A(-1) = A \cap P_{-1}$ are continuous convex Jordan curves. In the article [76] published in 1956, Max Shiffman proved three elegant theorems about a CBA. They are as follows:

Theorem 29.1 *If A is a CBA, then $A \cap P_t$ is a strictly convex Jordan curve for every $-1 < t < 1$. In particular, $X : A_R \hookrightarrow S(-1,1)$ is an embedding.*

Theorem 29.2 *If A is a CBA and $\Gamma = \partial A$ is a union of circles, then $A \cap P_t$ is a circle for every $-1 \le t \le 1$.*

Theorem 29.3 *If A is a CBA and $\Gamma = \partial A$ is symmetric with respect to a plane perpendicular to xy-plane, then A is symmetric with respect to the same plane.*

We are going to prove the three Shiffman's theorems by means of the Enneper-Weierstrass representation. We have already proved a weaker version of Theorem 29.1, namely Theorem 27.2

Let us first prove Theorem 29.1. We follow the proof of Shiffman. We will write the immersion as $X = (x, y, z)$. For any $\zeta = re^{i\theta} \in A_R$, since X is conformal, by (27.124) we have

$$x_\theta^2 + y_\theta^2 = r^2(x_r^2 + y_r^2) + \frac{1}{(\log R)^2}.$$

The immersion $X : A_R \hookrightarrow S(-1,1)$ satisfies

$$x_\theta^2 + y_\theta^2 \ge \frac{1}{(\log R)^2}. \tag{29.150}$$

Since X is continuous on A_R, $A(1)$ and $A(-1)$ are convex and hence rectifiable. Moreover, $x(R,\theta)$ and $y(R,\theta)$ are functions of bounded variation. Thus $x_\theta(R,\theta)$ and $y_\theta(R,\theta)$ exist almost everywhere. Let I denote the set on which $x_\theta(R,\theta)$ and $y_\theta(R,\theta)$ both exist. We will first prove that:

Lemma 29.4 *For any $\theta \in I$,*

$$\lim_{r\to R} x_\theta(r,\theta) = x_\theta(R,\theta), \quad \lim_{r\to R} y_\theta(r,\theta) = y_\theta(R,\theta), \tag{29.151}$$

and

$$x_\theta^2(R,\theta) + y_\theta^2(R,\theta) \ge \frac{1}{(\log R)^2}. \tag{29.152}$$

Proof. Let $\overline{x}$ and $\overline{y}$ be harmonic functions defined over the disk $D_R := r \leq R$ with boundary values given by $x(R,\theta)$ and $y(R,\theta)$ respectively. The functions $x(r,\theta) - \overline{x}(r,\theta)$ and $y(r,\theta) - \overline{y}(r,\theta)$, being harmonic in A_R and having the boundary value 0 on $r = R$, can be extended across $r = R$ by reflection. Thus

$$x_\theta(r,\theta) - \overline{x}_\theta(r,\theta) \to 0, \quad y_\theta(r,\theta) - \overline{y}_\theta(r,\theta) \to 0,$$

as $r \to R$.

Let P be the Poisson kernel of D_R,

$$P(Re^{i\phi}, re^{i\theta}) = \frac{1}{2\pi} \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\phi - \theta)}.$$

Then the harmonic function $\overline{x}$ can be expressed as

$$\overline{x}(r,\theta) = \int_0^{2\pi} x(R,\phi) P(Re^{i\phi}, re^{i\theta}) d\phi.$$

Differentiating, we have

$$\overline{x}_\theta(r,\theta) = \int_0^{2\pi} x(R,\phi) \frac{\partial P}{\partial \theta} d\phi = -\int_0^{2\pi} x(R,\phi) \frac{\partial P}{\partial \phi} d\phi = \int_0^{2\pi} P dx(R,\phi).$$

It follows, as in the proof of theorem of Fatou (see [59] pages 198 -200) that

$$\lim_{r \to R} \overline{x}_\theta(r,\theta) = x_\theta(R,\theta)$$

on I. Similarly for y_θ. From (29.150) it is obvious that (29.152) is true. □

Consider the harmonic function $\psi(r,\theta)$, the angle of the tangent vector of $A \cap P_t$ with the positive x-axis. We denote the angle defined by the tangent direction at $A(1)$ by $\psi(R,\theta)$ on I. Because of the convexity of $A(1)$, $\psi(R,\theta)$ is a monotonic function of θ on I of period $\pm 2\pi$. We can assume that the period is 2π, and we shall call the orientation described on $A(1)$ as θ varies from 0 to 2π the positive orientation of $A(1)$. The following lemma will be proved.

Lemma 29.5 *The period of $\psi(r,\theta)$ is exactly 2π, and*

$$\lim_{r \to R} \psi(r,\theta) = \psi(R,\theta) \quad \text{for } \theta \in I.$$

The single valued function $\psi(r,\theta) - \theta$ is a bounded harmonic function in A_R.

Proof. Consider the convex curve $A(1)$. Select a point Q_1 on $A(1)$ at which there is a unique supporting line L of $A(1)$, and let Q_3 be a point on $A(1)$ where a line parallel to L, but not coinciding with L, is a supporting line of $A(1)$. Select a direction not included among the directions of all the supporting lines of $A(1)$ at the point Q_3 and let Q_2 and Q_4 be two points of $A(1)$ at which there are supporting lines, distinct from each other, in this direction. The numbering is such that Q_1, Q_2, Q_3, Q_4 occur in the positive orientation around $A(1)$. Consider these four supporting lines as taken in the positive direction in describing $A(1)$, and let angles made by them with the positive x-axis be α_1, α_2, $\alpha_1+\pi$, $\alpha_2+\pi$, respectively, where

$$\alpha_1 < \alpha_2 < \alpha_1 + \pi < \alpha_2 + \pi.$$

Let the points on the circle $r = R$ which are mapped onto Q_1, Q_2, Q_3, Q_4, be denoted by q_1, q_2, q_3, q_4, respectively. On the circle $r = R$ denote the open arc from q_1 to q_3 (taken in the positive orientation and therefore including q_2) by B_1, the open arc from q_2 to q_4 by B_2, from q_3 to q_1 by B_3, and from q_4 to q_2 by B_4. Finally, let C_i be a closed arc on $r = R$ contained in B_i, $i = 1, 2, 3, 4$, such that the C_i together cover $r = R$.

Note that

$$\begin{aligned}(x_\theta, y_\theta) &= (x_\theta^2 + y_\theta^2)^{1/2}(\cos\psi, \sin\psi),\\ (y_\theta, -x_\theta) &= (x_\theta^2 + y_\theta^2)^{1/2}(\sin\psi, -\cos\psi).\end{aligned}$$

Consider first the function

$$Y_1(r,\theta) = y(r,\theta)\cos\alpha_1 - x(r,\theta)\sin\alpha_1, \tag{29.153}$$

which is a harmonic function of (r,θ) in A_R. Then

$$\begin{aligned}\frac{\partial Y_1(r,\theta)}{\partial\theta} &= y_\theta(r,\theta)\cos\alpha_1 - x_\theta(r,\theta)\sin\alpha_1\\ &= (y_\theta, -x_\theta)(r,\theta)\bullet(\cos\alpha_1, \sin\alpha_1)\\ &= (x_\theta^2 + y_\theta^2)^{1/2}(\sin\psi, -\cos\psi)\bullet(\cos\alpha_1, \sin\alpha_1)\\ &= (x_\theta^2 + y_\theta^2)^{1/2}\sin(\psi - \alpha_1).\end{aligned} \tag{29.154}$$

On the arc B_1 of $r = R$, the function $Y_1(R,\theta)$ is a monotonically increasing function of θ, since the arc B_1 corresponds to the portion of $A(1)$ from Q_1 to Q_3; thus $\alpha_1 \le \psi \le \alpha_1 + \pi$. In analogy to the proof of Lemma 29.4, the formula for $\frac{\partial \overline{Y}_1(r,\theta)}{\partial\theta}$ is

$$\frac{\partial \overline{Y}_1(r,\theta)}{\partial\theta} = \left(\int_{B_1} + \int_{CB_1}\right) P\,dY_1(R,\phi) \tag{29.155}$$

where CB_1 is the complement of B_1. The first integral in (29.155) is ≥ 0 for all (r,θ), since $Y_1(R,\phi)$ is an increasing function of ϕ in B_1; the second integral in (29.155) approaches 0 as (r,θ) approaches an interior point of B_1. Thus

$$\lim \inf_{(r,\theta)\to C_1} \frac{\partial \overline{Y}_1(r,\theta)}{\partial\theta} \ge 0.$$

It follows that likewise

$$\lim \inf_{(r,\theta)\to C_1} \frac{\partial Y_1(r,\theta)}{\partial\theta} \ge 0. \tag{29.156}$$

Take a positive ϵ and $\epsilon_1 = (\log R)^{-1}\epsilon$ such that

$$\delta = \arcsin \epsilon < \min\left(\frac{\alpha_2 - \alpha_1}{2}, \frac{\alpha_1 + \pi - \alpha_2}{2}\right).$$

By (29.156) there is a region R_1 in A_R, enclosing C_1, for which

$$\frac{\partial Y_1(r,\theta)}{\partial \theta} > -\epsilon_1, \quad (r,\theta) \in R_1.$$

From (29.150) and (29.154) we therefore see that

$$\sin(\psi - \alpha_1) > -\epsilon \quad \text{in} \quad R_1. \tag{29.157}$$

Selecting a determination of ψ at a particular point of R_1, we have

$$-\delta < \psi - \alpha_1 < \pi + \delta \quad \text{in} \quad R_1. \tag{29.158}$$

A similar argument applies to each of the other arcs B_2, B_3, B_4 of the circle $r = R$, with α_2, $\alpha_1 + \pi$, $\alpha_2 + \pi$, respectively, replacing α_1 in (29.153)-(29.158). On B_2 the function $Y_2 = y\cos\alpha_2 - x\sin\alpha_2$ is an increasing function of θ, leading to the result

$$\lim \inf_{(r,\theta)\to C_2} \frac{\partial Y_2(r,\theta)}{\partial \theta} \geq 0.$$

There is, therefore, a region R_2 of A_R, enclosing C_2, for which

$$\frac{\partial Y_2(r,\theta)}{\partial \theta} > -\epsilon_1, \quad (r,\theta) \in R_2.$$

And we have, analogously to (29.157),

$$\sin(\psi - \alpha_2) > -\epsilon \quad \text{in} \quad R_2.$$

But this means, from (29.158), begin with an already determined ψ in the region common to R_1, R_2, that

$$-\delta < \psi - \alpha_2 < \pi + \delta \quad \text{in} \quad R_2.$$

Similar arguments apply successively to the determination of the regions R_3, R_4 and of the corresponding inequalities for ψ:

$$-\delta < \psi - (\alpha_1 + \pi) < \pi + \delta \text{ in } R_3, \quad -\delta < \psi - (\alpha_2 + \pi) < \pi + \delta \text{ in } R_4. \tag{29.159}$$

Therefore, in the portion common to R_4 and R_1, the value of ψ returns to its initial value plus exactly 2π, or the period of ψ is exactly 2π.

The regions R_1, R_2, R_3, R_4, together form a neighbourhood of the circle $r = R$ in A_R.

A similar argument as the above applies to the inner circle $r = 1/R$ and $A(-1)$. By continuity, ψ has period 2π for every $1/R \leq r \leq R$.

Let θ be a value in the set I and take the limit of $\psi(r,\theta)$ as $r \to R$. By (29.151), (29.152) the limit of $\psi(r,\theta)$ is $\psi(R,\theta)$ modulo 2π. But the inequalities (29.157), (29.158), (29.159) show that the limit must be exactly $\psi(R,\theta)$. The lemma is proved. $\square$

We can now establish the inequality

$$\psi_\theta(r,\theta) > 0 \tag{29.160}$$

everywhere in the interior of A_R. Let $G = G(R,\phi,r,\theta)$ be the Green's function for the annular ring A_R, with singularity at (r,θ). In its dependence on ϕ and θ, G is a function of $\phi - \theta$. We have

$$\begin{aligned} \psi(r,\theta) - \theta &= \int_{\partial A_R} [\psi(r,\phi) - \phi] \frac{\partial G}{\partial \nu} ds \qquad (29.161) \\ &= \int_{r=R} [\psi(R,\phi) - \phi] R \frac{\partial G}{\partial \nu} d\phi + \int_{r=R^{-1}} [\psi(R^{-1},\phi) - \phi] R^{-1} \frac{\partial G}{\partial \nu} d\phi, \end{aligned}$$

where ν is the inward normal. This follows by considering the analogous formula for an interior annular ring, and performing the passage to the limit. Differentiating (29.162) with respect to θ, using $\partial(\partial G/\partial\nu)/\partial\theta = -\partial(\partial G/\partial\nu)/\partial\phi$, and intergrating by parts, we find

$$\psi_\theta - 1 = \int_{r=R} R \frac{\partial G}{\partial \nu} d[\psi(R,\phi) - \phi] + \int_{r=R^{-1}} R^{-1} \frac{\partial G}{\partial \nu} d[\psi(R^{-1},\phi) - \phi]$$

or

$$\psi_\theta = \int_{r=R} \frac{\partial G}{\partial \nu} R\, d\psi(R,\phi) + \int_{r=R^{-1}} \frac{\partial G}{\partial \nu} R^{-1} d\psi(R^{-1},\phi),$$

since

$$\int_{r=R} \frac{\partial G}{\partial \nu} R d\phi + \int_{r=R^{-1}} \frac{\partial G}{\partial \nu} R^{-1} d\phi = \int_{\partial A_R} \frac{\partial G}{\partial \nu} ds = 1.$$

Since $\partial G/\partial\nu > 0$ and $\psi(R,\phi)$, $\psi(R^{-1},\phi)$ are monotonic increasing functions of ϕ of period 2π, inequality (29.160) is obtained. Thus each $A(t)$ is a closed strictly convex curve and has total curvature 2π, so it must be a Jordan curve. Therefore, X must be an embedding. Theorem 29.1 is proved. $\square$

Theorem 29.2 is a special case of Theorem 30.1 in the next section, so we will postpone the proof until then. Instead we will prove Theorem 29.3 next.

Proof of Theorem 29.3 : We can assume that ∂A is symmetric with respect to the xz-plane. By Theorem 29.1, each $A(z)$ is a strictly convex Jordan curve for $-1 < z < 1$; hence there are exactly two points on $A \cap P_z$ at which the supporting lines of $A(z)$ are perpendicular to the xz-plane. Varying z we get two curves on A, say α_1 and α_2. Let P be the orthogonal projection on the xz-plane. The A consists of two pieces of graphs on the domain $\Omega = P(A) \subset xz$-plane, thus we have $(x, y_i(x,z), z)$, $i = 1, 2$. Moreover, $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where Γ_1 is the projection of $A(1) \cup A(-1)$ and $\Gamma_2 = P(\alpha_1 \cup \alpha_2)$. It is clear that on Γ_2 the graphs $(x, y_i(x,z), z)$ are perpendicular to the xz-plane.

Now assume that $A(1)$ and $A(-1)$ are strictly convex. Reflecting the graph generated by y_2 about the xz-plane we get a minimal graph generated by $\tilde{y}_2 = -y_2 : \Omega \to \mathbf{R}$. On Γ_1, we have $\tilde{y}_2 = y_1$ by the boundary symmetry. A theorem of Giusti ([22] Lemma 2.2) says that if $(x, y_1(x,z), z)$ and $(x, \tilde{y}_2(x,z), z)$ are perpendicular to the xz-plane on Γ_2 and $y_1 \geq \tilde{y}_2$ on Γ_1, then $y_1 \geq \tilde{y}_2$ on Ω. Since $y_1 = \tilde{y}_2$ on Γ_1, we have $y_1 = \tilde{y}_2$ in Ω.

If $A(1)$ or $A(-1)$ is not strictly convex, then by continuity of the surface, we know that for any $\epsilon > 0$ there is a $\delta > 0$ so small that $y_1(x, 1-t) \geq \tilde{y}_2(x, 1-t) - \epsilon$ and $y_1(x, -1+t) \geq \tilde{y}_2(x, -1+t) - \epsilon$ for $0 < t < \delta$. Thus on $\Omega \cap \{(x,z) \mid -1+\delta < z < 1-\delta\}$, $y_1 \geq \tilde{y}_2$. Letting $\epsilon \to 0$, we have $y_1 \geq \tilde{y}_2$ in Ω. Changing the role of y_1 and $\tilde{y}_2$, we have $y_1 = \tilde{y}_2$ in Ω.

But $y_1 = \tilde{y}_2$ means that A is symmetric about the xz-plane, the proof is complete. $\square$

30 A Generalisation of Shiffman's Second Theorem

Shiffman's second theorem says that if a minimal annulus is bounded by circles in parallel planes, then every level set is a circle.

In [25], it is proved that the same conclusion is true if we replace the boundary circles in Theorem 29.2 by parallel straight lines and assume A is properly embedded.

Furthermore, Toubiana [78] has proved that if two non-parallel straight lines lie in distinct parallel planes then they cannot bound any proper minimal annulus in the slab bounded by the planes.

In this section we will give a generalization of the results stated above, with a unified proof.

Theorem 30.1 *Suppose $A \subset S(-1,1)$ is a minimal annulus in a slab and $A(1) = A \cap P_1$, $A(-1) = A \cap P_{-1}$ are straight lines or circles.*

1. *If both $A(1)$ and $A(-1)$ are circles, then $A(t) = A \cap P_t$ is a circle for $-1 < t < 1$. In particular, A is embedded.*
2. *If at least one of the $A(1)$ and $A(-1)$ is a straight line and A is properly embedded, then $A(t) = A \cap P_t$ is a circle for $-1 < t < 1$.*

Remark 30.2 The first part of Theorem 30.1 is exactly Shffiman's second theorem, Theorem 29.2. We will see that the second part of theorem 30.1 implies the results in [25] and [78].

Let $A \subset S(-1,1)$ be a proper minimal annulus such that $A(1) = A \cap P_1$ and $A(-1) = A \cap P_{-1}$ are straight lines or circles and $\partial A = A(1) \cup A(-1)$. In the case that there is only one straight line, we will always assume that $A(1)$ is the straight line. Then the interior of A is conformally equivalent to the interior of

$$A_R = \{z \in \mathbf{C} : 1/R \leq |z| \leq R\},$$

for some $1 < R < \infty$. In fact the interior of A is conformally equivalent to

$$\{z \in \mathbf{C} : \rho < |z| < P,\ 0 \leq \rho < P \leq \infty\},$$

for some ρ and P. Since A has 1-dimensional boundary ∂A which is separated by the interior of A, it follows $0 < \rho$ and $P < \infty$. Hence if $R = \sqrt{P/\rho} > 1$ then $\mathrm{Int}(A) \cong \mathrm{Int}(A_R)$.

There is a conformal harmonic immersion

$$X : A_R - C \hookrightarrow S(-1,1),$$

where C is a subset of ∂A_R and $X(\{|z| = R\} - C) = A(1)$, $X(\{|z| = 1/R\} - C) = A(-1)$. If $A(1)$ and $A(-1)$ are both circles, then $C = \emptyset$; if only $A(1)$ is a straight line, then

$C \subset \{|z| = R\}$; if $A(1)$ and $A(-1)$ are both straight lines, then $C \cap \{|z| = R\} \neq \emptyset$ and $C \cap \{|z| = 1/R\} \neq \emptyset$. When $C \neq \emptyset$ we assume that X is a proper embedding.

The Enneper-Weierstrass representation of A is

$$X(z) = \Re \int_1^z (\omega_1, \ \omega_2, \ \omega_3) + V,$$

where $V = (a, b, 0) \in \mathbf{R}^3$, and

$$\begin{cases} \omega_1 = \frac{1}{2}(1 - g^2(z))f(z)dz, \\ \\ \omega_2 = \frac{i}{2}(1 + g^2(z))f(z)dz, \\ \\ \omega_3 = g(z)f(z)dz, \end{cases} \tag{30.162}$$

where g is the Gauss map and f is a holomorphic function. We first prove some facts about such a minimal immersion.

Lemma 30.3 *Suppose $X : \{1/R < |z| < R\} \to S(-1, 1)$ is a properly immersed minimal annulus and is embedded in a neighbourhood of $\{|z| = R\} \cup \{|z| = 1/R\}$. Let $g : \{1/R < |z| < R\} \to \mathbf{C}$ be the Gauss map of X. Let $A = X(\{1/R < |z| < R\})$. Suppose that $\partial A \subset P_1 \cup P_{-1}$ and $A(1) = \partial A \cap P_1$, $A(-1) = \partial A \cap P_{-1}$ are circles or straight lines. Let $C \subset \{|z| = 1/R\} \cup \{|z| = R\}$ be the set such that $|X(z_n)| \to \infty$ whenever $z_n \to z \in C$, then $C \cap \{|z| = R\} = p$ and $C \cap \{|z| = 1/R\} = q$ if they are not empty sets. The Gauss map g can be extended to a neighbourhood of A_R such that the extended g at p and q has either zero or pole. Moreover, the Gauss map g has neither zero nor pole in a neighbourhood of A_R except at p and q.*

Furthermore, the third coordinate function X^3 can be extended to the whole A_R such that $X^3|_{|z|=1/R} = -1$ and $X^3|_{|z|=R} = 1$.

Proof. Let $J = X(\{|z| = 1\})$ be the Jordan curve on A and let A_1 be the proper minimal annulus in A with boundary $A(1)$ and J. Suppose that $A(1)$ is a straight line, then let S be the rotation around $A(1)$ of angle π. By the Rotation Theorem (Theorem 8.2) and Extension Theorem (Theorem 4.2), $A_1 \cup S(A_1)$ is a smooth proper minimal surface with boundary $J \cup S(J)$. The conformal structure of $A_1 \cup S(A_1)$ is then $\{1 < |z| < R^2\} - C \cap \{|z| = R\}$ (with the mapping $Y(z) = X(z)$ for $z \in A_R - C$ and $Y(z) = S(X(R^2 z/|z|^2))$ for $z \in \{R < |z| < R^2\}$).

Since $\{|z| = R\} - C$ and $\{|z| = 1/R\} - C$ are homeomorphic to straight lines or circles, they are connected. It turns out that $C \cap \{|z| = R\}$ and $C \cap \{|z| = 1/R\}$ are also connected, hence simply connected as an interval.

Let $D \subset \{1 < |z| < R^2\}$ be a disk like neighbourhood of $C \cap \{|z| = R\}$ such that $z \in D$ if and only if $R^2 z/|z| \in D$ and ∂D is diffeomorphic to a circle, and the $Y(\partial D)$ is a Jordan curve on $A_1 \cup S(A_1)$ which bounds a properly embedded minimal annulus $\tilde{A} = Y(D - C \cap \{|z| = R\})$. Since $A_1 \cap S(A_1)$ is contained in the slab $S(-1, 3)$, by the Cone Lemma (Theorem 21.1), $\tilde{A}$ has finite total curvature. Then by Lemma 10.5, Propositions 10.7 and 10.6, this annular end has the conformal structure of a punctured disk, and the Gauss map of $\tilde{A}$ can be extended to the puncture. In particular, $C \cap \{|z| = R\}$ is a single point p and the Gauss map $g : D \to \mathbf{C}$ of Y can be extended to p, and $g(p)$ is either zero or ∞. Similarly, we can prove that $C \cap \{|z| = 1/R\} = \{q\}$ if it is not empty and $g(q)$ is either zero or ∞.

Since p corresponds to an embedded flat annular end, by Theorem 11.8 we know that there is a $\delta_1 > 0$ such that when $1 - \delta_1 < z < 1$, $P_z \cap A$ is compact. By Lemma 23.2, the tangent plane of A at any point of $A \cap P_z$ is not parallel to the xy-plane. In particular, $dX^3 \neq 0$ on $(X^3)^{-1}(z)$. Thus $(X^3)^{-1}(z)$ is a 1-dimensional submanifold of A_R consists of smooth loops. If it has more than one loop or any loop is homologically trivial, then using the maximum principle we can show that A is contained in a plane. Thus $(X^3)^{-1}(z)$ is a homologically non-trivial smooth Jordan curve. Similarly, if $A(-1)$ is a straight line, then there is a $\delta_2 > 0$ such that when $-1 < z < -1+\delta_2$, $(X^3)^{-1}(z)$ is a homologically non-trivial smooth Jordan curve. Let A'_z be the closed annulus bounded by $(X^3)^{-1}(z)$ and $(X^3)^{-1}(-z)$, for $0 < 1 - z < \min\{\delta_1, \delta_2\}$. Clearly A'_z is compact and $A'_z = X^{-1}(A \cap S(-z, z))$. Since $A \cap P_z$ is compact for $-1 < z < 1$, by Lemma 23.2, the extended Gauss map g of Y does not equal to zero or ∞ in a neighbourhood of A_R except at p or q.

For any sequence $z_n \to p$, since $p \notin A'_z$ for $1 - \delta_1 < z < 1$, $z_n \notin A'_z$ for almost all z_n. Thus $X^3(z_n)$ must converge to 1. Similarly, for any sequence $z_n \to q$, $X^3(z_n)$ must converge to -1. Thus the third coordinate function X^3 can be continuously extended to the whole A_R such that $X^3|_{|z|=1/R} = -1$ and $X^3|_{|z|=R} = 1$. □

The harmonic third coordinate function X^3 satisfies $X^3|_{|z|=1/R} = -1$ and $X^3|_{|z|=R} = 1$ and $-1 < X^3|_{\mathrm{Int}(A_R)} < 1$. Hence we have

$$X^3 = \frac{1}{\log R} \log |z|,$$

and

$$\omega_3 = f(z)g(z)dz = 2\frac{\partial X^3}{\partial z} dz = \frac{d}{dz}\left(\frac{1}{\log R} \log z\right) dz = \frac{1}{\log R}\frac{1}{z}dz.$$

Thus

$$f(z) = \frac{1}{\log R}\frac{1}{zg(z)},$$

and

$$\begin{cases} \omega_1 = \frac{1}{\log R}\frac{1}{2z}\left(\frac{1}{g} - g\right) dz \\ \omega_2 = \frac{1}{\log R}\frac{i}{2z}\left(\frac{1}{g} + g\right) dz \\ \omega_3 = \frac{1}{\log R}\frac{1}{z}dz, \end{cases}$$

and X can be represented as

$$X(p) = \frac{1}{\log R}\Re\int_1^p \left(\frac{1}{2z}(1/g - g), \frac{i}{2z}(1/g + g), \frac{1}{z}\right) dz + V. \tag{30.163}$$

Let

$$g(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad \frac{1}{g(z)} = \sum_{n=-\infty}^{\infty} b_n z^n. \tag{30.164}$$

Then by (27.126), (30.163) gives a minimal annulus if and only if

$$\Im(b_0) = \Im(a_0), \quad \Re(b_0) = -\Re(a_0). \tag{30.165}$$

Remark 30.4 Let S be the 180°-rotation around the straight line $A(1)$ in $\mathbf{R}^3$, and $\mathcal{S} = A \cup S(A)$. Then

$$\int_A K dA = \frac{1}{2}\int_{\mathcal{S}} K dA, \tag{30.166}$$

where K is the Gauss curvature, and dA is the area element of A.

As in the proof of Theorem 27.2, $\kappa = r^{-1}\Lambda^{-1}\Re(zg'/g)$ in the interior of A_R. We must prove that κ is a non-zero constant on each $\{|z| = r\}$, $1/R < r < R$. This is equivalent to prove that $\kappa_\theta \equiv 0$. For that we calculate

$$\begin{aligned}
r\kappa_\theta &= (\Lambda^{-1})_\theta \Re\left(\frac{zg'}{g}\right) + \Lambda^{-1}\left[\Re\left(\frac{zg'}{g}\right)\right]_\theta \\
&= -\frac{1}{2}\Lambda^{-1}\Lambda^{-2}(\Lambda^2)_\theta \Re\left(\frac{zg'}{g}\right) + \Lambda^{-1}\Re\left[iz\frac{d}{dz}\left(\frac{zg'}{g}\right)\right] \\
&= -\Lambda^{-1}\Re\left(\Lambda^{-2}\frac{\partial\Lambda^2}{\partial z}iz\right)\Re\left(\frac{zg'}{g}\right) - \Lambda^{-1}\Im\left[z\frac{d}{dz}\left(\frac{zg'}{g}\right)\right] \\
&= \Lambda^{-1}\Im\left(\frac{|g|^2-1}{1+|g|^2}\frac{zg'}{g}\right)\Re\left(\frac{zg'}{g}\right) - \Lambda^{-1}\Im\left[z\frac{d}{dz}\left(\frac{zg'}{g}\right)\right] \\
&= \Lambda^{-1}\frac{|g|^2-1}{|g|^2+1}\Im\left(\frac{zg'}{g}\right)\Re\left(\frac{zg'}{g}\right) - \Lambda^{-1}\Im\left[z\frac{d}{dz}\left(\frac{zg'}{g}\right)\right] \\
&= \frac{1}{2}\Lambda^{-1}\frac{|g|^2-1}{|g|^2+1}\Im\left(\frac{zg'}{g}\right)^2 - \Lambda^{-1}\Im\left[z\frac{d}{dz}\left(\frac{zg'}{g}\right)\right] \\
&= \Lambda^{-1}\Im\left[\frac{1}{2}\frac{|g|^2-1}{|g|^2+1}\left(\frac{zg'}{g}\right)^2 - z\frac{d}{dz}\left(\frac{zg'}{g}\right)\right].
\end{aligned}$$

Let

$$\begin{aligned}
H(z) &= \frac{1}{2}\frac{|g|^2-1}{|g|^2+1}\left(\frac{zg'}{g}\right)^2 - z\frac{d}{dz}\left(\frac{zg'}{g}\right) \\
&= -\frac{1}{|g|^2+1}\left(z\frac{g'}{g}\right)^2 + \frac{1}{2}\left(z\frac{g'}{g}\right)^2 - z\frac{d}{dz}\left(z\frac{g'}{g}\right).
\end{aligned} \tag{30.167}$$

Note that

$$v := r\Lambda\kappa_\theta = \Im H. \tag{30.168}$$

Since $r\Lambda > 0$, to prove $\kappa_\theta \equiv 0$ we only need prove that $v \equiv 0$.

Since

$$\frac{1}{2}\left(z\frac{g'}{g}\right)^2 - z\frac{d}{dz}\left(z\frac{g'}{g}\right)$$

is holomorphic, we have

$$\begin{aligned}
\triangle H &= 4\frac{\partial^2 H}{\partial z \partial \overline{z}} = -4\frac{\partial^2}{\partial z\partial\overline{z}}\left[\frac{1}{1+|g|^2}\left(\frac{zg'}{g}\right)^2\right] = -4\frac{\partial}{\partial z}\frac{-g\overline{g'}}{(1+|g|^2)^2}\left(\frac{zg'}{g}\right)^2 \\
&= 4\frac{|g'|^2(1+|g|^2) - 2g'\overline{g}g\overline{g'}}{(1+|g|^2)^3}\left(\frac{zg'}{g}\right)^2 + 8\frac{g\overline{g'}}{(1+|g|^2)^2}\left(z\frac{g'}{g}\right)\frac{d}{dz}\left(z\frac{g'}{g}\right) \\
&= \frac{-8|g'|^2}{(1+|g|^2)^2}\left[\frac{1}{2}\frac{|g|^2-1}{|g|^2+1}\left(\frac{zg'}{g}\right)^2 - z\frac{d}{dz}\left(z\frac{g'}{g}\right)\right].
\end{aligned}$$

By (7.28) and (7.30),

$$\Lambda^2 = \frac{1}{4}|f|^2\left(1+|g|^2\right)^2, \quad K = -\left[\frac{4|g'|}{|f|(1+|g|^2)^2}\right]^2,$$

hence we have

$$\frac{8|g'|^2}{(1+|g|^2)^2} = -\frac{1}{2}K|f|^2(1+|g|^2)^2 = -2K\Lambda^2.$$

Thus

$$\triangle H = 2K\Lambda^2 H. \tag{30.169}$$

Taking the imaginary part, we have

$$\triangle v = 2K\Lambda^2 v. \tag{30.170}$$

Remember that $\triangle_A = \Lambda^{-2}\triangle_{A_R} = \Lambda^{-2}\triangle$. If $\Gamma = A(1)$ and $A(-1)$ are straight lines or circles, then $\kappa_\theta \equiv 0$ on $\partial A_R - C$. Hence on A_R, v satisfies

$$\begin{cases} \triangle_A v - 2Kv = 0, \\ v|_{\partial A_R - C} = 0, \end{cases} \tag{30.171}$$

We want to prove that v is continuous on A_R and $v|_{\partial A_R} \equiv 0$, i.e., v is an eigenfunction corresponding to the eigenvalue zero. When $A(1)$ and $A(-1)$ are circles this is certainly true. The next lemma shows that it is always true.

Lemma 30.5 *Let A be as in Theorem 30.1, p, q be as in Lemma 30.3, and v be as defined in (30.168). Then v is continuous on A_R and $v|\partial A_R = 0$.*

Proof. Without loss of generality, we can assume that $p = R$. By Lemma 30.3, we can assume that the Gauss map g has limit zero at $p = R$ and g can be extended to a holomorphic function $\tilde{g}$. Let $\zeta = z - R$ on a disk D_ρ centered at $z = R$, we have

$$\tilde{g}(z) = (z - R)^n \tilde{h}(z) = \zeta^n h(\zeta),$$

where $\tilde{h}$ is a holomorphic function on D_ρ and $h(R) \neq 0$.

By definition, $v = \Im H$ and

$$H(z) = \frac{1}{2}\frac{|g|^2 - 1}{|g|^2 + 1}\left(\frac{zg'}{g}\right)^2 - z\frac{d}{dz}\left(\frac{zg'}{g}\right) = \left(1 - \frac{1}{|g|^2 + 1}\right)\left(\frac{zg'}{g}\right)^2 - \frac{1}{2}\left(\frac{zg'}{g}\right)^2 - z\frac{d}{dz}\left(\frac{zg'}{g}\right).$$

For convenience, we will write g and h instead of $\tilde{g}$ and $\tilde{h}$. Note that

$$\zeta^2\left(z\frac{g'(z)}{g(z)}\right)^2$$

is holomorphic on D_ρ and since $|g|^2 = |z - R|^4|h(z)|^2 = |\zeta|^4|h(z)|^2$,

$$\frac{1}{\zeta^2}\left(1 - \frac{1}{1 + |g|^2}\right) = \frac{1}{\zeta^2}\sum_{k=1}^{\infty}(-1)^{k+1}|g|^{2k} = \overline{\zeta^2}\sum_{k=1}^{\infty}(-1)^{k+1}|\zeta|^{4(k-1)}|h(z)|^{2k}$$

is a C^∞ complex function in a neighborhood of R. Thus

$$\Psi(z) := \left(1 - \frac{1}{|g|^2 + 1}\right)\left(\frac{zg'}{g}\right)^2 = \frac{1}{\zeta^2}\left(1 - \frac{1}{1 + |g|^2}\right)\zeta^2\left(z\frac{g'(z)}{g(z)}\right)^2$$

is a C^∞ complex valued function near $z = R$. If we can prove that

$$\Phi(z) := -\frac{1}{2}\left(\frac{zg'}{g}\right)^2 - z\frac{d}{dz}\left(\frac{zg'}{g}\right)$$

is holomorphic in a neighbourhood of R, then H is a C^∞ complex valued function in a neighbourhood of R. In particular, $v = \Im H$ is C^∞ in a neighbourhood of R, and thus $v(R) = 0$ since on $|z| = R$ and $z \neq R$ we already know that $v(z) = 0$.

Since R corresponds to an embedded flat end, and that end intersects P_1 at a straight line, we have $n = 2$ by Proposition 11.14. Hence

$$z\frac{g'(z)}{g(z)} = \frac{2R}{z - R} + 2 + z\frac{h'(z)}{h(z)}, \quad \text{or} \quad z\frac{g'(z)}{g(z)} = \frac{a_{-1}}{\zeta} + \sum_{k=0}^{\infty} a_k\zeta^k,$$

where

$$a_{-1} = 2R \quad \text{and} \quad a_0 = 2 + R\frac{h'(R)}{h(R)}.$$

Moreover

$$\left(z\frac{g'(z)}{g(z)}\right)^2 = \frac{a_{-1}^2}{\zeta^2} + \frac{2a_{-1}a_0}{\zeta} + \sum_{k=0}^{\infty} b_k\zeta^k,$$

$$z\frac{d}{dz}\left(z\frac{g'(z)}{g(z)}\right) = -\frac{a_{-1}R}{\zeta^2} - \frac{a_{-1}}{\zeta} + (\zeta+R)\sum_{k=1}^{\infty} ka_k\zeta^{k-1},$$

and

$$\begin{aligned}
&-\frac{1}{2}\left(z\frac{g'(z)}{g(z)}\right)^2 - z\frac{d}{dz}\left(z\frac{g'(z)}{g(z)}\right) \\
= &-\frac{1}{2}\frac{a_{-1}^2 - 2a_{-1}R}{\zeta^2} - \frac{a_{-1}a_0 - a_{-1}}{\zeta} - \frac{1}{2}\sum_{k=0}^{\infty} b_k\zeta^k - (\zeta+R)\sum_{k=1}^{\infty} ka_k\zeta^{k-1}.
\end{aligned}$$

Since $a_{-1} = 2R$,

$$a_{-1}^2 - 2a_{-1}R = 0.$$

We would like to prove that $a_{-1}a_0 - a_{-1} = 0$ and thus Φ is holomorphic near $z = R$.

The a_0 can be calculated as follows. The Weierstrass representation for the extended surface $\mathcal{S}$ is

$$\begin{cases}
\omega_1 = \frac{1}{\log R}\frac{1}{2z}\left(\frac{1}{g} - g\right)dz \\
\omega_2 = \frac{1}{\log R}\frac{i}{2z}\left(\frac{1}{g} + g\right)dz \\
\omega_3 = \frac{1}{\log R}\frac{1}{z}dz,
\end{cases}$$

as commented after Lemma 30.3. Let C be a loop around $z = R$ in a small disk. Then since $X : \{z : 1/R < |z| < R^3\} - \{R\} \to \mathbf{R}^3$ is well defined and

$$X(z) = \Re\int_{p_0}^{z}(\omega_1, \omega_2, \omega_3),$$

we must have

$$\Re\int_C \frac{1}{2z}\left(\frac{1}{g(z)} - g(z)\right)dz = 0, \quad -\Im\int_C \frac{1}{2z}\left(\frac{1}{g(z)} + g(z)\right)dz = 0,$$

and

$$\int_C \frac{1}{zg}dz = \overline{\int_C \frac{g}{z}dz} = 0,$$

since $g(z)/z$ is holomorphic at $z = R$. Hence we know that the residue of $1/zg(z)$ at $z = R$ is zero. Hence we have

$$\begin{aligned}
0 &= \lim_{z\to R}\left(\frac{(z-R)^2}{zg(z)}\right)' = \lim_{z\to R}\left(\frac{1}{zh(z)}\right)' \\
&= \lim_{z\to R}\left(-\frac{1}{z^2h(z)} - \frac{h'(z)}{zh^2(z)}\right) \\
&= -\frac{1}{R^2h(R)} - \frac{h'(R)}{Rh^2(R)}.
\end{aligned}$$

Thus

$$\frac{h'(R)}{h(R)} = -\frac{1}{R}, \quad \text{and} \quad a_0 = 2 + R\frac{h'(R)}{h(R)} = 1.$$

This shows that $a_{-1}a_0 - a_{-1} = 0$.

Note that by orientability, if $g(p) = 0$ then $g(q) = \infty$. Using

$$\frac{(1/g)'}{1/g} = -\frac{g'}{g},$$

we can prove that Φ is holomorphic near q exactly as above. □

Now by (30.170) and Lemma 30.5, v is a Jacobi field. Moreover, v satisfies

$$\begin{cases} \triangle_A v - 2Kv = 0, \\ \\ v|_{\partial A_R} = 0. \end{cases} \tag{30.172}$$

Recall that $v = \Im H = r\Lambda\kappa_\theta$.

If $v \not\equiv 0$, then the zero set of v divides A_R into connected subdomains, called *nodal domains*. As mentioned in Section refsec, any proper subdomain of a nodal domain is stable. Thus by Theorem 20.3, the total curvature of each nodal domain is less than or equal to -2π. Suppose that there are k nodal domains; the total curvature of A must be less than or equal to $-2k\pi$.

By our hypothesis that A is embedded and the proof of Lemma 30.3, $A(t)$ is a Jordan curve for $-1 < t < 1$. By the four-vertex-theorem, see [36], which says that if $\kappa_\theta \not\equiv 0$ then the zero set of κ_θ divides each $A(t)$ into at least four components, we know that there are at least four nodal domains. Thus if $v \not\equiv 0$, then $K(A) \leq -8\pi$.

The next lemma shows that in fact, $K(A) \geq -4\pi$. This contradiction then shows that $v \equiv 0$, which is equivalent to κ being constant along each $A(t)$ for $-1 < t < 1$. Since $A(t)$ is a Jordan curve, we know that $A(t)$ must be a circle.

Lemma 30.6 *Suppose that $A \subset S(-1,1)$ is a proper minimal annulus, and $\partial A = A(1) \cup A(-1)$. If $A(1) = A \cap P_1$ and $A(-1) = A \cap P_{-1}$ are circles or straight lines, then*

$$\int_A K dA \geq -4\pi.$$

Proof. If $A(1)$ and $A(-1)$ are both circles, then by Theorem 27.4 the Gauss map g is one-one onto a sphere domain. Hence $\int_A K dA > -4\pi$.

Now assume that $A(1)$ is a straight line and $C \cap \{|z| = R\} = \{p\}$, then $A(-1)$ is a circle. We will use the extended surface $\mathcal{S}$ in the proof of Lemma 30.3 to calculate the total curvature of A. Notice that $\mathcal{S}$ has an embedded flat annular end corresponding to the point p. Since the end is embedded, the order of Λ at that end is 2. Let

$D_\rho \subset \{1/R < |z| < R^3\}$ be a disk centred at p and radius ρ. Note that $\chi(\{1/R < |z| < R^3\} - \{p\}) = -1$. By the Gauss-Bonnet theorem

$$K(\mathcal{S}) = -2\pi - \int_{|z|=1/R} \kappa_g ds - \int_{|z|=R^3} \kappa_g ds - \int_{\partial D_\rho} \kappa_g ds.$$

Using the same argument as in Theorem 23.1 we have

$$\lim_{\rho\to 0} \int_{\partial D_\rho} \kappa_g ds = 2\pi.$$

Notice that the other two integrals are larger than -2π because $A(-1)$ and $R(A(-1))$ are circles and

$$\int_{|z|=1/R} \kappa_g ds = \int_{A(-1)} \kappa_g ds, \quad \int_{|z|=R^3} \kappa_g ds = \int_{R(A(-1))} \kappa_g ds.$$

We have

$$\int_S K dA > -8\pi.$$

By (30.166), we conclude that the total curvature of A is larger than -4π.

Assume $\{p\} = C \cap \{|z| = R\}$ and $\{q\} = C \cap \{|z| = 1/R\}$, i.e., $A(1)$ and $A(-1)$ are both straight lines. Then let D^1_ρ and D^2_ρ be two disks centered at p and q with radii ρ and let $M_\rho = A_R - (D^1_\rho \cup D^2_\rho)$. Since p and q correspond to embedded ends, Λ has order 2 at p and q. Thus

$$\int_{M_\rho} K\, dA + \int_{\partial M_\rho} \kappa_g\, ds + \sum_i (\alpha_i + \beta_i) = 0,$$

where α_i and β_i are the exterior angles at $\partial D^i_\rho \cap \partial A_R$, and obviously

$$\lim_{\rho\to 0} (\alpha_i + \beta_i) = \pi.$$

Again by the same argument as in Theorem 13.4, noting that Λ has poles at p and q, we have

$$\lim_{\rho\to 0} \int_{\partial D^i_\rho \cap A_R} \kappa_g\, ds = \pi.$$

Since $A(1)$ and $A(-1)$ are straight lines,

$$\lim_{\rho\to 0} \int_{\partial M_\rho - \cup \partial D^i_\rho} \kappa_g\, ds = \int_{\partial A_R} \kappa_g\, ds = 0.$$

Thus we have

$$K(A) = \lim_{\rho\to 0} \int_{M_\rho} K\, dA = -4\pi.$$

$\square$

The proof of theorem 30.1 is complete.

Note that the proof of $K(A) \geq -4\pi$ only used the fact that A is embedded in a neighbourhood of the straight line boundary. Thus we see immediately that

Corollary 30.7 *Suppose that $A \subset S(-1,1)$ is a proper minimal annulus. If $A(1) = A \cap P_1$ is a straight line and A is embedded in a neighborhood of $A(1)$, and $A(-1) = A \cap P_{-1}$ is a circle, then each $A(t) = A \cap P_t$ is a circle for $-1 < t < 1$. In particular, A is embedded.*

Proof. We only need point out that we can still use the four-vertex theorem, even though some level sets $A(t)$ may not be Jordan curves. It is shown in [36], that all curves which have exactly two vertices are curves which have exactly two simple loops, on each loop the curvature is positive or negative and hence its total curvature must be 0. Note that $A(-1)$ has total curvature 2π. Since $A(t)$ is a closed curve for $-1 \leq t < 1$, by continuity every $A(t)$ has total curvature 2π. Hence the four-vertex theorem is applicable to $A(t)$ for $-1 \leq t < 1$. □

Corollary 30.8 *Suppose that $A \subset S(-1,1)$ is a proper minimal annulus. If $A(1) = A \cap P_1$ and $A(-1) = A \cap P_{-1}$ are straight lines and A is embedded in neighbourhoods of $A(1)$ and $A(-1)$, then each $A(t) = A \cap P_t$ is a circle for $-1 < t < 1$. In particular, A is embedded.*

Proof. We have $X^3 = \log|z|/\log R$. Let $\epsilon > 0$ such that on $\{R - \epsilon < |z| \leq R\}$ X is an embedding. Then $A(t)$ is a Jordan curve when $\log(R-\epsilon)/\log R < t < 1$. Thus we can still use the four-vertex theorem. □

Remark 30.9 Corollaries 30.7 and 30.8 are slightly better than Corollary 1 in [17]. There do exist properly immersed minimal annuli in $S(-1,1)$ whose level sets are not circles, see [78].

Since all minimal surfaces foliated by circles must be a part of a Riemann's example, we have proved that:

Corollary 30.10 *Let $L_1 \subset P_1$, $L_{-1} \subset P_{-1}$ be two parallel straight lines. If $\Gamma = L_1 \cup L_{-1}$ is the boundary of a properly embedded minimal annulus A in $S(-1,1)$, then A is one of Riemann's examples.*

Finally, we have a non-existence theorem:

Corollary 30.11 *Let $L_1 \subset P_1$, $L_{-1} \subset P_{-1}$ be two non-parallel straight lines. Then $\Gamma = L_1 \cup L_{-1}$ cannot bound a properly embedded minimal annulus in $S(-1,1)$.*

Corollary 30.10 is the main theorem of [25], in which it is proved via elliptic function theory. Corollary 30.11 is a result of Toubiana [78]. The proof of Theorem 30.1 is adapted from [17].

31 Nitsche's Conjecture

Conjecture 31.1 (Nitsche) *Let $A \subset \mathbf{R}^3$ be an embedded complete minimal annulus such that $A \cap P_t$ is a Jordan curve for $t_1 < t < t_2$, where $-\infty \le t_1 < t_2 \le \infty$. Then A must be a catenoid. In particular, $t_1 = -\infty$ and $t_2 = \infty$.*

Nitsche made this conjecture in [62], it is still open. We only know that the conjecture is true under certain extra hypotheses, In this section we will give two such theorems. The first one, Theorem 31.2, is due to Nitsche [62]; the proof given here is essentially Nitsche's proof.

Theorem 31.2 *If each $A \cap P_t$ is a starshaped Jordan curve for $t_1 < t < t_2$, then A is a catenoid.*

Proof. By a translation we may assume that $t_1 < 0$, $t_2 > 0$. Let $0 < a < t_2$ and let $A \cap S(0,a)$ be a compact minimal annulus with Jordan curve boundary. By Lemma 9.1 and Proposition 9.2, its conformal structure is

$$A_{R(a)} = \{z \in \mathbf{C} \,\big|\, 1 \le |z| \le R(a)\},$$

where $R(a) > 1$.

Let $X(a) : A_{R(a)} \to \mathbf{R}^3$ be the conformal embedding. Then we know that the third coordinate $X(a)_3$ must be

$$X(a)_3 = \frac{a}{\log R(a)} \log |z|.$$

Let $0 < a < b < t_2$. The moduli of $A_{R(a)}$ and $A_{R(b)}$ are $R(a)$ and $R(b)$ respectively. Since $A \cap S(0,a) \subset A \cap S(0,b)$, we have $R(a) < R(b)$ and thus $A_{R(a)} \subset A_{R(b)}$. We have $X(b) : A_{R(b)} \to \mathbf{R}^3$ such that

$$X(b)_3 = \frac{b}{\log R(b)} \log |z|.$$

It must be that $X(b)_3|_{A_{R(a)}} = X(a)_3$, thus

$$\frac{b}{\log R(b)} \log R(a) = X(b)_3(R(a)e^{i\theta}) = X(a)_3(R(a)e^{i\theta}) = a,$$

which implies that

$$\frac{b}{\log R(b)} = \frac{a}{\log R(a)}. \tag{31.173}$$

Now let $0 < a_1 < a_2 < \cdots < a_n < \cdots t_2$ and $\lim_{n\to\infty} a_n = t_2$; we have $A_{R(a_1)} \subset \cdots \subset A_{R(a_n)} \subset \cdots$. Let $R = \lim_{n\to\infty} R(a_n) \le \infty$. Then the conformal structure of $\mathrm{Int}(A \cap S(0,t_2))$ is the interior of A_R and the conformal embedding is given by $X : \mathrm{Int}(A_R) \to \mathbf{R}^3$ and $X_3(z) = c \log |z|$, where

$$c = \frac{a_n}{\log R(a_n)} > 0$$

is well defined by (31.173).

Let $g : A_R \to \mathbf{C}$ be the Gauss map of $A \cap S(0,t_2)$. As before, we have $\eta = dz/zg(z)$ and the angle of the outward unit normal of $A \cap P_t$ with the x-axis is given by $\psi(r,\theta) = \Im \log g(z)$, where $z = re^{i\theta}$ such that $t = c \log r$. Thus ψ is a multivalued harmonic function. Since each $A \cap P_t$ is a Jordan curve for $0 \le t < t_2$, we have

$$\psi(r, \theta + 2\pi) = \psi(r,\theta) + 2\pi,$$

which implies that $\Im \log(g(z)/z)$ is a well defined harmonic function in A_R. Thus $h(z) := \log \frac{g}{z}$ is a well defined holomorphic function, and

$$g(z) = ze^{h(z)}. \tag{31.174}$$

The Laurent expansion of h is

$$h(z) = \sum_{-\infty}^{-1} a_n z^n + \sum_{n=0}^{\infty} a_n z^n = h_1(z) + h_2(z);$$

thus h_1 is holomorphic in $\{|z| > 1\} \cup \{\infty\}$ and h_2 is holomorphic in $|z| < R$.

If $\Im h_i$ is bounded for $i = 1, 2$, then $\Re h_i$ is also bounded, and thus $h = h_1 + h_2$ is bounded. In this case, if $R < \infty$, by the Enneper-Weierstrass representation we know that A is not complete. Thus if the $\Im h_i$ are bounded, then $R = \infty$.

Next we prove that indeed $\Im h_i$ are bounded.

Let $D_t \subset P_t$ be the bounded domain bounded by $A \cap P_t$. Let $\alpha(r,\theta) := (X_1, X_2)(re^{i\theta})$ be a parameter representation of $A \cap P_t$, $0 \le t < t_2$, where $c \log r = t$. For a point $x_0 \in D_t$, let $l(r,\theta)$ be the ray starting from x_0 and passing through $\alpha(r,\theta)$. Consider the angle $\phi(r,\theta)$ made by $l(r,\theta)$ and the x-axis in P_t. We can make ϕ a continuous function of θ such that $\phi(r,\theta + 2\pi) = \phi(r,\theta) + 2m\pi$, where m is an integer depending both on x_0 and α.

Since $A \cap P_t$ is starshaped, there is an $x_t \in D_t$ such that the $l(r,\theta)$ intersects α only at $\alpha(r,\theta)$. Thus for this x_t, $\phi(r,\theta+2\pi) = \phi(r,\theta) + 2\pi$, and ϕ is a non-decreasing function of θ.

Recall the angle $\psi(r,\theta) = \Im \log g(r^{i\theta})$. Fix $\psi(r,0) = \Im \log g(r)$ for $1 < r < R$. Comparing the angles ϕ and ψ, by their definitions we have

$$\phi(r,\theta) \le \psi(r,\theta) + 2n\pi \le \phi(r,\theta) + \pi/2, \tag{31.175}$$

where the integer n is decided by

$$\phi(r,0) \le \psi(r,0) + 2n\pi \le \phi(r,0) + \pi/2.$$

Now for any θ' and θ'' in $[0, 2\pi)$, we have

$$\phi(r,\theta') - \phi(r,\theta'') - \pi/2 \le \psi(r,\theta') - \psi(r,\theta'') \le \phi(r,\theta') - \phi(r,\theta'') + \pi/2. \tag{31.176}$$

Since $\psi(r,\theta) = \theta + \Im h(re^{i\theta})$, (31.176) gives that

$$\phi(r,\theta') - \phi(r,\theta'') - 5\pi/2 \le \Im h(r^{i\theta'}) - \Im h(r^{i\theta''}) \le \phi(r,\theta') - \phi(r,\theta'') + 5\pi/2,$$

or

$$|\Im h(r^{i\theta'}) - \Im h(r^{i\theta''})| \le 9\pi/2, \tag{31.177}$$

since $|\phi(r,\theta') - \phi(r,\theta'')| \le 2\pi$.

Now fix z_0 such that $|z_0| = r_0 \in (1, R)$. Define

$$m_i(r) = \min_{|z|=r} \Im h_i(z), \qquad M_i(r) = \max_{|z|=r} \Im h_i(z), \qquad i = 1,\ 2, \tag{31.178}$$

and

$$s_i(r) = \min_{|z'|=|z''|=r} (\Im h_i(z') - \Im h_i(z'')) = m_i(r) - M_i(r), \tag{31.179}$$

$$S_i(r) = \max_{|z'|=|z''|=r} (\Im h_i(z') - \Im h_i(z'')) = M_i(r) - m_i(r) = -s_i(r). \tag{31.180}$$

From the relation

$$\Im h_1(z'') - \Im h_1(z') = \Im h(z'') - \Im h(z') - [\Im h_2(z'') - \Im h_2(z')]$$

we find, using (31.177) and the maximum principle for harmonic functions ($S_2(r) \le S_2(r_0)$ for $0 < r \le r_0$), that

$$|\Im h_1(z'') - \Im h_1(z')| \le 9\pi/2 + S_2(r_0) \quad \text{for } 1 < |z'| = |z''| = r \le r_0. \tag{31.181}$$

On $|z| = r \le r_0$ we have, denoting by $\hat{z}$ a point with $|\hat{z}| = r$ and $\Im h_1(\hat{z}) = M_1(r)$,

$$\Im h_1(z) = \Im h_1(z) - \Im h_1(\hat{z}) + \Im h_1(\hat{z}) \ge M_1(r) - [9\pi/2 + S_2(r_0)].$$

By the minimum principle, applied to the harmonic function $\Im h_1$ in $1 < |z| \le r_0$, there must be a point z with $|z| = r$ for which $\Im h_1(z) \le \Im h_1(z_0)$, and consequently

$$M_1(r) \le \Im h_1(z_0) + [9\pi/2 + S_2(r_0)]. \tag{31.182}$$

This inequality, originally derived for $1 < |z| \le r_0$, holds automatically in $r_0 \le |z| < R$ as well because $M_1(r') \le M_1(r)$ for $r \le r'$. An argument similar to the one leading to (31.182) yields

$$m_1(r) \ge \Im h_1(z_0) - [9\pi/2 + S_2(r_0)] \quad \text{for } 1 < r < R. \tag{31.183}$$

Applying analogous reasoning to the function $\Im h_2$ we find

$$M_2(r) \le \Im h_2(z_0) + [9\pi/2 + S_1(r_0)], \tag{31.184}$$

$$m_2(r) \geq \Im h_2(z_0) - [9\pi/2 + S_1(r_0)], \tag{31.185}$$

for $1 < r < R$. These relations show that the harmonic functions $\Im h_i$ are bounded from both sides in $1 < |z| < R$ and thus $R = \infty$.

Similarly, consider $A \cap (t_1, 0)$. By the same argument, its conformal type is also $\{1 \leq |z| < \infty\}$.

Thus we know that the conformal type of A is $S^2 - \{p, q\}$. Without loss of generality, we can assume that it is $\mathbf{C} - \{0\}$. Similar argument shows that the third coordinate function can be written as

$$X^3(z) = c \log |z|,$$

where c is a real constant. Then the same argument shows that $g(z) = ze^{h(z)}$ and h is a bounded holomorphic function on $\mathbf{C} - \{0\}$. Passing to the universal covering $\mathbf{C}$ of $\mathbf{C} - \{0\}$ and using Liouville's theorem, h is a constant function. Then by the Enneper-Weierstrass representation, h must be a real constant. Thus $g(z) = az$, $a > 0$ is a real constant, and A must be a catenoid. The proof is complete. □

One observes that if A has finite total curvature, then $K(A) = 2\pi(\chi(A) - 2) = -4\pi$. Corollary 14.6 then tells us that A must be a catenoid. Since A has two annular ends, it is enough to prove that each end has finite total curvature. By Theorem 23.1, we know that if A is properly embedded and if one end of A is above a catenoid, then that end has finite total curvature. Thus if A is a counter-example to Nitsche's conjecture, either it is not properly embedded or one of its two ends is neither above nor below any standard catenoid type barrier. Given the level sets are Jordan curves, such a surface is very hard to imagine its existence.

The second theorem is due to G. D. Crow [11], which shows that uniformly bounded Gauss curvature implies finite total curvature for complete minimal surfaces of conformal type $S^2 - \{p, q\}$.

Theorem 31.3 *Let $X : M = S^2 - \{p, q\} \hookrightarrow \mathbf{R}^3$ be a minimal immersion satisfying:*

1. *$|K| < C$ (M is of bounded Gauss curvature);*
2. *The immersion is given by $X = (X^1, X^2, X^3)$ and is such that the limits as $z \to p$ and $z \to q$ of $X^3(z)$ exist uniformly as extended real numbers.*

Then M is of finite total curvature. In particular, if M is embedded then M is a catenoid.

Proof. We only need prove that M has finite total curvature.

First by Remark 16.4, $|K|$ bounded implies that the convex hull of $X(M)$ is $\mathbf{R}^3$. Thus $X^3(z) \to \pm\infty$ as $z \to p$ or q and the two limits must be different. So without loss of generality we may assume that $M = \mathbf{C} - \{0\}$ and $X^3(z) = c \log |z|$. The Weierstrass data for X then is g and $\eta = (1/zg(z))dz$ and the Gauss curvature is given by

$$K(z) = -\left(\frac{4|z||g||g'|}{(1 + |g|^2)^2}\right)^2.$$

To prove that M has finite total curvature it is enough to prove that g has no essential singularity at either 0 or ∞. Let $h = g^2$ and $r = |z|$, then K can be written as

$$K = -\left(\frac{2r|h'|}{(1+|h|)^2}\right)^2.$$

Now $|K|$ is bounded implies that

$$\frac{2r|h'|}{(1+|h|)^2} < C.$$

Since

$$\frac{1}{(1+|h|)^2} \leq \frac{1}{1+|h|^2} \leq \frac{2}{(1+|h|)^2},$$

$|K|$ is bounded implies that

$$\frac{|z||h'|}{1+|h|^2} < C.$$

The next lemma shows that if h has an essential singularity at ∞, then h cannot miss any value in $\mathbf{C} \cup \{\infty\}$. But $X^3(z) = c\log|z|$ means that the Gauss map g must miss 0 and ∞ in M, since if $g(z_0) = 0$ or ∞ then $|z| = |z_0|$ would not be a level set. Since $h = g^2$, we know that g does not have an essential singularity at ∞.

If h has an essential singularity at 0, using $\zeta = 1/z$, and observing that

$$\frac{|z||h'(z)|}{1+|h(z)|^2} < C, \quad \forall z \in \mathbf{C} - \{0\} \text{ if and only if } \frac{|z||h'(1/z)|}{1+|h(1/z)|^2} < C, \quad \forall z \in \mathbf{C},$$

Thus by the above argument, h and hence g could not have essential singularity at 0 either. Thus g is a meromorphic function on $\mathbf{C} \cup \{\infty\}$ and hence M has finite total curvature as mentioned in Remark 19.3. □

Lemma 31.4 *Let h be a meromorphic function in a neighbourhood U of ∞, and suppose h has an essential singularity at ∞. Suppose h satisfies the inequality*

$$\limsup_{z\to\infty} \frac{|z||h'(z)|}{1+|h(z)|^2} < \infty.$$

Then h cannot omit any value.

Proof. ([48], pages 7 and 8) Let γ be a simple divergent path in U tending to ∞. Then α is said to be an *asymptotic value* of h at ∞ if $h(z) \to \alpha$ as $z \to \infty$ along γ. Suppose h omits the value α. Then by Iversen's Theorem ([65], page 4), α is an asymptotic value at infinity along a simple divergent path γ. By the above theorem, h is normal in U slit along the path γ. By Theorem 2 of [47], page 53, and the remark that follows, h converges uniformly in $U - \gamma$ toward α, no matter of which way z goes to ∞. This contradicts the hypothesis that h has an essential singularity at $z = \infty$. Hence h cannot omit any value. □

Remark 31.5 Under the conditions of Theorem 31.3, if $A := X(M) \cap P_t$ is a Jordan curve, then X is an embedding, so A must be a catenoid. Moreover, by [51] and [85], if A satisfies the condition of Nitsche's conjecture and the Gauss curvature is bounded, then conformally A is $S^2 - \{p, q\}$. Thus Nitsche's conjecture is true if the Gauss curvature is bounded.

32 Appendix The Eigenvalue Problem

In order to discuss the stabilities of a minimal surface, we need some general knowledge of the (Dirichlet) eigenvalues of a self-adjoint second order elliptic operator.

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain and L be a self-adjoint second order elliptic operator

$$Lu = D_i(a^{ij}D_ju + b^iu) - b^iD_iu + cu,$$

where (a^{ij}) is symmetric. We suppose that L satisfies

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \quad \forall x \in \Omega, \quad \xi \in \mathbf{R}^n, \tag{32.186}$$

$$\sum |a^{ij}(x)|^2 \leq \Lambda^2, \quad 2\lambda^{-2}\sum_{i=1}^{n}\left(|b^i(x)|^2 + \lambda^{-1}|c(x)|\right) \leq \nu^2, \quad \forall x \in \Omega, \tag{32.187}$$

for some constants λ, Λ, $\nu > 0$.

Define $(u,v) = \int_\Omega uv\,dx$, and a quadratic form on $H = H(\Omega) = W_0^{1,2}(\Omega)$ by

$$\mathcal{L}(u,v) = \int_\Omega (a^{ij}D_iuD_jv + b^iuD_iv + b^ivD_iu - cuv)dx = -(Lu,v).$$

The ratio

$$J(u) = \frac{\mathcal{L}(u,u)}{(u,u)}, \quad u \not\equiv 0, \quad u \in H,$$

is called the Rayleigh quotient of L.

By (32.186) and (32.187) we see that J is bounded from below. In fact, writing $b = (b^1, \cdots, b^n)$ and $|b|^2 = \sum_i |b^i|^2$, we have

$$\begin{aligned}
\mathcal{L}(u,u) &= \int_\Omega (a^{ij}D_iuD_ju + 2b^iuD_iu - cu^2)dx \\
&\geq \int_\Omega \left[\lambda|Du|^2 - \left(\frac{1}{2}\lambda|Du|^2 + 2\lambda^{-1}|b|^2u^2 + cu^2\right)\right] dx \\
&\quad \text{(by (32.186) and Schwarz's inequality)} \\
&\geq \int_\Omega \left(\frac{1}{2}\lambda|Du|^2 - \lambda\nu^2|u|^2\right) dx \text{ (by (32.187))} \qquad (32.188) \\
&\geq \left(\frac{\lambda}{2}C^{-1} - \lambda\nu^2\right)\int_\Omega |u|^2\,dx \text{ (by Poincaré's inequality).}
\end{aligned}$$

Hence we may define

$$\lambda_1 = \inf_H J. \tag{32.189}$$

We claim now that λ_1 is the minimum eigenvalue of L on H; that is, there exists a non-trivial $u \in H$ such that $Lu + \lambda_1u = 0$ and λ_1 is the smallest number for which this is possible. To show this we choose a minimizing sequence $\{u_m\} \subset H$ such that $\|u_m\|_{L^2} = 1$ and $J(u_m) \to \lambda_1$.

By (32.188) and $\|u_m\|_{L^2} = 1$, we have

$$\frac{\lambda}{2}\int_\Omega |Du_m|^2 dx \le 2\lambda\nu^2 + \mathcal{L}(u_m, u_m) \le 2(\lambda\nu^2 + |\lambda_1|),$$

hence $\{u_m\}$ is bounded in H. Thus by the compactness of the embedding $H \to L^2(\Omega)$, a subsequence, which we still note as $\{u_m\}$ itself, converges in $L^2(\Omega)$ to a function u with $\|u\|_{L^2} = 1$. Since $Q(u) = \mathcal{L}(u, u)$ is quadratic, we also have for any l, m,

$$Q\left(\frac{u_l - u_m}{2}\right) + Q\left(\frac{u_l + u_m}{2}\right) = \frac{1}{2}(Q(u_l) + Q(u_m)).$$

Since

$$\begin{aligned} Q\left(\frac{u_l + u_m}{2}\right) &= \mathcal{L}\left(\frac{u_l + u_m}{2}, \frac{u_l + u_m}{2}\right) \\ &\ge (\inf_H J)\left(\frac{u_l + u_m}{2}, \frac{u_l + u_m}{2}\right) = \lambda_1\left(\frac{u_l + u_m}{2}, \frac{u_l + u_m}{2}\right), \end{aligned}$$

we have

$$Q\left(\frac{u_l - u_m}{2}\right) \le \frac{1}{2}\left(Q(u_l) + Q(u_m)\right) - \lambda_1\left\|\frac{u_l + u_m}{2}\right\|_{L^2}^2 \to 0.$$

Again by (32.188),

$$\begin{aligned} \frac{\lambda}{2}\int_\Omega |D(u_l - u_m)|^2 dx &\le \mathcal{L}(u_l - u_m, u_l - u_m) + 2\lambda\nu^2\int_\Omega |u_l - u_m|^2 dx \\ &\le 4Q\left(\frac{u_l - u_m}{2}\right) + 2\lambda\nu^2\|u_l - u_m\|_{L^2}^2 \to 0, \end{aligned}$$

and so we see that $\{u_m\}$ is a Cauchy sequence in H. Hence $u_m \to u$ in H, and moreover $Q(u) = \lambda_1$.

Let $v \in H$ and consider

$$J(u + tv) = \frac{\mathcal{L}(u + tv, u + tv)}{(u + tv, u + tv)} = \frac{Q(u) + 2t\mathcal{L}(u, v) + t^2 Q(v)}{(u, u) + 2t(u, v) + t^2(v, v)}.$$

By (32.189), we have

$$0 = \frac{dJ(u + tv)}{dt}\Big|_{t=0} = \frac{2\mathcal{L}(u, v)(u, u) - 2(u, v)Q(u)}{(u, u)^2} = 2[\mathcal{L}(u, v) - \lambda_1(u, v)],$$

i.e.,

$$\int_\Omega (a^{ij}D_i u D_j v + b^i u D_i v + b^i v D_i u - cuv - \lambda_1 uv)dx = 0.$$

Integrating by parts we obtain

$$\int_\Omega \left[D_j\left(a^{ij}D_i u + b^j u\right) - b^i D_i u + cu + \lambda_1 u\right] v dx = \int_\Omega (Lu + \lambda_1 u) v dx = 0.$$

By the arbitrariness of $v \in H$, we must have $Lu + \lambda_1 u = 0$.

On the other hand, suppose $v \in H$ satisfies $Lv + \sigma v = 0$ (such a σ is called an *eigenvalue* and v is called an *eigenfunction* corresponding to σ). Then

$$0 = \int_\Omega (Lv + \sigma v) v dx = -\mathcal{L}(v, v) + \sigma(v, v).$$

We have

$$\sigma = J(v) \geq \inf_H J(u) = \lambda_1,$$

and thus λ_1 is the minimum eigenvalue.

Let λ be an eigenvalue, the *eigenspace* V_λ corresponding to λ is defined by

$$\{u \in H \mid Lu + \lambda u = 0\}.$$

If we arrange (as we will always do) the eigenvalues of L in increasing order λ_1, λ_2, $\cdots$, and designate their corresponding eigenspaces by V_1, V_2, $\cdots$, we may characterize the eigenvalues of L through the formula

$$\lambda_m = \inf\{J(u) \,\Big|\, u \not\equiv 0, \ \ (u, v) = 0, \ \ \forall v \in \{V_1, \cdots, V_{m-1}\}\}. \tag{32.190}$$

We summarize the above in the following result. Readers can refer to the books [21] (Theorem 8.37, p 214) and [10] (Chapter V, especially page 424).

Theorem 32.1 *Let L be a self-adjoint operator satisfying* (32.186) *and* (32.187). *Then L has a countably infinite discrete set of eigenvalues, $\Sigma = \{\lambda_m\}$, given by* (32.190). *Whose eigenfunctions span $W_0^{1,2}(\Omega)$. Furthermore,* $\dim V_m < \infty$ *and* $\lim_{m\to\infty} \lambda_m = \infty$.

We also need the Harnack inequality,

Theorem 32.2 (See [21] Corollary 8.21, page 199) *Assume L satisfies* (32.186) *and* (32.187), *$u \in W^{1,2}(\Omega)$ satisfies $u \geq 0$ in Ω, and $Lu = 0$ in Ω. Then for any $\Omega' \subset\subset \Omega$ we have*

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u,$$

where $C = C(n, \Lambda/\lambda, \nu, \Omega', \Omega)$.

Theorem 32.3 *Given v_1, $\cdots$, $v_{k-1} \in H$, let*

$$\mu = \inf\{J(u) \mid u \in H, \ u \not\equiv 0, \ \ (u, v_i) = 0, \ \ 1 \leq i \leq k-1\}.$$

Then we have $\lambda_k \leq \mu$.

Proof. Take ϕ_i as the i-th eigenfunction corresponding to the i-th eigenvalue λ_i, $1 \leq i \leq k$. We can assume that ϕ_i's are orthonormal in $L^2(\Omega)$. We can select k constants $d_1, \cdots, d_k$, not all zero, such that

$$\sum_{i=1}^{k} d_i \int_\Omega \phi_i v_j dx = 0, \quad 1 \leq j \leq k-1.$$

Let $c_i = d_i(\sum_{j=1}^{k} d_j^2)^{-1/2}$ and define $f = \sum_{i=1}^{m} c_i\phi_i$. Then $(f, f) = \sum_{i=1}^{k} c_i^2 = 1$, and $(f, v_i) = 0$ for $1 \leq i \leq k-1$. By the definition of μ we have

$$\mu \leq J(f) = \mathcal{L}(f, f) = \sum_{i=1}^{k} c_i^2\lambda_i \leq \lambda_k \sum_{i=1}^{k} c_i^2 = \lambda_k.$$

□

Theorem 32.4 *Let $\Omega_1, \cdots, \Omega_m$ be pairwise disjoint domains in Ω. Considering the eigenvalue problem for each Ω_i and arrange all the eigenvalues of $\Omega_1, \cdots, \Omega_m$ in an increasing sequence*

$$v_1 \leq v_2 \leq \cdots$$

then we have

$$\lambda_k \leq v_k \quad \text{for} \quad k \geq 1.$$

Proof. Choose ψ_i to be the eigenfunction corresponding to v_i in the related domain and extend ψ_i by 0 such that $\psi_i \in H = W_0^{1,2}(\Omega)$ for $1 \leq i \leq k$. We can assume that the ψ_i's are orthonormal. For any $h_1, \cdots, h_{k-1} \in H$, as in the proof of Theorem 32.3 we can select c_i not all zero, and $f = \sum_{i=1}^{k} c_i\psi_i$ such that $(f, f) = 1$ and $(f, h_j) = 0$ for $1 \leq j \leq k-1$. If we select h_i as the i-th eigenfunction corresponding to λ_i, then by Theorem 32.3 and (32.190),

$$\lambda_k \leq J(f) = \mathcal{L}(f, f) = \sum_{i=1}^{k} c_i^2 v_i \leq v_k.$$

□

Combining the above with the Harnack inequality, we have an immediate corollary:

Corollary 32.5 *If $\Omega' \subset \Omega$, and the eigenvalues of L on $H(\Omega')$ are λ_1', λ_2', $\cdots$, then*

$$\lambda_m' \geq \lambda_m, \quad m = 1, \; 2, \; 3 \cdots.$$

If $\Omega' \subset \Omega$ is a proper subdomain, i.e., $\Omega - \overline{\Omega'}$ contains an non-empty open set, then

$$\lambda_m' > \lambda_m, \quad m = 1, \; 2, \; 3 \cdots.$$

Remark 32.6 We have neglected the boundary regularity of subdomains in the theorems, but it is true that if u on Ω' satisfies $Lu + \lambda u = 0$ and $u|_{\partial\Omega'} = 0$, then $u \in W_0^{1,2}(\Omega') \subset H$. See [5], page 21.

Let $\Sigma = S^2$ and $L = \Delta_\Sigma$ be the sphere Laplacian, $\Omega = \Sigma$ and $\partial\Omega = \emptyset$. Then it is well known that $\lambda_1 = 0$ and $\lambda_2 = 2$. Hence we have

Corollary 32.7 *Let $\Omega \subset \Sigma$ be a proper domain, then the second eigenvalue of the sphere Laplacian on Ω is larger than* 2.

Let u_m be the m-th eigenfunction corresponding to the m-th eigenvalue λ_m. Define the *nodal set* of u_m as $Z_m = \{x \in \Omega : u_m(x) = 0\}$.

Theorem 32.8 ([10], p 452) *Z_m divides the domain Ω into no more than m subdomains.*

Proof. Suppose Z_m divides Ω into more than m subdomains; label them as Ω_1, Ω_2, $\cdots$, Ω_k, $k > m$, and let $Z_m \bigcup \bigcup_{i=1}^k \Omega_i = \Omega$.

Since u_m does not change sign on each Ω_i, $1 \le i \le k$, Harnack's inequality tells us that $u_m \not\equiv 0$ on Ω_i (in fact, the nodal set has measure zero). Hence for each Ω_i, $1 \le i \le m$, we can define a $v_i \in H$ by $v_i = u_m$ on Ω_i, and $v_i = 0$ elsewhere. Define $w_i = \|v_i\|_{L^2}^{-1} v_i$, then $(w_i, w_i) = 1$. We see that w_i satisfies $Lw_i + \lambda_m w_i = 0$. Since $\int_\Omega w_i w_j dx = \delta_j^i$, $\{w_i\}_{i=1}^m$ is linearly independent.

For the $m-1$ eigenfunctions u_1, $\cdots$, u_{m-1} in H corresponding to the first $m-1$ eigenvalues, as in the proof of Theorem 32.3, we can select m constants c_1, $\cdots$, c_m, not all zero, such that

$$\sum_{i=1}^m c_i \int_\Omega w_i u_j dx = 0, \quad 1 \le j \le m-1,$$

and $\sum_{j=1}^m c_j^2 = 1$. Define $\phi = \sum_{i=1}^m c_i w_i$; then $(\phi, \phi) = \sum_{i=1}^m c_i^2 = 1$ and $(\phi, u_i) = 0$ for $1 \le i \le m-1$. Let $\Omega' = \operatorname{Int} \bigcup_{i=1}^m \overline{\Omega}_i$; then $\phi \in H(\Omega') \subset H(\Omega)$. Notice that Ω' is a proper subdomain of Ω, since the Ω_i are nonempty subdomains of Ω for $m+1 \le i \le k$.

By (32.190) we have

$$\begin{aligned}\lambda_m \le J(\phi) &= \mathcal{L}(\phi, \phi) = -\int_\Omega \phi L\phi dx = -\sum c_i c_j \int_\Omega w_i L w_j dx \\ &= -\sum_{i=1}^m c_i^2 \lambda_m \int_\Omega w_i^2 dx = \sum_{i=1}^m c_i^2 \lambda_m = \lambda_m.\end{aligned}$$

Hence ϕ is an eigenfunction corresponding to the m-th eigenvalue, but $\phi|(\Omega - \Omega') \equiv 0$ contradicts Harnack's inequality. This contradiction proves the theorem. □

Corollary 32.9 *The first eigenfunction ϕ_1 corresponding to the first eigenvalue does not change sign in Ω. All other eigenfunctions must change sign in Ω. Moreover,* $\dim V_{\lambda_1} = 1$.

Proof. ϕ_1 does not change sign by Theorem 32.8. This also shows that the eigenfunctions corresponding to the first eigenvalue must be either positive or negative, but two of them cannot orthogonal to each other, thus $\dim V_{\lambda_1} = 1$. Let ϕ_i be the i-th eigenfunction where $i > 1$, then by $(\phi_1, \phi_i) = 0$ we know that ϕ_i has to change sign in Ω. $\square$

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