NON-DIFFERENTIABLE INVEX

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Abstract. It is well known that various properties of constrained optimization, such as converse Karush-Kuhn-Tucker and duality, remain valid when *convex* hypotheses are much relaxed, e.g. to *invex*. But *convex* does not need derivatives, whereas *invex* does. However, there is a property intermediate between *convexifiable* (by transformation of the domain) and *invex*, which gives a nondifferentiable extension of *invex*. Its properties will be surveyed.

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1. Introduction. This survey describes the relations between *invex* functions and some other related functions, namely functions *convexifiable* by a diffeomorphism of the domain space, and an intermediate class of *protoconvex* functions, which give an invex analog of nondifferentiable convex functions. *Protoconvex* functions satisfy a *basic alternative theorem*, from which follow necessary and sufficient conditions for a class of constrained optimization problems. Under some restrictions, a local protoconvex property follows from *invex*. Jeyakumar and Mond's *V-invex* generalization of *invex* is shown to relate to a scaling of a constraint system.

A differentiable vector function $F : \mathbf{R}^n \to \mathbf{R}^k$ is *invex* if

$$(\forall x, p)\ F(x) - F(p) \geq F'(p)\eta(x,p), \tag{1.1}$$

defining $\geq$ by an order cone $K \subset \mathbf{R}^k$. For the minimization problem:

$$\text{MIN } f(x) \text{ subject to } -g(x) \in S, \tag{1.2}$$

let $f = (f, g)$ and $K := \mathbf{R}_+ \times S$ (or $K := Q \times S$) if f is vector-valued, and MIN denotes weak minimum with order cone Q). It is well known [6] that the invex property makes necessary Karush-Kuhn-Tucker (KKT) conditions sufficient for a minimum, and also suffices for duality results. Derivatives can be relaxed to Clarke differentials for Lipschitz functions.

Now F is *convex* if $\eta(x,p) = x-p$, and a convex function need not be differentiable. There are several variants of *invex* that do not require derivatives. Current progress is described. With suitable definitions,

$$\begin{array}{ccccc}
\text{(without derivatives)} & & & & \text{(with derivatives)} \\
\textit{convexifiable} \Rightarrow \textit{protoconvex} & & \Rightarrow & & \textit{invex} \\
\Downarrow & & ?\Leftarrow & & \Downarrow \\
\textit{Basic Alternative Theorem} & & & & \textit{Converse KKT} \\
 & & \Downarrow & & \\
 & & \textit{Necessary \& Sufficient Lagrangian Conditions} & &
\end{array}$$

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2. Main Definitions and Results. F is *convexifiable* if $H := F \circ \phi^{-1}$ is convex, for some invertible transformation ϕ. From H convex, for $0 < \alpha < 1$,

$$\begin{aligned}(1-\alpha)F(p) + \alpha F(x) &= (1-\alpha)H(\phi(p)) + \alpha H(\phi(x)) \\ &\geq H((1-\alpha)\phi(p) + \alpha\phi(x)) \\ &= F(\xi(\alpha, x, p)), \end{aligned} \tag{2.1}$$

where

$$\xi(\alpha, x, p) := \alpha^{-1}((1-\alpha)\phi(p) + \alpha\phi(x))$$

$$(= (1-\alpha)p + \alpha x \text{ if } F \text{ is convex}).$$

If ϕ is differentiable, then there exists

$$(\partial/\partial\alpha)\xi(\alpha, x, p)|_{\alpha=0} = \phi^{-1\,\prime}(\phi(p))[\phi(x) - \phi(\alpha)] \equiv \eta(x, p) \tag{2.2}$$

The combination of the *convexlike* property (2.1) (see [7]) with (2.2) has been called *protoconvex* (see [5], also [4] where it was called *miniconvex*).

If F is also differentiable, then *invex* follows from *protoconvex* by letting $\alpha \to 0$ in (2.2) If F is Lipschitz, then $F'(p)\eta(x,p)$ is replaced by Clarke's generalized directional derivative $F^\circ(p, \eta(x,p))$, [1].

From (2.1) there follows the *Basic Alternative Theorem* [7] (see also 2, [3]) for a convexlike function $F : \Gamma \to Y$, where Γ is convex, and an ordering defined by a closed convex cone in Y, namely :

$$(\nexists\, x \in \Gamma)\ F(x) < 0 \Rightarrow (\exists 0 \neq \rho)\rho F(\Gamma) \geq 0.$$

Applied to problem (1.2), with $intS \neq \emptyset$, and $f(p) = 0$, it gives :

$$\text{MIN at } p \Leftrightarrow F(x) \notin -intK \Leftrightarrow (\exists 0 \neq \rho \in K*)\ \rho F(.) \geq 0.$$

So $(\tau f + \lambda g)(.) \geq 0$, with $\tau \neq 0$ if a constraint qualification (such as Slater's : $(\exists x_0) - g(x_0) \in intS$) is assumed.

If f and g are Lipschitz, then *Wolfe's dual problem* is:

$$\text{MAX } f(u) + vg(u) \text{ such that } u \in S*,\ (f + vg)^\circ(u, .) \geq 0. \tag{2.3}$$

Then *weak duality* follows from protoconvex , since

$$(f + vg)(x) - (f + vg)(u) \geq (f + vg)^\circ(u; \eta(x, u)) \geq 0$$

if x is feasible for (1.2), and u, v is feasible for (2.3), so that $f(x) \geq f(u) + vg(u)$.

3. Relation of *invex* to *protoconvex*.

Proposition 1. *Let $F \in C^2$ be invex at p with C^2 scale function η. If quadratic terms dominate higher-order terms, then F is protoconvex near p.*

Proof. By shift of origin, $p = 0$ and $F(p) = 0$ may be assumed. Then the invex property is expressed by $(\forall x)F(x) \geq F'(0)\eta(x, 0)$. It is required to prove that

$$F(x) \geq F'(0)\eta(x, 0) \Rightarrow (\forall \alpha \in (0, 1))\ \alpha F(x) \geq F(\xi(\alpha, x, 0).$$

To do this, expand $F(x) = Ax + x^T B_. x$ and $\eta(x, 0) = x + x^T D_. x$ up to quadratic terms. The dot subscript means a matrix for each component. Then *invex* requires that $B_. - AD_. \geq 0$, where here ≥ 0 for matrices means positive semidefinite. Substituting the trial function

$$\xi(\alpha, x, 0) := \alpha(x + xTD_. x) - \alpha^2 x^T D_. x$$

leads to the requirement that

$$A(x + x^T D_. x - \alpha x^T D_x) + \alpha x^T B_. x \leq Ax + x^T B_. x \ .$$

and thus to

$$(\forall \alpha \in (0,1)) \ (1 - \alpha)(B_. - AD_.) \geq 0$$

which is true from invex. □

REMARK 1. Calculations with quadratic functions can only show that *invex* holds locally. Unless the functions are positive definite, which gives convexity, the inequalities can only hold in a restricted domain, until the function 'turns over'.

One approach towards a global property is by a preliminary transformation of the domain, to map it into a local region. By shift of origin, $p = 0$ can be assumed. Choosing polar coordinates $x = (r, \theta)$, where $r = ||x||$ and θ lies on the unit sphere, a possible transformation of the domain is given by

$$\hat{x} = \kappa(x) \Leftrightarrow \hat{r} = \tanh kr, \hat{\theta} = \theta.$$

Suppose that F is a C^2 vector function, and $F \circ \kappa^{-1}$ is invex over a local domain (in which quadratic terms dominate). Since invex is invariant to a diffeomorphism of the domain, it follows that F is also invex, over a larger domain.

4. V-invex. Jeyakumar & Mond [8] defined a relaxation of *invex*, called *V-invex*. In the present notation, a weight function $\beta_j(.) > 0$ is assumed for each constraint $g_j(x) \leq 0$, and the property is:

$$(\forall x) g_j(x) - g_j(p) \geq \beta_j(x) g_j'(p) \eta(x, p).$$

It suffices to assume this for constraints active at p. From this, converse KKT readily follows.

However, if the real function $r_j(.) > 0$, then

$$g_j(.) \leq 0 \Rightarrow G_j(.) := r_j(.) g_j(.) \leq 0.$$

Thus, given positive functions r_j, the constraints $g_j(.) \leq 0$ are equivalent to the constraints $G_j(.) \leq 0$.

Suppose that $g_j(.)$ is invex with scale function $\eta(.,.)$. If $g_j(p) = 0$ then

$$G_j(x) - G_j(p) = G_j(x) = r_j(x)[g_j(x) - g_j(p)]$$

$$\geq r_j(x) g_j'(p) \eta(x, p)$$

$$= [r_j(x)/r_j(p)] G_j'(p) \eta(x, p).$$

Thus $G_j(.)$ is *V-invex* with weight function $\beta_j(x, p) = r_j(x)/r_j(p)$.

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