LIMITATIONS ON ACHIEVABLE TRACKING ACCURACY FOR SINGLE-INPUT TWO-OUTPUT SYSTEMS.

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Abstract. A fundamental limitation exists in the achievable tracking performance of systems in which the plant has a single input and two outputs (SITO). For a SITO plant in a unity feedback configuration, we consider the optimal reference tracking controller in the $\mathcal{L}_2$ sense. The results are compared to the limiting cost of cheap control problems for SITO plants.

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1. Introduction. One of the main aims of feedback control theory is to ensure that the output of a plant behaves as specified; the output should track reference signals satisfactorily. To achieve this objective feedback is used; the error signal (the difference between the reference and the plant output) is fed back as the input to a controller.

One method of choosing a controller so that the closed loop system acts as specified is to perform an optimization. One cost functional that is often minimized in control theory is the integral of the error signal squared. The advantage of performing the optimization using the integral squared error (ISE) as the cost functional is that the optimization is carried out in $\mathcal{L}_2$. As $\mathcal{L}_2$ is a Hilbert Space, the optimization problem is comparatively tractable. The minimization of the ISE is often carried out in the context of the Internal Model Control (IMC) design paradigm [9]; several applications of this theory have been documented, [8] [11].

The ISE for linear time invariant systems using unity feedback control has been studied by several authors [2] [9]. Expressions for the infimal ISE have been obtained for single-input single-output systems[9]. Expressions for the infimal ISE were obtained for square multivariable systems in terms of the non-minimum phase zeros and unstable poles of the plant[2].

The cheap control tracking problem minimizes the linear quadratic cost of tracking a reference signal whilst the control cost goes to zero[6]. The minimization of the ISE using unity feedback is closely related to the cheap control tracking problem. The difference between the two methods is that the optimal linear quadratic controller uses full state feedback to track the reference. The cheap control tracking problem has been intensively studied[1][3][7]. The minimal cheap control tracking cost functional for square multivariable systems is dependent on the non-minimum phase (NMP) zeros of the plant [10].

Systems with a single input and two outputs (SITO) have recently been studied to examine their fundamental limitations [16]. For a SITO plant in steady state, the outputs of the closed loop system must lie in the range space of the plant. If the direction of the plant gain does not change with frequency and the plant is minimum phase, then the linear quadratic cost functional for a unit step change in the reference will approach zero as the control cost approaches zero [13]. If the direction of the plant gain varies with frequency, then the cheap control cost for a unit step change in the reference is strictly positive [13]. For SITO systems, the cheap control cost is

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dependent on the NMP zeros of the plant, and the change in the direction of the plant with frequency.

In this paper we consider controllers which minimize the ISE using unity feedback for SITO systems. We consider step changes in the reference signal. For stable plants, we show that the infimal value of the ISE is equal to the limiting value of the cheap control cost functional. We also show that if the plant has unstable poles, the infimal ISE cost may be larger than if the plant was stable. Practically implemented controllers will not be optimal with respect to the ISE criterion in general, due to robust stability and performance considerations. However, a large value of the optimal ISE for a given plant implies that there are fundamental limitations in tracking a step change in the reference signal.

An outline of the paper is as follows, In Section 2 we review preliminaries necessary for the paper's development. We provide a coordinate transformation to expedite our study of plants in which the direction changes with frequency in section 3. In Section 4 we provide an expression for the infimal ISE given a stable SITO plant. In Section 5 we provide an expression for the infimal ISE for an unstable plant. Section 6 concludes the paper.

2. Preliminaries. We require the following definitions from linear algebra. Given a vector $v \in \mathbf{C}^n$, let $\|v\|$ denote the Euclidean vector norm of v. Given a matrix $\Gamma \in \mathbf{C}^{m\times n}$, denote the range (or column space) of Γ by $\mathcal{R}(\Gamma)$, the orthogonal complement of $\mathcal{R}(\Gamma)$ by $\mathcal{R}^{\perp}(\Gamma)$, the nullspace of Γ by $\mathcal{N}(\Gamma)$, the rowspace of Γ by $\mathcal{R}_{row}(\Gamma)$, and the left nullspace of Γ by $\mathcal{N}_{left}(\Gamma)$. Denote the Euclidean matrix norm of Γ by $\|\Gamma\|$. Denote the complex conjugate transpose of Γ as Γ^{H}; denote the transpose of Γ as Γ^{T}. Denote the open and closed left and right half planes of $\mathbf{C}$ as OLHP, CLHP, ORHP, and CRHP respectively.

As usual, $\mathcal{L}_2(j\mathbf{R})$ is the Hilbert Space of matrix valued functions G on $j\mathbf{R}$ such that

$$\int_{-\infty}^{\infty} \operatorname{trace}\left(G^{\mathrm{H}}(j\omega)G(j\omega)\right) d\omega < \infty. \tag{2.1}$$

For any $G(s) \in \mathcal{L}_2(j\mathbf{R})$, the norm of G is defined as

$$\|G\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{trace}\left(G^{\mathrm{H}}(j\omega)G(j\omega)\right) d\omega}. \tag{2.2}$$

A transfer function matrix $G \in \mathcal{L}_2(j\mathbf{R})$ is said to be in $\mathcal{H}_2$ if G is analytic in the ORHP. $\mathcal{H}_2$ is a closed subspace of $\mathcal{L}_2(j\mathbf{R})$. A transfer function matrix $G \in \mathcal{L}_2(j\mathbf{R})$ is said to be in $\mathcal{H}_2^{\perp}$ if G is analytic in the OLHP. $\mathcal{H}_2^{\perp}$ is the orthogonal complement of $\mathcal{H}_2$ in $\mathcal{L}_2(j\mathbf{R})$. The norms of the spaces $\mathcal{H}_2$ and $\mathcal{H}_2^{\perp}$ are defined as [18]

$$\|G\|_2 = \sqrt{\sup_{\sigma>0}\frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{trace}\left(G^{\mathrm{H}}(\sigma+j\omega)G(\sigma+j\omega)\right) d\omega} \tag{2.3}$$

and

$$\|G\|_2 = \sqrt{\sup_{\sigma<0}\frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{trace}\left(G^{\mathrm{H}}(\sigma+j\omega)G(\sigma+j\omega)\right) d\omega} \tag{2.4}$$

respectively. Because $\mathcal{H}_2$ and $\mathcal{H}_2^{\perp}$ are orthogonal,

$$||G_1 + G_2||_2^2 = ||G_1||_2^2 + ||G_2||_2^2, \quad \forall G_1 \in \mathcal{H}_2, G_2 \in \mathcal{H}_2^{\perp}. \tag{2.5}$$

Denote the set of all stable proper rational transfer functions as $\mathbf{R}\mathcal{H}_\infty$. For any transfer function matrix G, denote the conjugate system of G by $G^{\sim}(s) = G^{\mathrm{T}}(-s)$. A transfer function $G(s) \in \mathbf{R}\mathcal{H}_\infty$ is said be *inner* if $G^{\sim}(s)G(s) = I$. A transfer function is said to be *outer* if $G(s)$ has no zeros in the ORHP. If a scalar transfer function $G(s)$ has a zero z_1 in the ORHP so that $G(z_1) = 0$, the zero z_1 is said to be a non-minimum phase (NMP) zero. If the plant has no NMP zeros, it is said to be minimum phase; otherwise it is termed a non-minimum phase plant.

We consider the case in which both the plant and the controller are linear and time invariant. Consider the control of a SITO plant by a unity feedback controller, as shown in Figure 2.1. Denote the transfer function matrices of the plant and the controller as $P(s) = \begin{bmatrix} p_1(s) & p_2(s) \end{bmatrix}^{\mathrm{T}}$ and $C(s) = \begin{bmatrix} c_1(s) & c_2(s) \end{bmatrix}$ respectively. The reference signal and the error signal are denoted by $r(t)$ and $e(t)$ respectively, where $r(t) = \begin{bmatrix} r_1(t) & r_2(t) \end{bmatrix}^{\mathrm{T}}$ and $e(t) = \begin{bmatrix} e_1(t) & e_2(t) \end{bmatrix}^{\mathrm{T}}$.

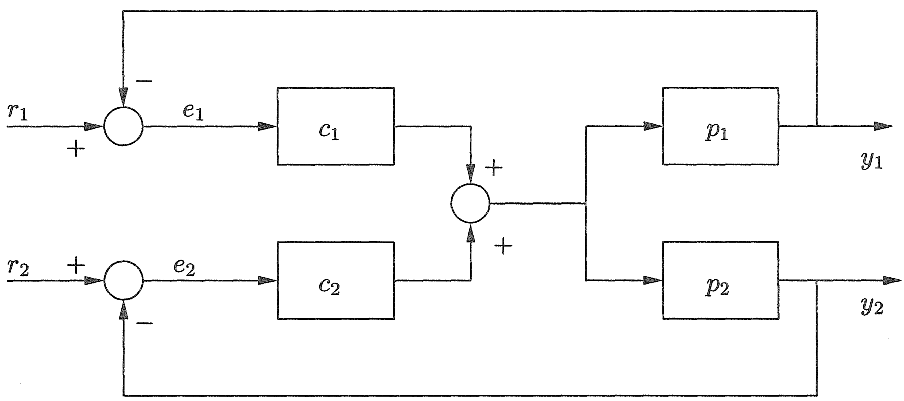

FIG. 2.1. *Single Input Two Output System*

Define the output open loop transfer function as $L_O(s) = P(s)C(s)$. The output sensitivity and complementary sensitivity functions are defined as $S_O(s) = (I + L_O(s))^{-1}$ and $T_O(s) = L_O(s)(I + L_O(s))^{-1}$ respectively. Similarly, define the the input open loop transfer function as $L_I(s) = C(s)P(s)$, and the input sensitivity and complementary sensitivity functions as $S_I(s) = (1 + L_I(s))^{-1}$ and $T_I(s) = L_I(s)(1 + L_I(s))^{-1}$ respectively.

Let the right and left coprime factorizations of $P(s)$ be given by

$$P(s) = N_p(s)D_p^{-1}(s) = \bar{D}_p^{-1}(s)\bar{N}_p(s), \tag{2.6}$$

where $N_p(s)$, $D_p(s)$, $\bar{N}_p(s)$, and $\bar{D}_p(s) \in \mathbf{R}\mathcal{H}_\infty$ satisfy the double Bezout identity

$$\begin{bmatrix} \bar{X}(s) & -\bar{Y}(s) \\ -\bar{N}_p(s) & \bar{D}_p(s) \end{bmatrix} \begin{bmatrix} D_p(s) & Y(s) \\ N_p(s) & X(s) \end{bmatrix} = I, \tag{2.7}$$

for some $X(s)$, $Y(s)$, $\bar{X}(s)$, $\bar{Y}(s) \in \mathbf{R}\mathcal{H}_\infty$. Then the set of all stabilizing compensators

$C(s)$ is characterised by [12]

$$\mathcal{C} \triangleq \left\{C(s) : C(s) = (Y(s) - D_p(s)Q(s))(N_p(s)Q(s) - X(s))^{-1}, Q(s) \in \mathbf{R}\mathcal{H}_\infty\right\}. \tag{2.8}$$

If $P(s)$ is stable, then we can select $N_p(s) = \bar{N}_p(s) = P(s)$, $\bar{X}(s) = D(s) = I$, $X(s) = \bar{D}(s) = I$, and $Y = \bar{Y} = 0$. In this case, the parameterization of all stabilizing controllers is

$$\mathcal{C} = \left\{C(s) : C(s) = Q(s)(I - P(s)Q(s))^{-1}, Q(s) \in \mathbf{R}\mathcal{H}_\infty\right\}. \tag{2.9}$$

Denote the ISE cost functional with zero initial conditions as

$$J = \int_0^\infty \|e(t)\|^2 dt. \tag{2.10}$$

It is well known that the output of the system due to the reference signal is $Y(s) = T_O(s)R(s)$. It follows that $E(s) = S_O(s)R(s)$. ¿From Parseval's identity

$$J = \|S_O(s)R(s)\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \|S_O(j\omega)R(j\omega)\|^2 d\omega. \tag{2.11}$$

We will compute

$$J^* = \inf_{Q(s)\in\mathbf{R}\mathcal{H}_\infty} J, \tag{2.12}$$

where $Q(s)$ is the free parameter of (2.8) and (2.9).

3. DC Coordinates. We use a special coordinate system, termed "DC Coordinates" [15], to expedite our study of the effects of plant direction varying with frequency. DC Coordinates are useful because the expressions which we will develop for the infimal value of J include zeros of polynomials of the plant transfer function in DC coordinates.

DEFINITION 3.1. (*Plant Direction* [4]) Let $N(s)D^{-1}(s)$ be a right co-prime polynomial factorization of $P(s)$ ($N(s)D^{-1}(s)$ is also termed a matrix fraction description of the plant). We define the direction of the plant at $s \in \mathbf{C}$ as $\mathcal{R}(P(s))$. If $s \in \mathbf{C}$ is a pole of $P(s)$, we define the direction of the plant as $\mathcal{R}(N(s))$.

To describe the plant direction, we also introduce the plant direction vector $\vec{P}(s)$. Given $s \in \mathbf{C}$, let $\vec{P}(s) \in \mathbf{C}^{2\times 1}$ be a unit vector such that

$$\mathcal{R}(N(s)) = \mathcal{R}(\vec{P}(s)). \tag{3.1}$$

DEFINITION 3.2. (*DC Coordinates*) Choose a unit vector $\vec{P}(0)$ to span $\mathcal{R}(P(0))$ and a unit vector $\vec{P}^\perp(0)$ to span $\mathcal{N}_{left}(P(0))$. Then the DC coordinates of $P(s)$ are the unique transfer functions $\tilde{p}_1(s)$, $\tilde{p}_2(s)$, such that

$$P(s) = \vec{P}(0)\tilde{p}_1(s) + \vec{P}^\perp(0)^{\mathrm{T}}\tilde{p}_2(s). \tag{3.2}$$

The plant, controller and output sensitivity function in DC coordinates are denoted as

$$\tilde{P}(s) = \begin{bmatrix} \tilde{p}_1(s) \\ \tilde{p}_2(s) \end{bmatrix}, \quad \tilde{C}(s) = \begin{bmatrix} \tilde{c}_1(s) & \tilde{c}_2(s) \end{bmatrix}, \quad \tilde{S}_O(s) = (I + \tilde{P}(s)\tilde{C}(s))^{-1}. \tag{3.3}$$

Define an orthogonal matrix

$$\Lambda \triangleq \begin{bmatrix} \vec{P}(0) & \vec{P}^{\perp}(0)^{\mathrm{T}} \end{bmatrix}. \tag{3.4}$$

The plant, controller, and output sensitivity transfer functions in DC coordinates are related to those in standard coordinates by

$$P(s) = \Lambda\tilde{P}(s), \qquad C(s) = \tilde{C}(s)\Lambda^{-1}, \qquad S_O(s) = \Lambda\tilde{S}_O\Lambda^{-1}. \tag{3.5}$$

Denote a matrix fraction description of the plant in DC coordinates by $\tilde{P} = \tilde{N}\tilde{D}^{-1}$. Denote the elements of the transfer function $\tilde{N}$ by $\tilde{n}_{p1}$ and $\tilde{n}_{p2}$, so that $\tilde{N}(s) = \begin{bmatrix} \tilde{n}_{p1}(s) & \tilde{n}_{p2}(s) \end{bmatrix}^T$. By construction $\tilde{n}_{p2}(0) = 0$. If the plant direction vector does not change with frequency, then $\tilde{n}_{p2}(s) \equiv 0$.

Lemma 3.1. *Let $\tilde{R}(s)$ be the reference signal of the closed loop system in DC coordinates, and let the ISE cost functional of the system in DC coordinates be $\tilde{J}$, where*

$$\tilde{J} = \frac{1}{2\pi}\int_{-\infty}^{\infty} ||\tilde{S}_O(j\omega)\tilde{R}(j\omega)||^2 d\omega. \tag{3.6}$$

Then $\tilde{J} = J$.

Proof. The reference signal in standard coordinates is related to the reference signal in DC coordinates by $R(s) = \Lambda\tilde{R}(s)$. Therefore

$$J = \frac{1}{2\pi}\int_{-\infty}^{\infty} ||S_O(j\omega)R(j\omega)||^2 d\omega, \tag{3.7}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} ||\Lambda\tilde{S}_O(j\omega)\Lambda^{-1}\Lambda\tilde{R}(j\omega)||^2 d\omega = \tilde{J}. \tag{3.8}$$

□

In view of Lemma 3.1 we assume, without loss of generality, that all plants are given in DC coordinates. Since there is no risk of confusion, we will no longer denote plants in DC coordinates using "∼".

4. Tracking Step Reference Signals with a Stable Plant. Let us consider the control of stable SITO plants such that J is infimized. We consider first the case of stable plants, because we will show that if the plant is stable then there is no inherent $\mathcal{H}_2$ cost associated with using a unity feedback controller compared with using a full state feedback controller.

We make the following standing assumption.

ASSUMPTION 4.1.
(i). The transfer function $p_1(s)$ does not have a zero at $s = 0$.
(ii). The reference signal is a unit step in the direction of the plant, that is

$$R(s) = \frac{v}{s}, \tag{4.1}$$

where $v = \begin{bmatrix} 1 & 0 \end{bmatrix}^{\mathrm{T}} = \vec{P}(0)$.

If $p_1(0) = 0$ then $P(0) = 0$ and the plant will not be able to track any steady state change in the reference. As SITO systems are not right invertible, a SITO plant cannot track all reference signals with zero steady state error. Therefore, the reference signal applied to the plant must be in the range space of the plant, so that the integral in (2.10) converges. It is also possible to study reference signals other than unit step changes [2]. However, the analysis in other cases (such as sinusoidal and ramp signals) is similar to the case studied here and is therefore omitted.

Factor the plant as

$$P(s) = \begin{bmatrix} n_{p1}(s) \\ n_{p2}(s) \end{bmatrix} K_P(s), \tag{4.2}$$

where $n_{p1}(s), n_{p2}(s)$ are coprime polynomials and $K_P(s)$ is a rational transfer function.

NOTATION . Denote the
(i). NMP zeros of $K_P(s)$ as $\{\alpha_i : i = 1, \ldots, N_\alpha\}$
(ii). Zeros of $n_{p1}(s)$ as $\{\gamma_i : i = 1, \ldots, N_\gamma\}$

We now provide an expression for the infimal ISE cost achievable using a unity feedback configuration.

Theorem 4.1. *For a stable plant controlled as shown in Figure 2.1, the infimal value of J is given by*

$$J^* = \inf_{Q(s) \in \mathbb{R}\mathcal{H}_\infty} J = \sum_{i=1}^{N_\alpha} \frac{2}{\alpha_i} + \sum_{i=1}^{N_\gamma} \frac{1}{\gamma_i} - \sum_{i=1}^{N_\delta} \frac{1}{\delta_i}. \tag{4.3}$$

where $\{\delta_i : i = 1, \ldots, N_\delta\}$ are the OLHP zeros of

$$\Delta(s) \triangleq n_{p1}(s)n_{p1}(-s) + n_{p2}(s)n_{p2}(-s). \tag{4.4}$$

Proof. See Woodyatt [14]. □

For stable plants, the infimal value of J is equal to the cheap control cost of tracking a unit step change in the reference[13]. The infimal value of J is dependent both on the non-minimum phase zeros of the plant, and upon the change in the direction of the plant with frequency. If the plant does not change direction with frequency and the plant is minimum phase, then the infimal ISE cost is zero.

5. Tracking Step Reference Signals with an Unstable Plant. Let us now consider controlling a SITO plant $P(s)$ which has a non-empty set of unstable poles denoted as $\{p_i : i = 1, \ldots, N_u\}$. We will develop an expression for the infimal value of J in this case.

Denote the Blaschke product of the NMP zeros of $P(s)$ as $B_\alpha(s)$, that is,

$$B_\alpha(s) = \prod_{i=1}^{N_\alpha} \frac{\alpha_i - s}{\bar{\alpha}_i + s}. \tag{5.1}$$

Lemma 5.1. *A transfer matrix $P(s)$ can always be factorized as $P(s) = P_1(s)P_2(s)$, such that $P_1(s)$ is inner, $P_2(s)$ is minimum phase and right invertible and the unstable poles of $P_2(s)$ are equal to the unstable poles of $P(s)$.*

Proof. See [10]. □

For a plant $P(s)$, denote an inner outer factorization of the plant as $P_1(s)P_2(s)$, where $P_1(s)$ is inner and $P_2(s)$ is outer. Denote

$$F(s) = \begin{bmatrix} f_1(s) \\ f_2(s) \end{bmatrix} = B_\alpha^{-1}(s)P_1(s). \tag{5.2}$$

Note that $B_\alpha^{-1}(-s)B_\alpha^{-1}(s) = 1$, and so $F(s)$ is an inner transfer function. For a SITO system, all the inner factors $P_1(s)$ of the plant $P(s)$ differ only by a factor of ± 1, [14]. We therefore assume that $P_1(s)$ has been normalised so that $f_1(0) = 1$, without loss of generality.

We now provide a characterization of the infimal value of J for the case in which the plant has at least one unstable pole.

Theorem 5.1. *Let $P(s)$ be a plant which has a non-empty set of simple unstable poles $\{p_i : i = 1, \ldots, N_u\}$, such that there are no unstable pole zero cancellations. Then the infimal value of J is given by*

$$J^* = \inf_{Q(s) \in \mathbf{R}\mathcal{H}_\infty} J = \sum_{i=1}^{N_\alpha} \frac{2}{\alpha_i} + \sum_{i=1}^{N_\gamma} \frac{1}{\gamma_i} - \sum_{i=1}^{N_\delta} \frac{1}{\delta_i} + J_u^* \tag{5.3}$$

where J_u^* is finite and given by

$$J_u^* = \sum_{i,j \in \mathbf{I}} \frac{4\mathbf{Re}(p_i)\mathbf{Re}(p_j)}{(\bar{p}_i + p_j)p_i\bar{p}_j\bar{b}_i b_j}(1 - f_1(-p_i)B_\alpha^{-1}(p_i))^H(1 - f_1(-p_j)B_\alpha^{-1}(p_j)), \tag{5.4}$$

$$b_i = \prod_{\substack{j \in \mathbf{I} \\ j \neq i}} \frac{\bar{p}_j}{p_j}\frac{p_j - p_i}{\bar{p}_j + p_i}, \tag{5.5}$$

and $\mathbf{I}$ *is an index set such that*

$$\mathbf{I} = \{i : \bar{D}_p(p_i)v = 0\}. \tag{5.6}$$

Proof. See Woodyatt [14]. □

REMARK 5.1. In the case in which the plant does not change direction with frequency, $f_1(s) \equiv 1$ and the results reduce to those of a SISO plant [2].

Compare Theorems 4.1 and 5.1. Suppose we consider controlling two plants, for which the sets $\{\alpha_i : i = 1, \ldots, N_\alpha\}$, $\{\delta_i : i = 1, \ldots, N_\delta\}$, and $\{\gamma_i : i = 1, \ldots, N_\gamma\}$ are respectively equal. If one plant is stable, and the other is unstable, then the infimal ISE cost of the unstable plant will be greater than or equal to that of the stable plant.

Since $\bar{D}_p(p_i)v = 0$, it follows that [5]

$$S_O(p_i)v = 0, \forall i \in \mathbf{I}. \tag{5.7}$$

Therefore, for any unstable plant pole p_i such that $i \in \mathbf{I}$,

$$E(p_i) = S_O(p_i)R(p_i) = S_O(p_i)v\frac{1}{p_i} = 0. \tag{5.8}$$

So the condition $\bar{D}_p(p_i)v = 0$ implies an interpolation constraint on the Laplace transform of the error signal. Moreover, the condition $\bar{D}_p(p_i)v = 0$ is equivalent to the plant direction vector at the pole p_i being in the direction v.

Lemma 5.2. *Let* p *be an unstable pole of* $P(s)$. *Then* $\bar{D}_p(p_i)v = 0$ *if and only if*

$$\vec{P}(p_i) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = v. \tag{5.9}$$

Proof. See Woodyatt [14]. □

Therefore, unless the plant direction vector at the unstable pole is v, it will not imply an additional constraint on the infimal ISE cost compared with a stable plant. Thus, the effect of unstable plant poles is not robust with respect to small plant perturbations. Unstable poles which do not have the direction v also imply inherent constraint on achievable performance, however; constraints can also be derived using integral sensitivity relations [17].

To simplify the expression for J_u^*, consider the case in which $P(s)$ has one unstable pole $s = p_1$ satisfying the condition $\bar{D}_p(p_1)v = 0$.

Corollary 5.1. *Let* $P(s)$ *be a plant with a single unstable pole at* $s = p_1$, *such that* $\bar{D}_p(p_1)v = 0$. *Suppose further that* $p_1(0) \neq 0$. *Then the infimal value of* J *as defined in* (2.10) *is*

$$J^* = \inf_{Q(s) \in \mathbf{R}\mathcal{H}_\infty} J = \sum_{i=1}^{N_\alpha} \frac{2}{\alpha_i} + \sum_{i=1}^{N_\gamma} \frac{1}{\gamma_i} - \sum_{i=1}^{N_\delta} \frac{1}{\delta_i} + \frac{2}{p_1} \left|1 - f_1(-p_1)B_\alpha^{-1}(p_1)\right|^2. \tag{5.10}$$

Note that in Theorem 4.1, the minimal ISE cost is made up of two parts: the first term depends only on the NMP zeros of the plant; the last two terms depend on zeros that exist only if the plant direction varies with frequency. From Corollary 5.1 it is clear that J_u^*, which is given by the last term in (5.10), depends on two terms also: $f_1(s)$ and $B_\alpha(s)$. The Blaschke product $B_\alpha(s)$ depends only on the NMP zeros of the plant; the transfer function $f_1(s)$ depends on the variation of the plant with frequency.

6. Conclusions. For SITO plants, we have developed expressions for the infimal ISE cost of the closed loop system. These expressions yield insight into the achievable performance of SITO systems using unity feedback. If the plant is stable, the infimal ISE cost equals the cheap control cost of the SITO system. Thus in this case there is no additional cost of using output feedback compared with full state feedback. If the plant has unstable poles, then there is an additional term in the expression for the infimal ISE cost. However, the additional term will be non-zero only if at least one of the unstable poles is in a specific direction.

REFERENCES

[1] B. D. O. ANDERSON AND J. B. MOORE, *Optimal Control: Linear Quadratic Methods*, Prentice-Hall, 1989.

[2] J. CHEN, L. QIU, AND O. TOKER, *Limitations on maximal tracking accuracy*, tech. rep., University of California, Riverside, 1997.

[3] B. A. FRANCIS, The optimal linear-quadratic time-invariant regulator with cheap control, *IEEE Transactions on Automatic Control*, AC-24 (1979), pp. 616–621.

[4] J. S. FREUDENBERG AND R. H. MIDDLETON, *Properties of single input, two output feedback systems.* Submitted to the International Journal of Control, 1998.

[5] G. I. GÓMEZ, *On Feedback Control of Linear Multivariable Systems*, PhD thesis, University of Newcastle, May 1997.

[6] A. JAMESON AND R. E. O'MALLEY JR., Cheap control of the time-invariant regulator, *Applied Mathematics and Optimization*, 1 (1975), pp. 337–354.

[7] H. KWAKERNAAK AND R. SIVAN, *Linear Optimal Control Systems*, Wiley-Interscience, 1972.

[8] T. H. LEE, T. S. LOW, A. AL-MAMUM, AND C. H. TAN, Internal model control (IMC) approach for designing disk drive servo-controller, *IEEE Transactions on Industrial Electronics*, 42 (1995), pp. 248–256.

[9] M. MORARI AND E. ZAFIROU, *Robust Process Control*, Prentice-Hall, 1989.

[10] L. QIU AND E. J. DAVISON, Performance limitations of non-minimum phase systems in the servomechanism problem, *Automatica*, 29 (1993), pp. 337–349.

[11] Y.-Y. TZOU, DSP-based robust control of an AC induction servo drive for motion control, *IEEE Transactions on Control Systems Technology*, 4 (1996), pp. 614–626.

[12] M. VIDYASAGAR, *Control System Synthesis: A Factorization Approach*, MIT Press, 1985.

[13] A. WOODYATT, J. FREUDENBERG, AND M. SERON, Cheap control of non-square systems with more outputs than inputs, *Tech. Rep. EE9818, University of Newcastle*, 1998.

[14] A. R. WOODYATT, Limitations on achievable tracking accuracy for single-input two-output systems., *Tech. Rep. EE9841, University of Newcastle*, 1998.

[15] A. R. WOODYATT, J. S. FREUDENBERG, AND R. H. MIDDLETON, A design tradeoff for single input two output feedback systems, *in Proceedings of the 35th Allerton Conference on Communication, Control and Computing*, October 1997.

[16] ———, Integral constraints for single input two output feedback systems, *Tech. Rep., University of Newcastle*, 1997. EE9703.

[17] A. R. WOODYATT, R. H. MIDDLETON, AND J. S. FREUDENBERG, Fundamental constraints for the inverted pendulum problem, *Proceedings of the IEEE Singapore International Symposium on Control Theory and Applications*, 1997, pp. 321–325.

[18] K. ZHOU, J. C. DOYLE, AND K. GLOVER, *Robust and Optimal Control*, Prentice Hall, 1996.