

# A Configuration Space for Solutions of Maxwell's Equations

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## Abstract

In joint work with Robin Balean, we have constructed a parameterisation of the space of all solutions of Maxwell's equations in an exterior domain in Minkowski space. The domain has boundary consisting of a null cone and a timelike cylinder, and the construction arises from uniqueness and existence theorems, which we establish within a certain "Lorentz" gauge. The resulting free boundary data are unconstrained.

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Consider the problem of constructing a simple parameterisation of the space of solutions of Maxwell's equations. A natural attempt is to use a class of "well-posed" boundary conditions, with the success of such an endeavour providing our *definition* of the meaning of "well-posed". What conditions should such a class of boundary data fulfill?

Firstly, the boundary data and conditions should determine a solution<sup>1</sup> *uniquely*: if two solutions have the same boundary data then they should agree. Secondly, for every boundary datum in the class, there should *exist* a solution of Maxwell's equations; and thirdly, *every* solution of Maxwell's equations should be constructible from some datum in the class of boundary data.

A final requirement, arising from the gauge degeneracy of Maxwell's equations, is that the boundary data should be *unconstrained*. By this we mean

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<sup>1</sup>For the purposes of this paper, "solution" means  $C^\infty$ , with  $C^\infty$  boundary values. It should not be difficult to extend our results to more general regularity conditions.

that the boundary data should not be required to satisfy any constraint differential equations. This condition leads to a transparent parameterisation of the space of admissible boundary data.

However, such constraints seem to be an inevitable feature of formulations of Maxwell's equations based on a Cauchy problem. Maxwell's equations may be expressed either as  $dF = 0$ ,  $d * F = 0$  where  $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$  is a spacetime 2-form, or in terms of the electric field  $E_i = F_{0i}$  and magnetic field  $B_i = \frac{1}{2} \epsilon_{ijk} F_{jk}$ , as

$$\begin{aligned} \frac{\partial}{\partial t} \begin{bmatrix} E \\ B \end{bmatrix} &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \text{curl } E \\ \text{curl } B \end{bmatrix} \\ \text{div } E &= 0 \\ \text{div } B &= 0 \end{aligned}$$

In particular, this shows that any 3+1 formulation must have boundary data satisfying the constraint  $\text{div } E = 0$  (note that  $\text{div } B = 0$  is satisfied automatically once the parameterisation

$$\begin{aligned} E_i &= \partial_t A_i - \partial_i \phi \\ B_i &= \epsilon_{ijk} \partial_j A_k \end{aligned}$$

of the curvature  $(E, B)$  in terms of the Maxwell potentials  $(A_\mu) = (\phi, A_i)$  is adopted). For example, if we impose the temporal gauge  $\phi = A_0 = 0$ , then the potential 3-vector  $A_i$  satisfies  $(\partial_t^2 - \Delta)A_i = 0$  together with the initial value constraints

$$\text{div } A = 0, \quad \text{div}(\partial_t A_i) = 0.$$

In particular the Cauchy data for  $A_i$  cannot be arbitrarily chosen functions, since these differential conditions must also be satisfied. Other gauge choices lead to similar constraints, and we note also that imposing a constraint like  $\text{div } E = 0$  typically involves solving an elliptic boundary value problem, with consequent nonlocal complications arising if the domain is non-compact (decay conditions) or has unspecified boundary conditions.

We have established [3] a simple classification satisfying all four of these conditions, for solutions of Maxwell's equations in an exterior domain

$$\Omega = \{(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3 \mid t \geq r - r_0, r \geq r_0\}$$

where  $r = |x|$  and  $r_0 \in \mathbb{R}^+$ , with boundary consisting of a timelike tube

$$\mathcal{T} = \{(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3 \mid r = r_0, t \geq 0\}$$

and a characteristic truncated null cone

$$\mathcal{C} = \{(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3 \mid t - r + r_0 = 0, r \geq r_0\}.$$

Rather surprisingly, our formulation of the null-timelike boundary value problem involves arbitrary fields without any constraints, neither algebraic nor differential. Moreover, decay conditions are not needed – the total energy may be unbounded. Central to our constructions is a careful choice of boundary conditions, starting with the *Lorentz* gauge:

$$(1) \quad L(A) := \nabla^\mu A_\mu = 0$$

Introducing the null-polar coordinates  $(z, \rho, \vartheta, \varphi)$ ,

$$z = t - r + r_0, \quad \rho = r$$

with the usual polar coordinates  $(\vartheta, \varphi)$  on  $S^2$ , we may decompose the Maxwell potential  $A$  by

$$(2) \quad A = A_\mu dx^\mu = A_z dz + A_\rho d\rho + \rho A_\vartheta d\vartheta + \rho A_\varphi \sin \vartheta d\varphi,$$

whereupon the Lorentz gauge equation becomes

$$(3) \quad L(A) = \rho^{-2} \partial_\rho(\rho^2(A_z - A_\rho)) + \partial_z A_\rho - \rho^{-2} \operatorname{div}_2 A_2.$$

Here  $A_2 := A_\vartheta d\vartheta + A_\varphi \sin \vartheta d\varphi$  is the angular component of  $A$ , and

$$\operatorname{div}_2 A_2 = \partial_\vartheta A_\vartheta + \cot \vartheta A_\vartheta + \csc \vartheta \partial_\varphi A_\varphi$$

is the spherical divergence.

Our *Lorentz boundary conditions* are now

$$(4) \quad A_\rho|_{\mathcal{C}} = 0$$

$$(5) \quad A_z|_{\mathcal{T}} = 0$$

$$(6) \quad \partial_\rho A_z|_{\mathcal{S}} = 0$$

where  $\mathcal{S} = \mathcal{C} \cap \mathcal{T} = \{t = 0, r = r_0\}$  is the corner sphere.

Note that in order to express Maxwell's equations globally in  $\Omega$  in terms of the potential 1-form  $A$ , it is necessary (and sufficient) to assume that the magnetic charge vanishes:

$$k := \frac{1}{4\pi} \oint_{\mathcal{S}} B_r = \frac{1}{4\pi} \oint_{\mathcal{S}} F = \frac{1}{4\pi} \oint_{\mathcal{S}} dA = 0$$

Applying the Lorentz gauge and boundary conditions along the corner sphere  $\mathcal{S}$  shows that the radial electric field  $E_r = F_{0r} = r^{-1}x^i F_{0i}$  satisfies

$$E_r := \partial_z A_\rho - \partial_\rho A_z = \rho^{-1} \operatorname{div}_2 A_2 \quad \text{on } \mathcal{S}.$$

Thus we must also require that the electric charge also vanish,

$$e = \frac{1}{4\pi} \oint_{\mathcal{S}} *F = \frac{1}{4\pi} \oint_{\mathcal{S}} E_r = 0.$$

This consequence of the Lorentz gauge and boundary conditions proves to play an important role in the uniqueness and existence theorems.

If  $\Lambda \in C^\infty(\Omega)$  then  $A$  and  $A + d\Lambda$  are gauge equivalent potentials, and we eliminate this gauge freedom by requiring the Lorentz gauge and boundary conditions:

**Theorem 1** *Let  $B$  be any gauge 1-form in  $\Omega$  having vanishing electric charge on  $\mathcal{S}$ . Then there exists a unique gauge-equivalent potential  $A = B + d\Lambda$  satisfying the Lorentz gauge (3) and boundary conditions (4–6).*

The proof comes down to constructing  $\Lambda$  as a solution of the wave equation  $\square\Lambda = f$  in  $\Omega$  with Dirichlet boundary conditions on  $\partial\Omega = \mathcal{C} \cup \mathcal{T}$ . Suitable existence results for this wave equation BVP were established in [1, 2], together with existence results for the Neumann boundary conditions on  $\mathcal{T}$  for the wave equation and the modified wave operator  $\tilde{\square}u = 2\partial_\rho\partial_z u - \partial_\rho^2 u - \rho^{-2}\Delta_2 u$ .

This gauge-fixing result provides a slice of the gauge-equivalence classes in the space of all potentials on  $\Omega$ , and we now consider the effect of imposing Maxwell's equations. As mentioned, we seek boundary conditions which are sufficiently stringent as to ensure *uniqueness*, but not so restrictive that some genuine solutions of Maxwell's equations are excluded. Naturally enough, we start with the Lorentz conditions (4–6).

**Theorem 2** *If  $A, B$  are potentials satisfying Maxwell's equations with the Lorentz gauge and boundary conditions (4–6), and if*

$$(7) \quad A_2|_{\mathcal{C} \cup \mathcal{T}} = B_2|_{\mathcal{C} \cup \mathcal{T}},$$

*then  $A = B$  in  $\Omega$ .*

*Outline of proof:* By considering  $A - B$ , it will suffice to show that if  $A$  satisfies the Lorentz gauge and boundary conditions, then  $A_2|_{\mathcal{C} \cup \mathcal{T}} = 0$  implies that  $A = 0$ . From the general identity

$$\begin{aligned} \partial_\rho(\rho\partial_\rho(\rho A_z)) &= \frac{1}{2}\partial_\rho^2(\rho^2 A_\rho) - \frac{1}{2}\Delta_2 A_\rho + \partial_\rho(\rho \operatorname{div}_2 A_{S^2}) \\ &\quad - \rho \left( \frac{1}{2}\rho \square A_z + \frac{1}{2}x^i \square A_i - \partial_\rho(\rho L(A)) \right) \end{aligned}$$

and the boundary conditions, we find

$$\partial_\rho(\rho\partial_\rho(\rho A_z)) = 0 \quad \text{on } \mathcal{C}.$$

Since  $\partial_\rho A_z = 0$  on  $\mathcal{S}$ , it follows that  $A_z = 0$  on  $\mathcal{C} \cup \mathcal{T}$  and thus (since  $\square A_z = 0$  by Maxwell's equations and the Lorentz gauge condition),  $A_z = 0$  in  $\Omega$ . A similar argument working through the Neumann problem for  $\square(\rho^2 A_\rho)$  shows that  $A_\rho = 0$  in  $\Omega$ .

Now introducing the Hodge-Helmholtz representation of  $A_2$  in terms of functions  $v, w$ ,

$$A_2 = d_2 v + *_2 d_2 w$$

(ie.  $A_\vartheta = \partial_\vartheta v + \text{csc } \vartheta \partial_\varphi w$ ,  $A_\varphi = -\partial_\vartheta w + \text{csc } \vartheta \partial_\varphi v$ , with  $v, w$  normalised to have vanishing spherical averages  $\oint_{S^2} v = \oint_{S^2} w = 0$ ), we find that

$$\square v = 0, \quad \square w = 0$$

since  $A_\rho = 0 = A_z$ . Since  $A_2|_{\mathcal{C} \cup \mathcal{T}} = 0$ , the boundary relations (4-6) show that  $v = w = 0$  on  $\mathcal{C} \cup \mathcal{T}$ , and uniqueness for the wave equation gives  $v = w = 0$  in  $\Omega$  and thus  $A = 0$  as required. QED.

The boundary condition implicit in Theorem 2 also appears in our existence theorem. Let  $\Gamma^\infty(\mathcal{C}, T^*S^2)$  denote the space of  $C^\infty$  sections of the bundle  $T^*S^2 \times \mathbb{R}^+$  over  $\mathcal{C} = S^2 \times \mathbb{R}^+$  (that is, the space of angular 1-forms over  $\mathcal{C}$ ), and similarly define  $\Gamma^\infty(\mathcal{T}, T^*S^2)$ ,  $\Gamma^\infty(\Omega, \Lambda^2 T^*S^2)$  etc.

**Theorem 3** *Let  $H_{\mathcal{C}} \in \Gamma^\infty(\mathcal{C}, T^*S^2)$ ,  $H_{\mathcal{T}} \in \Gamma^\infty(\mathcal{T}, T^*S^2)$  satisfy the corner continuity condition*

$$H_{\mathcal{C}}|_{\mathcal{S}} = H_{\mathcal{T}}|_{\mathcal{S}}.$$

*Then there exists  $A \in \Gamma^\infty(\Omega, T^*\Omega)$ , a smooth solution of Maxwell's equations in the Lorentz gauge with boundary conditions (4-6), such that*

$$(8) \quad A_2|_{\mathcal{C}} = H_{\mathcal{C}}, \quad A_2|_{\mathcal{T}} = H_{\mathcal{T}}$$

*and the electric and magnetic charges of  $A$  both vanish.*

The proof is by explicit construction of the null-polar components  $A_z, A_\rho$  and the Hodge-Helmholtz potentials  $v, w$ , using a decomposition of the coupled tensorial Maxwell equations into a sequence of partially decoupled linear wave equations with reconstructible sources and boundary data. An important final step establishes that the constructed potential  $A$  satisfies the Lorentz gauge condition, since the Lorentz field  $L = L(A)$  can be shown to satisfy a linear wave equation with zero boundary data. Full details of this argument are given in [3].

Hence we obtain the classification theorem

**Theorem 4** *There is a one to one correspondence between the sets*

$$\begin{aligned}\mathcal{M} &= \{F \in \Gamma^\infty(\bar{\Omega}, \Lambda^2 T^* \Omega) : dF = 0, d * F = 0\} \\ \mathcal{D} &= \{(e, k, H_C, H_T) : e \in \mathbb{R}, k \in \mathbb{R}, \\ &\quad H_C \in \Gamma^\infty(\mathcal{C}, T^* S^2), H_T \in \Gamma^\infty(\mathcal{T}, T^* S^2), H_C|_S = H_T|_S\}\end{aligned}$$

*Proof:* If  $F \in \Gamma^\infty(\Omega, \Lambda^2 T^* \Omega)$  is a 2-form satisfying Maxwell's equations, then there are  $e, k \in \mathbb{R}$  such that  $F_0 = F - F_e^{elec} - F_k^{mag}$  satisfies Maxwell and has vanishing electric and magnetic charges, where

$$F_e^{elec} = e/r^2 dt \wedge dr, \quad F_k^{mag} := k \sin \vartheta d\vartheta \wedge d\varphi.$$

Since  $F_0$  has vanishing magnetic charge, it admits a representation in terms of a potential 1-form  $A$ ; since the electric charge also vanishes, by Theorem 1 we may assume that  $A$  satisfies the Lorentz gauge and boundary conditions; then the boundary restrictions of the angular part  $A_2$  give the boundary functions  $H_C = A_2|_C$ ,  $H_T = A_2|_T$  and uniquely determine a “point” in  $\mathcal{D}$ .

Conversely, given  $(e, k, H_C, H_T) \in \mathcal{D}$  we construct  $F_0$  from  $(H_C, H_T)$  via Theorem 3, and Theorem 2 ensures that this is the unique solution of Maxwell's equations satisfying the conditions of Theorem 3 with boundary fields  $(H_C, H_T)$ . We finally define  $F = F_0 + F_e^{elec} + F_k^{mag}$ , the required element of  $\mathcal{M}$ . QED.

*Remarks:*

(i) If we wish  $F$  to represent the curvature of a  $U(1)$  line bundle, then the magnetic charge must satisfy the Dirac quantisation condition  $k \in 2\pi\mathbb{Z}$ .

(ii) The requirement that the electric and magnetic charges vanish means that our constructions admit a *duality* invariance: if  $F$  satisfies Maxwell's equations and has zero charges, then  $F_\lambda := \cos \lambda F + \sin \lambda * F$  also satisfies Maxwell's equations, for any constant  $\lambda \in \mathbb{R}$ . By Theorem 4 there must be a corresponding action on the space of classifying boundary data,  $H = (H_C, H_T) \mapsto H_\lambda$ , but it seems not to be a simple matter to explicitly describe this transformation of boundary data.

(iii) The boundary data and conditions determine some components of the Maxwell 2-form  $F$ , namely

$$F_{\rho\vartheta} = \partial_\rho(\rho A_\vartheta), \quad F_{\rho\varphi} = \partial_\rho(\rho \sin \vartheta A_\varphi), \quad F_{\vartheta\varphi} = \partial_\vartheta(\rho \sin \vartheta A_\varphi) - \partial_\varphi(\rho A_\vartheta)$$

on  $\mathcal{C}$ , and similarly  $(F_{z\vartheta}, F_{z\varphi}, F_{\vartheta\varphi})$  on  $\mathcal{T}$ . These curvature components cannot be arbitrarily specified, but must satisfy the constraint relations

$$(9) \quad \partial_\rho F_{\vartheta\varphi} - \partial_\vartheta F_{\rho\varphi} + \partial_\varphi F_{\rho\vartheta} = 0 \quad \text{on } \mathcal{C}$$

$$(10) \quad \partial_z F_{\vartheta\varphi} - \partial_\vartheta F_{z\varphi} + \partial_\varphi F_{z\vartheta} = 0 \quad \text{on } \mathcal{T}.$$

Even so, the boundary curvatures do not determine the boundary data uniquely — more information is required:

**Theorem 5** ([3, Proposition 6.1]) *Given the fields  $(F_{\rho\partial}, F_{\rho\varphi}, F_{\partial\varphi})$  on  $\mathcal{C}$  and  $(F_{z\partial}, F_{z\varphi}, F_{\partial\varphi})$  on  $\mathcal{T}$  satisfying (9), (10), and given  $f \in C^\infty(\mathcal{S})$  with  $\oint_{\mathcal{S}} f = 0$ , there exists a unique potential  $A_2 \in \Gamma^\infty(\mathcal{C} \cup \mathcal{T}, T^*S^2)$  satisfying  $\operatorname{div}_2 A_2|_{\mathcal{S}} = f$  and determining the given boundary curvatures. Moreover, there is a unique extension of  $A_2$  to a smooth solution of Maxwell's equations in  $\Omega$  in the Lorentz gauge with boundary conditions (4–6), such that the given function  $f$  represents the radial electric field  $E_r$  along  $\mathcal{S}$ .*

## References

- [1] R. Baean, *The Null-Timelike Boundary Problem*, thesis, University of New England 1996.
- [2] ———, *The Null-Timelike Boundary Problem for the Linear Wave Equation*, to appear in *Commun. in PDE* 1997.
- [3] ——— and R. Bartnik, *The Null-Timelike Boundary Problem for Maxwell's Equations in Minkowski Space*, preprint, January 1997.

