DYNAMICAL SYSTEMS AND OPERATOR ALGEBRAS

IAIN RAEBURN *

Abstract. We study the algebraic consequences of the traditional descriptions of dynamics based on differential equations. We thus show how C^*-dynamical systems consisting of an action of a locally compact group on a C^*-algebra provide a common framework for studying the time evolution and symmetry groups of systems in classical and quantum physics.

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One thing which emerged clearly at the workshop in March is that there are many different ideas about what constitutes a "dynamical system." In my talk, therefore, I tried to explain how the dynamical systems operator-algebraists study arise from the more classical notion of dynamical system as a system of ordinary differential equations. In discussions afterwards, I became aware that many workers in my own area are unaware of the circle of ideas that led to the general acceptance of C^*-dynamical systems as an important field of research. So I have included in this written version a discussion of how these also arise in quantum mechanics.

Dynamical systems were originally mathematical formulations of dynamics — how physical systems change in time. By exploring the algebraic properties of dynamics, we shall arrive at a notion of dynamical system which provides a common framework for studying time evolutions and symmetry groups in classical mechanics. We then seek a parallel framework for Hilbert-space models of quantum mechanics; carrying out this program raises a variety of interesting mathematical issues. We finish by showing how C^*-algebraic dynamical systems bring these ideas together in one neat conceptual package.

1. Classical dynamical systems. We consider a physical system in which the different states of the system are described by points of a state space X. The time evolution is given by an ordinary differential equation $x' = f(x)$, in which the unknown is a function $x : \mathbf{R} \mapsto X$, and the value $x(t)$ represents the state of the system at time t. To see what happens to a system which starts in a given state x_0, we solve the initial-value problem

$$x' = f(x), \qquad x(0) = x_0. \tag{1.1}$$

We know from experience that the general solution of $x' = f(x)$ will involve an arbitrary constant $c \in X$, which should be determined by the initial condition; thus we expect the initial-value problem (1.1) to have a unique solution. More formally, if $f : X \to \mathbf{R}$ satisfies a Lipschitz condition

$$|f(x) - f(y)| \leq K|x - y|,$$

then the global existence and uniqueness theorem implies that (1.1) has a unique solution valid on all of $\mathbf{R}$.

EXAMPLE 1.1. Consider a particle of mass m moving in $\mathbf{R}^3$ according to Newton's laws; we assume that there are no external influences, so that the system is

*Department of Mathematics The University of Newcastle, NSW 2308, Australia, email:iain@frey.newcastle.edu.au. This research was supported by the Australian Research Council.

conservative in the sense that energy is conserved. Then the position of the particle is described by a vector-valued function $q : \mathbf{R} \to \mathbf{R}^3$ which satisfies an equation of the form $q'' = g(q, q')$ (really a system of 3 ordinary differential equations). If we introduce the momentum $p := mq'$ and set $f(q,p) := (p/m, g(q,p))$, then the function $x := (q,p) : \mathbf{R} \to \mathbf{R}^6$ satisfies the first-order equation $x' = f(x)$, and the motion of the particle is determined by its initial position and momentum $(q(0), p(0)) \in \mathbf{R}^6$. Thus we can make this system fit our model by taking $X := \mathbf{R}^6$ for the state space. The conservative nature of the system is reflected in our implicit assumption that the right-hand side $f(x)$ depends only on the position and momentum of the particle and not directly on time; this is crucial for the analysis which follows.

Solving the initial-value problem (1.1) gives us a state $x(t)$ which depends both on the time t and the initial value x_0; to emphasise this, we write $t \cdot x_0 := x(t)$. This pairing $(t, x_0) \mapsto t \cdot x_0$ of $\mathbf{R} \times X$ into X has a key algebraic property:

$$s \cdot (t \cdot x_0) = (s + t) \cdot x_0. \tag{1.2}$$

To establish this, note that $t \cdot x_0 := x(t)$ and $(s+t) \cdot x_0 := x(s+t)$ are defined using the solution x of (1.1). To define $s \cdot (t \cdot x_0)$, we have to solve instead

$$y' = f(y), \qquad y(0) = t \cdot x_0, \tag{1.3}$$

and take $s \cdot (t \cdot x_0) := y(s)$. But the function $y(r) := x(r + t)$ satisfies (1.3), and the initial-value problem (1.3) has a unique solution, so $y(s) = x(s + t) = (s + t) \cdot x_0$.

The algebraic property (1.2) says that we have an *action* of the additive group $(\mathbf{R}, +)$ on the state space X. This action is continuous, in the sense that $(t, x_0) \mapsto t \cdot x_0$ is a continuous map of $\mathbf{R} \times X$ into X; this amounts to saying that the solution $x(t)$ to (1.1) depends continuously on the initial data x_0, and follows from the inequality

$$|x(t) - y(t)| \leq |x(t_0) - y(t_0)| e^{K|t-t_0|}$$

enjoyed by any pair of solutions to $x' = f(x)$. (See, for example, [7, p.169]; the constant K appearing here is the Lipschitz constant of f.) We can sum up the discussion so far by saying that solving the initial-value problem gives us a continuous action of $\mathbf{R}$ on the state space X.

To reach this conclusion we have had to impose some non-trivial hypotheses; for example, the existence of the global Lipschitz constant K is not much weaker than linearity. Nevertheless, now that we have this description, we can see that it is physically reasonable: we expect conservative systems to have the additive property (1.2) (it shouldn't matter when we start the clock), and we want physical models to be stable in the sense that small perturbations in the initial data don't change the system much, which is precisely what the continuity says. So one is naturally led to wonder why we bother with the differential equation at all:

DEFINITION 1.1. The time evolution in a classical dynamical system is given by a continuous action of $\mathbf{R}$ on the state space.

One immediate advantage of such a formulation is that it can be trivially modified to handle discrete-event systems, in which the only times of interest are multiples of a fixed unit t_0: the time evolution is given by an action $(n, x_0) \mapsto n \cdot x_0 = (nt_0) \cdot x_0$ of $\mathbf{Z}$ on the state space. In other words, we simply look at actions of a different group.

By pushing this idea a little further we can accommodate symmetries in the same formalism.

A symmetry of a geometric object is a transformation which does not change the shape of the object. For example, if the object is a square, we can rotate it through multiples of $\pi/2$ or reflect it in one of several axes; if it is a circle, we can rotate it through any angle or reflect it in any diameter. These symmetries always form a group under composition. For the square, the symmetries form a group D of order 8 called the dihedral group: D is generated by an element a of order 4 (rotation through $\pi/2$, say) and an element b of order 2 (reflection in the y-axis, say) satisfying $ba = a^3b$. In any model of a physical system in which identical objects are located at the vertices of a square, we would expect to find an action of the group D on the state space.

Symmetries of a physical system can arise in other ways. For example, if a particle moves in $\mathbf{R}^3$ under the gravitational attraction of an object at the origin, the system will be invariant under rotations about the origin, and there should be an action of the group $SO(3, \mathbf{R})$ of rotations on the state space.

In general, we think of the group operation as multiplication rather than addition (because we have been trained through linear algebra not to expect multiplications to commute), and an *action* of a group G on a space X is a continuous pairing $(g, x) \mapsto g \cdot x$ such that $g \cdot (h \cdot x) = (gh) \cdot x$. We arrive at a notion of dynamical system which allows us to study time evolution and symmetry simultaneously:

DEFINITION 1.2. A classical dynamical system consists of an action of a group G on a state space X.

At this point we have a very broad definition, which could be tightened up in several ways by adding structure. Thus, for example, a differential geometer would naturally insist that the group be a Lie group, that the state space be a manifold and that the action be smooth; a topologist might ask that G be a topological group and that X be a polyhedron. Our own prejudices would lead us to suppose that G is a locally compact group and that X is a locally compact Hausdorff space, and these assumptions will help to ensure that what follows can be rigorously justified.

At this stage, in preparation for the discussion in the next sections, we introduce a couple of mathematical devices, both of which are knee-jerk reactions for an operator-algebraist.

Our first device is to replace the space X by the algebra $C_0(X)$ of continuous functions $a : X \to \mathbf{C}$ which vanish at infinity. (If $X = \mathbf{R}^n$, for example, "a vanishes at infinity" means that $a(x) \to 0$ as $|x| \to \infty$.) Here we use the word algebra in a technical sense, to say that $C_0(X)$ has a multiplication which is compatible with the linear space structure: $(ab)(x) := a(x)b(x)$. The point is that $C_0(X)$ has a great deal more structure than the space X: it is a *C^*-algebra*. Thus it also has an involution $a \mapsto a^*$ (given by pointwise complex conjugation) and a norm (defined by $\|a\| := \sup\{|a(x)| : x \in X\}$) which satisfies the C^*-identity $\|a^*a\| = \|a\|^2$ and is complete, in the sense that Cauchy sequences converge. In fact, $C_0(X)$ is the the prototypical *commutative C^*-algebra*: the Gelfand-Naimark Theorem says that every commutative C^*-algebra is isomorphic to one of the form $C_0(X)$.

A continuous action of a locally compact group G on X lifts to an action τ of G by automorphisms of $C_0(X)$; the automorphism τ_g corresponding to $g \in G$ is given by $\tau_g(a)(x) := a(g^{-1} \cdot x)$. The presence of the inverse is just bookkeeping to ensure that $g \mapsto \tau_g$ is a homomorphism of G into the group $\operatorname{Aut} C_0(X)$ of automorphisms

under composition:

$$\begin{aligned}\tau_{gh}(a)(x) &= a((gh)^{-1}\cdot x) = a((h^{-1}g^{-1})\cdot x) = a(h^{-1}\cdot(g^{-1}\cdot x))\\ &= \tau_h(a)(g^{-1}\cdot x) = \tau_g(\tau_h(a))(x).\end{aligned}$$

The action τ is strongly continuous in the sense that $g \mapsto \tau_g(a)$ is continuous for each fixed a, and every strongly continuous action of G on a commutative C^*-algebra arises this way. (See [10, Lemma 7.1] for a more detailed statement.) So now a classical dynamical system consists of a strongly continuous action τ of a locally compact group G by automorphisms of a commutative C^*-algebra $C_0(X)$.

Our second mathematical device is to represent everything by operators on a Hilbert space $\mathcal{H}$. The algebra $B(\mathcal{H})$ of bounded operators on $\mathcal{H}$ is itself a C^*-algebra: the multiplication is given by composition, the adjoint is the map $T \mapsto T^*$ characterised by $(Th \mid k) = (h \mid T^*k)$, and the norm is the radius of $\{Th : \|h\| \leq 1\}$. A *representation* of a C^*-algebra A is a linear map $\pi : A \to B(\mathcal{H})$ which preserves the multiplication and adjoint; all C^*-algebras have lots of them. For example, one can obtain representations of $C_0(X)$ by fixing a measure μ on X, and defining $\pi : C_0(X) \to B(L^2(X, d\mu))$ by $(\pi(a)h)(x) = a(x)h(x)$ for $h \in L^2(X, d\mu)$. To represent groups, one uses unitary operators: the linear operators V on $\mathcal{H}$ satisfying $VV^* = V^*V = 1$. These are the automorphisms of Hilbert space, and form a group $U(\mathcal{H})$ under composition. A *unitary representation* of a group G on $\mathcal{H}$ is a homomorphism $U : G \to U(\mathcal{H})$. If the group is locally compact, we insist that unitary representations are strongly continuous: $g \mapsto U_g h$ is continuous for every $h \in \mathcal{H}$.

The automorphisms of $B(\mathcal{H})$ all have the form $\operatorname{Ad} V : T \mapsto VTV^*$ for some $V \in U(\mathcal{H})$; it is easy to check that each $\operatorname{Ad} V$ is an automorphism, and not ridiculously hard to see that they all have this form. (The proof of [10, Proposition 1.6] carries over, for example.) When we want to represent the whole system $(C_0(X), G, \tau)$ by operators on Hilbert space, we ask that the automorphisms τ_g take the form $\operatorname{Ad} U_g$. Thus a *covariant representation* of the dynamical system $(C_0(X), G, \tau)$ consists of a representation $\pi : C_0(X) \to B(\mathcal{H})$ and a unitary representation $U : G \to U(\mathcal{H})$ on the same space such that

$$\pi(\tau_g(a)) = U_g \pi(a) U_g^* \quad \text{for } a \in C_0(X) \text{ and } g \in G.$$

A third mathematical device which we shall not follow up here is to encapsulate the covariant representation theory of the dynamical system in one C^*-algebra, called the crossed product. We shall discuss this briefly in Section 4.

2. Quantum mechanical systems on Hilbert space. In the most basic mathematical models of quantum mechanics, the states are given by unit vectors h in a Hilbert space $\mathcal{H}$, with the proviso that if $\lambda \in \mathbf{C}$ and $|\lambda| = 1$, then λh and h represent the same state. We write $(\cdot \mid \cdot)$ for the inner product, which we assume to be linear in the first variable (we apologise to any physicist who has stumbled across this). The time evolution in such a quantum mechanical system is given by a differential equation $\psi' = -iH(\psi)$ called *Schrödinger's equation*, in which the unknown is a function $\psi : \mathbf{R} \to \mathcal{H}$ and the *Hamiltonian* H is a linear operator on $\mathcal{H}$. If the system is in state h at time 0, then its state at time t is obtained by solving the initial-value problem

$$\psi' = -iH(\psi), \qquad \psi(0) = h, \tag{2.1}$$

and evaluating at t to obtain $\psi_t = \psi(t)$. The Hamiltonian H is typically a partial differential operator, so Schrödinger's equation is a partial differential equation; nevertheless, provided H is self-adjoint in a technical sense, the initial-value problem (2.1) has a unique solution valid on all of $\mathbf{R}$. The operator H is typically unbounded and defined only on a dense subspace $\mathcal{D}$ of $\mathcal{H}$, so as well as the formal self-adjointness condition $(Hh \,|\, k) = (h \,|\, Hk)$ for $h, k \in \mathcal{D}$, one needs to impose conditions on the domain $\mathcal{D}$: loosely speaking, one insists that $\mathcal{D}$ is as large as possible (see [11, Chapter VIII]). In the Example 2.1 below, we try to give some idea of what this means. However, if H is self-adjoint, the maps U_t which send the initial value $\psi_0 = h$ to the solution $\psi(t)$ form a unitary representation $U : \mathbf{R} \to U(\mathcal{H})$ [11, Theorem VIII.7]. We write $U_t = e^{-itH}$.

DEFINITION 2.1. The time evolution in a quantum mechanical system with underlying Hilbert space $\mathcal{H}$ is given by a unitary representation of $\mathbf{R}$ on $\mathcal{H}$.

The notion of self-adjointness turns out to be intrinsic to this view of quantum mechanics. First of all, if we are to use this definition of time evolution, H has to be self-adjoint: a theorem of Stone asserts that every unitary representation $U : \mathbf{R} \to U(\mathcal{H})$ has the form e^{-itT} for some self-adjoint operator T [11, Theorem VIII.8]. The operator T is called the infinitesimal generator of U, and is defined by

$$Th := i\frac{d}{dt}(U_t h)\Big|_{t=0} = i \lim_{t \to 0} \frac{U_t h - h}{t}$$

with domain consisting of the set of vectors h for which this limit exists. Second, the observable quantities of the system correspond to self-adjoint operators on $\mathcal{H}$. The Hamiltonian H, for example, corresponds to the energy of the system. Third, the hypothesis of self-adjointness has deep mathematical implications: spectral theory allows one to reconstruct the operator from a projection-valued measure on the Borel subsets of $\mathbf{R}$ called the *spectral measure* of the operator (see [11, Chapter VIII.3]). To see how this is useful physically, note that it is in the nature of quantum mechanics that one cannot expect to measure things exactly, and for each observable T and state h there is a probability distribution describing the values of the observable T in the state h. The number $(Th \,|\, h)$ represents the expected value of this distribution; the distribution itself involves the spectral measure of the self-adjoint operator T. Indeed, if E^T is the spectral measure of T, so that $E^T(B)$ is a projection on $\mathcal{H}$ for each Borel set $B \subset \mathbf{R}$, then $(E^T(B)h \,|\, h)$ is the probability that measuring the observable quantity T in the state h gives an answer lying in the set B.

We have described the Schrödinger picture of quantum mechanics, in which the states evolve; one can also use the Heisenberg picture, in which the states are viewed as fixed and the observables evolve. If U_t describes the evolution in the Schrödinger picture, then the observable T at time 0 moves to the observable $U_t^* T U_t$ at time t. To prove this, one merely has to fix an initial state h, and verify that the measures $B \mapsto (E^{U_t^* T U_t}(B)h \,|\, h)$ and $B \mapsto (E^T(B)(U_t h) \,|\, U_t h)$ coincide, which amounts to showing that $E^{U_t^* T U_t} = U_t^* E^T U_t$.

EXAMPLE 2.1. Consider a particle of mass m moving in $\mathbf{R}^3$ under a potential $V : \mathbf{R}^3 \to \mathbf{R}$; in classical mechanics, $V(x, y, z)$ would be the potential energy of the particle when it is located at (x, y, z). The underlying Hilbert space is $L^2(\mathbf{R}^3)$, and the unit vectors are functions $\psi : \mathbf{R}^3 \to \mathbf{C}$ satisfying $\int |\psi|^2 = 1$. If $B \subset \mathbf{R}^3$, then $\int_B |\psi|^2$ is

the probability that a particle in the state ψ is located in the set B. The time evolution is described by the map $\psi_0 \mapsto \psi_t$, where the function $\psi : (t,x,y,z) \mapsto \psi_t(x,y,z)$ satisfies Schrödinger's equation

$$i\hbar\frac{\partial\psi}{\partial t}(t,x,y,z) = -\frac{\hbar^2}{m}\Delta(\psi)(t,x,y,z) + V(x,y,z)\psi(t,x,y,z), \quad \psi(0,x,y,z) = \psi_0(x,y,z),$$

where Δ is the usual Laplacian $\Delta : f \mapsto f_{xx} + f_{yy} + f_{zz}$ and $\hbar$ is Planck's constant. Because they have no impact on the mathematical ideas, authors often choose units so that $\hbar = 1$ and $m = 1$.

The operator M_x of multiplication by the variable x is self-adjoint on the domain

$$\left\{\phi \in L^2(\mathbf{R}^3) : \iiint |x\phi(x,y,z)|^2\, dxdydz < \infty\right\};$$

this operator corresponds to the observable in which we measure the x-coordinate of the position of the particle. For constant coefficient differential operators L such as the Laplacian Δ, we can write down suitable domains by using the Fourier transform to convert L to a multiplication operator (see [12, Theorem IX.27]). On the other hand, it can be a very delicate matter to decide if there is a self-adjoint version of the Hamiltonian $-\Delta + M_V$; there is if, for example, V is the sum of an L^2-function and a bounded function (see [12, Theorem X.15]).

Symmetries of quantum mechanical systems should be bijections on the set of states. One would expect these symmetries to preserve some of the algebraic structure we have imposed, and Wigner identified the transition probabilities as the key feature. If h and k are unit vectors, the *transition probability* is the absolute value $|(h \mid k)|$ of the inner product; one interprets this physically as the probability of a positive answer when we ask whether the state h is in the state k. Notice that multiplying h or k by scalars of absolute value one does not affect $|(h \mid k)|$, so transition probabilities are well-defined on the state space $P(\mathcal{H})$ of unit rays, which we can view as the quotient space of the unit sphere $S(\mathcal{H}) := \{h \in \mathcal{H} : \|h\| = 1\}$ by the action of $\mathbf{T} := \{\lambda \in \mathbf{C} : |\lambda| = 1\}$. We write $[h]$ for the state in $P(\mathcal{H})$ corresponding to $h \in S(\mathcal{H})$.

Theorem 2.1 (Wigner). *If $f : P(\mathcal{H}) \to P(\mathcal{H})$ is a bijection which preserves transition probabilities, then there is a unitary or antiunitary operator U on $\mathcal{H}$ such that $f([h]) = [Uh]$ for all $h \in S(\mathcal{H})$. The operator U is unique up to multiplication by a scalar $\lambda \in \mathbf{T}$.*

This theorem was formulated by Wigner in 1931; a proof in the spirit of his original argument was given by Bargmann in 1964 [1]. Other authors have started from different definitions of symmetry, but, remarkably, all seem to arrive at a theorem of this sort. (For surveys of these ideas, see [14] or [4].) So to exploit symmetry in quantum mechanics one has to study representations of the appropriate group as unitary or antiunitary operators on Hilbert space. Since mathematicians instinctively prefer linearity to antilinearity, we immediately look for excuses to stick with unitary representations, and this leads to some interesting mathematical ideas.

Suppose that G is a symmetry group of a quantum mechanical system, so that we have a homomorphism T of G into the group $\operatorname{Symm} P(\mathcal{H})$ of transition-probability-preserving bijections of the state space $P(\mathcal{H})$. By Wigner's theorem, each symmetry

T_g is implemented by a unitary or antiunitary operator. If $g = h^2$ for some $h \in G$, then $T_g = T_h \circ T_h$ is implemented by a unitary, because the composition of two antilinear maps is linear. Thus if every element of G is a square, then every symmetry T_g is implemented by a unitary operator. This is the case if $G = \mathbf{R}$ (because the group operation is $+$, and every $x = \frac{1}{2}x + \frac{1}{2}x$), and more generally whenever G is a connected Lie group. So for such G we can always choose unitary operators U_g which implement the symmetries T_g.

Can we choose the unitaries U_g so that $U : G \to U(\mathcal{H})$ is a (strongly continuous) unitary representation? Let us first make an arbitrary choice $g \mapsto U_g$. Since two unitaries U, V induce the same symmetry if and only if $U = \lambda V$ for some $\lambda \in \mathbf{T}$ (this is the uniqueness assertion in Wigner's theorem), and since the product $U_g U_h$ implements $T_{gh} = T_g \circ T_h$, there is a scalar $\omega(g,h) \in \mathbf{T}$ such that $U_g U_h = \omega(g,h) U_{gh}$. A comparison of the unitaries implementing $T_{(gh)k} = T_{g(hk)}$ shows that

$$\omega(g,h)\omega(gh,k) = \omega(g,hk)\omega(h,k);$$

such functions $\omega : G \times G \to \mathbf{T}$ are called 2-*cocycles*. Trivial examples of 2-cocycles are the *coboundaries*, which are given by $\partial\lambda : (g,h) \mapsto \lambda(g)^{-1}\lambda(h)^{-1}\lambda(gh)$ for some map $\lambda : G \to \mathbf{T}$. It turns out that ω is a coboundary $\partial\lambda$ precisely when we can choose $g \mapsto U_g$ to be a homomorphism: just replace the U_g we first chose by $\lambda(g)U_g$. The 2-cocycles form an abelian group under pointwise multiplication, and the quotient by the subgroup of coboundaries is called the *second cohomology group* of G with coefficients in $\mathbf{T}$, and denoted $H^2(G, \mathbf{T})$. In fancy language, we have just shown that there is a complete obstruction in $H^2(G, \mathbf{T})$ to implementing $T : G \to \operatorname{Symm} P(\mathcal{H})$ by a homomorphism $U : G \to U(\mathcal{H})$. The group $H^2(G, \mathbf{T})$ is the 2-dimensional component in a well-developed cohomology theory, which includes powerful tools for computation, so at this stage we have reduced our problem to a well-understood one. Before we get too excited, though, we have to remember that we expect our unitary representations to be strongly continuous.

We shall have to impose a continuity condition on our action T of G by symmetries, and we shall do this by appealing to Wigner's theorem. The set $G(\mathcal{H})$ of unitary and antiunitary operators on $\mathcal{H}$ has a natural topology called the strong operator topology, which is determined by the seminorms $\|U\|_h := \|Uh\|$ for $h \in \mathcal{H}$, and $G(\mathcal{H})$ is a topological group in this topology. Wigner's theorem says that $\operatorname{Symm} P(\mathcal{H})$ is the quotient of $G(\mathcal{H})$ by the subgroup $\mathbf{T}1$ of multiples of the identity operator 1. We give $\operatorname{Symm} P(\mathcal{H})$ the quotient topology, and ask that T be continuous for this topology. This is reasonable: one symmetry S is near another one T if for each finite collection $[h_i]$ of states we can choose implementing operators U for S and V for T such that $\|Uh_i - Vh_i\|$ is small for all i.

DEFINITION 2.2. The symmetries of a quantum mechanical system with underlying Hilbert space $\mathcal{H}$ are given by a continuous homomorphism of a topological group G into $\operatorname{Symm} P(\mathcal{H})$.

We now have to ask if we can lift a continuous homomorphism $T : G \to U(\mathcal{H})/\mathbf{T}$ to a (strongly continuous) unitary representation $U : G \to U(\mathcal{H})$. It turns out that there are topological obstructions to doing this: not every continuous map f into $U(\mathcal{H})/\mathbf{T}$ lifts to a continuous map into $U(\mathcal{H})$. It is, however, possible to do this locally. (For example, one can identify $U(\mathcal{H})/\mathbf{T}$ with the group $\operatorname{Aut}\mathcal{K}(\mathcal{H})$ of automorphisms of the C^*-algebra of compact operators, and appeal to [10, Proposition 1.6].) We can piece

these locally defined maps together to give a Borel lifting $g \mapsto U_g$, and then the cocycle ω is a Borel map of $G \times G$ into $\mathbf{T}$. If $\omega = \partial\lambda$ for some Borel map $\lambda : G \to \mathbf{T}$, then we have a Borel lifting $g \mapsto \lambda(g)U_g$ which is a homomorphism, and hence automatically strongly continuous by, for example, [9, Proposition 5]. The relevance of the group $\mathbf{H}^2(G, \mathbf{T})$ of Borel 2-cocycles modulo Borel coboundaries was pointed out by Mackey (see, for example, [8, Chapter 18], where 2-cocycles are called multipliers). Nowadays $\mathbf{H}^2(G, \mathbf{T})$ is called the second Moore cohomology group of G, because C.C. Moore has developed systematic tools for studying this version of group cohomology with Borel cochains ([9]; see [10, Section 7.4] for more recent references).

We can sum up our progress so far:

Theorem 2.2. *Suppose G is a connected Lie group such that $\mathbf{H}^2(G, \mathbf{T}) = 0$. Then every continous action $T : G \mapsto \mathrm{Symm}\, P(\mathcal{H})$ is implemented by a strongly continuous unitary representation $U : G \to U(\mathcal{H})$.*

This theorem applies, for example, if $G = \mathbf{R}$ (see [14, Theorem 8.1]), giving independent confirmation of our Definition 2.1 of time evolution. We could now *deduce* the existence of Schrödinger's equation: Stone's Theorem implies that the time evolution is given by the differential equation $\psi'(t) = -itH(\psi(t))$ where H is the infinitesimal generator of U. (This route to the foundations of quantum mechanics is followed in [15].) Theorem 2.2 also applies to any connected and simply-connected semisimple Lie group [15]. It does not apply to the group $G = SO(3, \mathbf{R})$ of rotations, where we would have to pass to the simply-connected covering group $\widetilde{G}$ to get $H^2(\widetilde{G}, \mathbf{T}) = 0$. Thus not every action of $SO(3, \mathbf{R})$ is implemented by a unitary representation of $SO(3, \mathbf{R})$; this observation leads to the concept of spin (and to the name $\mathrm{Spin}(3, \mathbf{R})$ for the covering group).

3. Quantum mechanical systems on C^*-algebras. As in Example 2.1, a quantum mechanical system with n particles is modelled using the Hilbert space $L^2(\mathbf{R}^{3n})$. When there are infinitely many particles — for example, in statistical mechanics, where one considers infinite lattices — there is no obvious Hilbert space to use. Some models avoid this problem by focusing on the observables, as in the Heisenberg picture, and using the self-adjoint elements in a C^*-algebra A as the observables of the system. The states are then given by the linear functionals ω on A of norm one which are *positive* in the sense that $\omega(a^*a) \geq 0$ for every $a \in A$; if $a = a^*$ is self-adjoint and ω is a state, then $\omega(a)$ is a real number which is the expected value of the observable a in the state ω. For an interesting discussion of how C^*-algebraic models were developed and why the context seems appropriate, see the introduction to [2].

When there is a natural Hilbert space $\mathcal{H}$, we take $A = B(\mathcal{H})$, and each unit vector $h \in \mathcal{H}$ determines a state $\omega_h : T \mapsto (Th \,|\, h)$. Conversely, we can always represent an abstract C^*-algebra as operators on a Hilbert space $\mathcal{H}$, and then each state ω will extend (in a noncanonical way) to a state on $B(\mathcal{H})$. This state will not necessarily have the form ω_h: see [11, Theorem VI.26] for a description of the states on $B(\mathcal{H})$. But if we start with a given state ω of a C^*-algebra A, then the GNS-construction gives us a Hilbert space $\mathcal{H}_\omega$, a representation $\pi_\omega : A \to B(\mathcal{H}_\omega)$, and a unit vector $h \in \mathcal{H}_\omega$ such that $\omega(a) = (\pi_\omega(a)h \,|\, h)$. Thus we can always tailor a Hilbert-space system to study any given state.

The time evolution of a C^*-algebraic quantum system should be given by a one-parameter group $\{\alpha_t : t \in \mathbf{R}\}$ of symmetries of the set of observables. We have

to decide what "symmetry" should mean: a symmetry should certainly preserve the physical structure, but it is not clear how much of the C^*-algebra structure is a mathematical convenience. For example, note that if $a, b \in A$ are self-adjoint but do not commute, then the product ab will not be self-adjoint, and hence will not represent an observable quantity. On the other hand, the *Jordan product* $\{a, b\} := ab + ba$ is self-adjoint, and one could ask that symmetries be given by Jordan automorphisms of A: linear isomorphisms which preserve the Jordan product. It turns out that these are precisely the linear isomorphisms $\phi : A \to A$ such that $\phi^*(\omega) := \omega \circ \phi$ is a state whenever ω is [2, Theorem 3.2.3], so this definition is intuitively reasonable. Automorphisms and antiautomorphisms of the C^*-algebra A give Jordan automorphisms of A, but there can be many more. However, if $\{\alpha_t : t \in \mathbf{R}\}$ is a strongly continuous one-parameter group of Jordan automorphisms, then each α_t is a C^*-algebra automorphism [2, Theorem 3.2.12]. Thus:

DEFINITION 3.1. The time evolution in a quantum mechanical system whose observables are the self-adjoint elements of a C^*-algebra A is given by a strongly continuous action $\alpha : \mathbf{R} \to \operatorname{Aut} A$ of $\mathbf{R}$ by automorphisms of A. The observable represented by the self-adjoint element $a = a^* \in A$ at time $t = 0$ evolves to the observable $\alpha_t(a)$ at time t; alternatively, a state ω at time 0 becomes the state $\omega \circ \alpha_t$ at time t.

When we realise our C^*-algebraic system by operators on Hilbert space via a representation $\pi : A \to B(\mathcal{H})$, we want this notion of time evolution to match up with the one we had in the previous section, given by a unitary representation $U : \mathbf{R} \to U(\mathcal{H})$. Since representations of C^*-algebras are $*$-preserving, the operator $\pi(a)$ corresponding to an observable $a = a^* \in A$ is a (bounded) self-adjoint operator on $\mathcal{H}$ representing an observable in the Hilbert-space system, which in the Heisenberg picture evolves according to the rule $t \mapsto U_t^* \pi(a) U_t$. Thus the pair (π, U) must satisfy

$$\pi(\alpha_t(a)) = U_t^* \pi(a) U_t;$$

in other words, the pair (π, U^*) is a covariant representation of the dynamical system $(A, \mathbf{R}, \alpha)$ in the sense of Section 1. If we start with a state ω of A, then as before we can represent A on a Hilbert space $\mathcal{H}_\omega$ using the GNS-representation π_ω: loosely speaking, $\mathcal{H}_\omega$ is the completion of A with respect to the inner product $(a \,|\, b)_\omega := \omega(b^* a)$, and $\pi_\omega(a) b = ab$ for $b \in A \subset \mathcal{H}_\omega$. If the state ω is invariant in the sense that $\omega \circ \alpha_t = \omega$ for all $t \in \mathbf{R}$, then $b \mapsto \alpha_t(b)$ extends to a unitary operator U_t on $\mathcal{H}_\omega$ such that

$$\pi_\omega(\alpha_t(a)) = U_t \pi_\omega(a) U_t^*,$$

and the whole system is realised by a covariant representation.

REMARKS 3.1. (1) In motivating Definition 3.1, we claimed it was reasonable that symmetries should take states to states. However, we should point out that there are important models where this does not happen. For example, in [3, Chapter 5.2] several different statistical mechanical models are considered; in some of these the time evolution is given by an automorphism group, and in some it is not. So one should not interpret this as a universal rule. It is nevertheless interesting to see the standard notion of C^*-dynamical system emerging from physically plausible assumptions.

(2) In describing the time evolution in C^*-algebraic systems, we have done exactly what we advocated in Section 1: we haven't bothered with differential equations at all.

In fairness, though, we should mention that strongly continuous one-parameter groups of automorphisms do have densely-defined infinitesimal generators δ, which satisfy the relation $\delta(ab) = \delta(a)b + a\delta(b)$ for a, b in a dense $*$-subalgebra $\mathcal{D}$ of A; loosely speaking, this relation comes from differentiating the function $t \mapsto \alpha_t(ab) = \alpha_t(a)\alpha_t(b)$ using the product rule. Such maps $\delta : \mathcal{D} \to A$ are called (unbounded) derivations, and an enormous effort has gone into their study (see, for example, [2, Chapter 3] and [13]).

To require that more general symmetry groups act by automorphisms of the underlying C^*-algebras seems to be more an act of faith. But it is certainly true that the original impetus for studying actions of locally compact groups on noncommutative C^*-algebras came from exactly these considerations (see [5], [2, Chapter 4.3]).

DEFINITION 3.2. Consider a quantum mechanical system whose observables are the self-adjoint elements of a C^*-algebra A. A symmetry group of the system is a strongly continuous action $\alpha : G \to \mathrm{Aut}\, A$ of G by automorphisms of A.

When we represent the C^*-algebraic quantum system on Hilbert space via a representation $\pi : A \to B(\mathcal{H})$, we want to implement the symmetries α_g by symmetries of the Hilbert-space system. For actions of $\mathbf{R}$, these are given by a unitary representation of $\mathbf{R}$, but we saw earlier, for other groups more complicated things could happen, and these complications can be of physical significance. Nevertheless, the most important representations are the covariant representations (π, U), in which U is a unitary representation of G and

$$\pi(\alpha_g(a)) = U_g \pi(a) U_g^* \quad \text{for } a \in A \text{ and } g \in G.$$

As in our discussion of time evolution, any invariant state gives rise to a covariant representation (π_ω, U) extending the GNS-representation π_ω.

4. Concluding remarks. We have now arrived at the standard notion of C^*-dynamical system: a triple (A, G, α) consisting of a C^*-algebra A, a locally compact group G, and an action of G by automorphisms of A. We have shown that this formalism encompasses in a mathematically precise way the time evolution and symmetry groups of classical mechanics, via actions on commutative C^*-algebras, and the time evolution of C^*-algebraic models in quantum mechanics, and that it is also useful in exploiting symmetry in quantum mechanics. Our operator-algebraic instinct of representing everything by operators on Hilbert space is particularly relevant: covariant representations relate the C^*-algebraic models of quantum mechanics to the more traditional Hilbert-space ones.

We shall close by saying briefly why covariant representations are important in other areas of mathematics. They arose first in Mackey's theory of induced representations of locally compact groups: a representation U of a locally compact group G is induced from a given closed subgroup H if and only if it forms part of a covariant representation of the system $(C_0(G/H), G, \tau)$ in which G acts by left translation on G/H (this formulation of Mackey's Imprimitivity Theorem was pointed out by Glimm; see [10, Section C.5] for more details). Takesaki subsequently showed that Mackey's machine for computing the irreducible unitary representations of G carries over to a large class of dynamical systems involving noncommutative algebras A, and a deep and powerful version of the Mackey machine was subequently developed by Green based on ideas of Rieffel [6].

Green's analysis of (A, G, α) focused on the structure of the *crossed product* $A \rtimes_\alpha G$ — which, interestingly, had been introduced earlier with applications to physics in mind [5]. This C^*-algebra is generated by a covariant representation of (A, G, α), and is universal for such representations: there is a one-to-one correspondence between representations of $A \rtimes_\alpha G$ and covariant representations of (A, G, α) (see [10, Section 7.1] for further references). Over the past 20 years, these crossed products have turned up all over the place, and provide many of the most important examples of C^*-algebras. For example, the irrational rotation algebras which were the key example in Connes' noncommutative differential geometry are by definition the crossed products of $C(\mathbf{T})$ by actions of $\mathbf{Z}$ by rotations. Crossed products have been instrumental in noncommutative topology, where they allow one to deal with group actions in a systematic way. And our knowledge of the relationship between the structure of dynamical systems and their crossed products has led via nonabelian duality to the modern theory of Hopf C^*-algebras (or quantum groups) and their actions on C^*-algebras.

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