SPECTRAL FLOW IN BREUER-FREDHOLM MODULES

A.L. CAREY[1] AND JOHN PHILLIPS[2]

ABSTRACT. This review discusses work in progress and related earlier studies by many authors. We have attempted to place our results in their broad context beginning with the L^2 index theorem of Atiyah and Singer, subsequent extensions and the motivation for our results and conjectures. The geometric setting is the analysis of L^2 invariants of non-compact covering spaces, several of which are not present (or are trivial) on compact manifolds. These invariants use the von Neumann algebra of the covering transformation group in an essential way.

1. INTRODUCTION

The story begins with Atiyah and Singer [At][Si] in the mid-seventies, who were investigating a generalization of the Atiyah-Singer index theorem to (non-compact) covering spaces of compact Riemannian manifolds and with the work of Atiyah, Patodi and Singer on the index theorem for manifolds with boundary, spectral flow and the eta invariant [APS]. The optimistic viewpoint on our work is that it would lead to a real synthesis of these two directions in the sense of giving, for naturally defined geometric operators on bundles over these covering spaces, a useful theory of spectral flow and its relation to the Breuer-Fredholm index and the L^2 index theorem. Indeed a careful reading of [APS] seems to indicate an intention to pursue this line by those authors. However results of this nature did not eventuate. One can only surmise that there were technical problems not the least of which is the obvious one: the whole idea seems unlikely because the operators which arise in examples can have continuous spectrum and so there is no notion of spectral flow as an intersection number. Our aim in this paper is to explain our solution to this conundrum and to place the general problem in its geometric setting. The proofs are lengthy and heavily functional analytic in character and we refer the interested reader to [CP1] for the details.

We shall assume that $\Gamma \to \widehat{M} \to M$ is the universal cover for a compact Riemannian manifold (M, g) of dimension n, with the Riemannian metric on M lifting to a Riemannian metric on the cover $\widehat{M}$. (All manifolds are assumed to be oriented.) While the geometric situations we have in mind require us to consider bundles over $\widehat{M}$ for the purposes of this introduction, which are mainly analytic, it will suffice to consider the situation on $L^2(\widehat{M})$. We have an isomorphism

$$L^2(\widehat{M}) \cong L^2(M) \otimes \ell^2(\Gamma). \tag{1.1}$$

Now Γ acts on $\widehat{M}$ by deck transformations, and these lift to a unitary representation on $L^2(\widehat{M})$. The isomorphism in (1.1) can be chosen to intertwine this representation with $1 \otimes R$ where R is the right regular representation on $\ell^2(\Gamma)$. The commutant of $1 \otimes R$ is $\mathcal{N} = \mathcal{B}(L^2(M)) \otimes \mathcal{A}_0$ where $\mathcal{A}_0$ is the von Neumann algebra generated by the left regular representation of Γ. Both $\mathcal{B}(L^2(M))$ and $\mathcal{A}_0$ have canonical traces denoted Tr and $\tau_{\mathcal{A}_0}$. These combine to give a trace on the type II von Neumann algebra $\mathcal{N}$ defined by

$$\tau = \mathrm{Tr} \otimes \tau_{\mathcal{A}_0}.$$

The ideal $\mathcal{T}_{\mathcal{N}}$ of operators with finite trace has norm closure $\mathcal{K}_{\mathcal{N}}$ and the quotient

$$\mathcal{Q} = \mathcal{N}/\mathcal{K}_{\mathcal{N}}$$

is the type II Calkin algebra. A Breuer-Fredholm operator in $\mathcal{N}$ is an operator which is invertible in $\mathcal{Q}$. Such operators have finite dimensional kernel and co-kernel in the von Neumann sense and the index of a Breuer-Fredholm operator F is given by $i(F) = \tau(P_{\ker F}) - \tau(P_{\mathrm{coker} F})$ where $P_{\ker F}$ and $P_{\mathrm{coker} F}$ denote the orthogonal projections onto the kernel and cokernel. The quantity $\tau(P_{\ker F})$ is often referred to as von Neumann's 'continuous' dimension: it replaces the ordinary notion of dimension in the type II setting.

One of the objects of study initiated by the L^2 theory of [At][Si] is the analysis of the spectral properties of operators in $\mathcal{N}$ especially Breuer-Fredholm operators. Also of interest is the study of the Laplacian on the complex of L^2 differential forms. There is now a long history on this and we refer the reader to [Do], [AtSc], [G2], [ChGr3], [ChGr1], [ChGr2], [Lu] [DM], [DM2]. In the late eighties, Novikov and Shubin [ES] defined some new invariants using the von Neumann spectral density function of the Laplacian on differential forms. (This measures the von Neumann spectral multiplicity.) They reasoned that the germ of the spectral density function near zero should also be a homotopy invariant. This important result was established in [GS] and [L]; see also [ES], [NS2], [BMW], [F]. There is an outstanding conjecture that the Novikov-Shubin invariants are always positive.

One way of thinking about these invariants is that they constitute an analytic obstruction to generalising a number of results which work in the compact case because zero is an isolated point in the spectrum. In the L^2 theory zero need not be an isolated point and this leads to the study of Novikov-Shubin invariants. We are able to avoid these in the discussion of spectral flow for finitely summable Breuer-Fredholm modules however they are critical for the so-called θ-summable case. We elaborate on this in subsequent sections.

The technical difficulties associated with type II spectral invariants are explored both in the study of the Novikov-Shubin invariants (references above) and also in the discussion of the so called L^2 torsion. We refer the reader to [RS] for the classical theory of torsion and to [M],[L], [CM], [CFM], [LuR], [BFKM] for the extension to the L^2 setting. See also the L^2 spectral invariants discussed in [CCMP].

After this brief review of the literature we move on to the main issues.

2. Definitions and statement of results

The main objects of interest in our work are Fredholm modules. We discuss first the type I situation. An *odd unbounded* (respectively, *p-summable*) *Fredholm module* for a unital C^*-algebra, $\mathcal{A}$, is a pair (H, D) where $\mathcal{A}$ is represented on the Hilbert space, H, and D is an unbounded self-adjoint operator on H satisfying:

1. $\{a \in A \mid [D,a] \text{ is bounded}\}$ is a dense $*-$subalgebra of A, and
2. $(1+D^2)^{-1}$ is compact (respectively, $\mathrm{Tr}\left((1+D^2)^{-(p/2)}\right) < \infty$).

Here Tr is used to represent the type I trace.

If u is a unitary in the dense $*-$subalgebra mentioned in point 1 of the defintion then

$$uDu^* = D + u[D, u^*] = D + B \tag{2.1}$$

where B is a bounded self-adjoint operator. The path

$$D_t^u := (1-t)\,D + tuDu^* = D + tB \tag{2.2}$$

is a "continuous" path of unbounded self-adjoint "Fredholm" operators. More precisely in [CP1] we establish that

$$F_t^u := D_t^u \left(1 + (D_t^u)^2\right)^{-\frac{1}{2}} \tag{2.3}$$

is a norm-continuous path of (bounded) self-adjoint Fredholm operators. The *spectral flow* of this path $\{F_t^u\}$ (or $\{D_t^u\}$) in the type I case is roughly speaking the net number of eigenvalues that pass through 0 in the positive direction as t runs from 0 to 1. This integer,

$$sf\{D_t^u\} := sf\{F_t^u\}, \tag{2.4}$$

recovers the pairing of the K-homology class $[D]$ with the K-theory class $[u]$.

To go further that this we use an idea due to Ezra Getzler [G]. He outlined a method of exhibiting spectral flow as the integral of a one-form in the context of unbounded θ-summable ($\mathrm{Tr}(e^{-tD^2}) < \infty$ for all $t > 0$) Fredholm modules. Following this approach we consider the operator B as a parameter in the Banach manifold, $B_{sa}(H)$, so that spectral flow can be exhibited as the integral of a closed 1-form on this manifold. Now, for B in our manifold, any $X \in T_B\ (B_{sa}(H))$ is given by an X in $B_{sa}(H)$ as the derivative at B along the curve $t \mapsto B + tX$ in the manifold. Then we show [CP1] that for m a sufficiently large half-integer:

$$\alpha(X) = \frac{1}{\tilde{C}_m} \mathrm{Tr}\left(X(1 + (D+B)^2)^{-m}\right) \tag{2.5}$$

is a closed 1-form. For any piecewise smooth path $\{D_t = D + B_t\}$ with D_0 and D_1 unitarily equivalent we show that

$$sf\{D_t\} = \frac{1}{\tilde{C}_m} \int_0^1 \mathrm{Tr}\left(\frac{d}{dt}(D_t)(1 + D_t^2)^{-m}\right) dt \tag{2.6}$$

the integral of the 1-form α. If D_0 and D_1 are not unitarily equivalent, we must add a pair of correction terms to the right-hand side. We also prove a bounded finitely summable version of the form:

$$sf\{F_t\} = \frac{1}{C_n} \int_0^1 \mathrm{Tr}\left(\frac{d}{dt}(F_t)(1 - F_t^2)^n\right) dt \tag{2.7}$$

for $n \geq \frac{p}{2}$ an integer. The unbounded case is proved by reducing to the bounded case via the map $D \mapsto F = D(1 + D^2)^{-\frac{1}{2}}$.

One of the interesting features of our proofs are that they apply similtaneously to the type II situation as well. In that case we are dealing with a real-valued spectral flow. To be more precise, we need a definition.

Definition. An *odd unbounded* (respectively, *p-summable*) *Breuer-Fredholm module* for a unital C^*-algebra, $\mathcal{A}$, is a pair $(\mathcal{N}, D)$ where $\mathcal{N}$ is a semifinite factor with fixed trace, τ (on a separable Hilbert space), $\mathcal{A}$ is unitally $*$-represented in $\mathcal{N}$, and D is an unbounded self-adjoint operator affiliated with $\mathcal{N}$ satisfying

1. $(1+D^2)^{-1}$ is in the "compact" ideal $\mathcal{K}_N$ (respectively, $\mathrm{Tr}\left((1 + D^2)^{-\frac{p}{2}}\right) < \infty$) and
2. $\mathcal{A}_D := \{a \in A \mid [D, a]$ is bounded$\}$ is a dense $*$-subalgebra of $\mathcal{A}$.

If u is a unitary in the subalgebra $\mathcal{A}_D$, then

$$uDu^* = D + u[D, u^*] = D + B \quad \text{where} \quad B \in N_{sa}. \tag{2.8}$$

The path

$$D_t^u := (1 - t)\, D + tuDu^* = D + tB \in D + \mathcal{N}_{sa} \tag{2.9}$$

is a continuous path (in the obvious sense) of unbounded self-adjoint "Breuer-Fredholm" operators. That is, we show that

$$F_t^u := D_t^u \left(1 + (D_t^u)^2\right)^{-\frac{1}{2}} \tag{2.10}$$

is a continuous path of self-adjoint Breuer-Fredholm operators in $\mathcal{N}$, [B1;B2]. We denote by $sf\{D_t^u\} = sf\{F_t\}$ the *spectral flow* of this path.

Here, we again borrow Getzler's idea of considering the operator B defined above as a parameter in the real Banach manifold, N_{sa}, so that spectral flow can be obtained as the integral of a one-form on this manifold. That is, our manifold is $M := D + N_{sa}$ and the tangent space to M at D_1 is $T_{D_1}(M) = N_{sa}$. So, $X \in T_{D_1}(M)$ is the derivative at D_1 along the curve $t \mapsto D_1 + tX$ in M. It is easy to see that for any $m \geq \frac{p}{2}$

$$\alpha(X) = \frac{1}{\tilde{C}_m} \mathrm{Tr}\left(X(1 + D_1^2)^{-m}\right) \tag{2.11}$$

is a 1-form on M. In fact, we show that for m a sufficiently large half-integer, this one-form is *exact* and that for any piecewise-C^1 continuous path $\{D_t\}$

in M with D_0 and D_1 unitarily equivalent (*eg.* $\{D_t^u\}$),

$$sf\{D_t\} = \frac{1}{\tilde{C}_m} \int_0^1 \operatorname{Tr}\left(\frac{d}{dt}(D_t)(1+D_t^2)^{-m}\right) dt \tag{2.12}$$

which is the integral of the 1-form, α.

If D_0 and D_1 are not unitarily equivalent, we must add a pair of correction terms to the right hand side of the equation. These correction terms take account of the spectral asymmetry of the end-points. They are related to the regularised eta invariant introduced in [G]. We will not discuss this refinement here. Note that the normalizing constant is

$$\tilde{C}_m = \int_{-\infty}^{\infty} (1+x^2)^{-m}\, dx = \frac{\Gamma\left(m-\frac{1}{2}\right)\sqrt{\pi}}{\Gamma(m)} = \frac{n!\; 2^{n+1}}{1\cdot 3\cdots(2n+1)} = C_n \tag{2.13}$$

where $m = n + \frac{3}{2}$ and n is an integer.

If (N, F_0) is a bounded p-summable Breuer-Fredholm module, $u \in U(\mathcal{A})$ and

$$F_t^u = (1-t)\, F_0 + t\, uF_0u^*,$$

then for any positive integer $n \geq \frac{p}{2}$ we get

$$\alpha(X) = \frac{1}{C_n} \operatorname{Tr}\left(X(1-F^2)^n\right) \tag{2.14}$$

is an exact 1-form on a suitable manifold and

$$sf\{F_t^u\} = \frac{1}{C_n} \int_0^1 \operatorname{Tr}\left(\frac{d}{dt}(F_t^u)\left(1-(F_t^u)^2\right)^n\right) dt. \tag{2.15}$$

Part of this was already done in [P2] where spectral flow for arbitrary continuous paths of Breuer-Fredholm operators in a II_∞ factor was first defined. In the next section we will explain the connection between [P2] and the formulae introduced above.

There is also the unbounded finitely summable version obtained by reducing to the bounded case *via* the map

$$D \mapsto F = D(1+D^2)^{-\frac{1}{2}}. \tag{2.16}$$

Many technical analytic difficulties arise in implementing this simple idea. A number of these problems involve the operator-norm continuity and trace-norm continuity of functions of unbounded self-adjoint operators. In the final section we will mention one of these continuity results and develop some consequences of it.

Example 2.1. Let M be a compact spin manifold of dimension n. Let $\widehat{M}$ be the universal cover and assume the fundamental group of M has a type II regular representation which is a factor. Choosing metrics as in the introduction we find that the Dirac operator D_0 on the Hilbert space of L^2 sections of the spinor bundle over $\widehat{M}$ satisfies $(1+D_0^2)^{-p} \in \mathcal{T}_{\mathcal{N}}$ for any $p > n/2$ [Se]. Thus we have a p-summable Fredholm module to which the considerations of this section will apply.

3. Strategy

To our knowledge it was Mathai [M0] who first suggested using the Breuer-Fredholm index to define spectral flow and provided examples where it is not trivial. There is a special case of our overall setting in which this is also possible and to which we can (eventually) reduce the theorems mentioned in Section 1. This special case was discovered in [P2].

Suppose B_0 and B_1 are self adjoint in $\mathcal{N}$ with square one and all eigenspaces of infinite von Neumann dimension. Suppose that

$$\tau(|B_0 - B_1|^p) < \infty. \tag{3.1}$$

Then (3.1) implies that PQ is a Breuer-Fredholm operator where $P = (B_0 + 1)/2$ and $Q = (B_1 + 1)/2$. The spectral flow along the path

$$B_t = (1-t)B_0 + tB_1 \tag{3.2}$$

can be seen to be equal to the index $i(PQ)$. This Breuer-Fredholm index is computed by regarding PQ as a map from the range of Q to the range of P. The index here is measuring the amount of spectrum gained minus the amount of spectrum lost as we move along the path.

For more general paths $\{B_t\}$ in the space of self-adjoint Breuer-Fredholm operators in $\mathcal{N}$, one has to break the path into finitely many pieces along which $\chi(Bt)$ (χ is the characteristic function of $\mathbb{R}^+$) varies little modulo "compacts" so that at the endpoints B_{t_j} and $B_{t_{j+1}}$ the projections $P_j = \chi(B_{t_j})$ and $Q_j = \chi(B_{t_{j+1}})$ are close modulo $\mathcal{K}_{\mathcal{N}}$ forcing P_jQ_j to be Breuer-Fredholm. Then by adding the contributions $i(P_jQ_j)$ one obtains the spectral flow of the path.

Theorem 3.1 of [P2] gives the spectral flow along the path (3.2) for any integer $n \geq p/2$ as

$$\mathrm{sf}(\{B_t\}) = \frac{1}{C_n}\int_0^1 \tau\left(\frac{d}{dt}(B_t)(1-B_t^2)^n\right)dt \tag{3.3}$$

¿From this (2.15) follows by connecting $F_j, j = 0, 1$ by a linear path to the isometry B_j in its polar decomposition, joining B_0 and B_1 by a path as in (3.2) and showing that the new path which joins F_j via these isometries gives the same number as (2.15).

4. The θ-summable case and asymptotics

It is not difficult to see that p-summable (Breuer-)Fredholm modules are θ-summable. Thus one might ask about the connection between our formulas for spectral flow in the former case and those in [G] in the latter case. The connection is provided by the following calculation.

The formula in [G] for spectral flow in the θ-summable case when the endpoints are unitarily equivalent is:

$$\mathrm{sf}(D_0, D_1) = \frac{1}{\sqrt{\pi}}\int_0^1 \tau\left(\frac{d}{du}(D_u)\exp(-\epsilon D_u^2)\right)du. \tag{4.1}$$

Observe that

$$\text{sf}(D_0, D_1) = \frac{1}{\Gamma(n-1/2)} \int_0^\infty \epsilon^{n-3/2} e^{-\epsilon} \text{sf}(D_0, D_1) d\epsilon.$$

$$= \frac{1}{\Gamma(n-1/2)\sqrt{\pi}} \int_0^\infty \epsilon^{n-3/2} e^{-\epsilon} \int_0^1 \tau(\frac{d}{du}(D_u) \exp(-\epsilon D_u^2)) du d\epsilon \tag{4.2}$$

Now using the formula

$$\int_0^\infty s^n \exp(-(1+D^2)s) ds = \Gamma(n+1)(1+D^2)^{-n-1} \tag{4.3}$$

(4.2) becomes

$$\frac{\Gamma(n)}{\Gamma(n-1/2)\sqrt{\pi}} \int_0^1 \tau(\frac{d}{du}(D_u)(1+D_u^2)^{-n}) du. \tag{4.4}$$

This gives our finitely summable formula:

$$\text{sf}_n(D_0, D_1) = \frac{\Gamma(n)}{\Gamma(n-1/2)\sqrt{\pi}} \int_0^1 \tau(\frac{d}{du}(D_u)(1+D_u^2)^{-n}) du. \tag{4.5}$$

One might ask why we did not use this strategy to prove our formula. The answer is that the formula in [G] depends for its proof on the geoemtric definition of spectral flow as an intersection number and on some analytic details which fail in the type II setting.

We aim eventually to provide a proof of the θ summable spectral flow formula in both the type I and type II settings which is entirely independent of the geometric definition (and uses only index theory). However we do not anticipate that our proof will avoid some of the technical questions which complicate the type II case. To explain this assertion and to highlight some of the analytic results in [CP1] which contribute to a resolution of these complications is our next task.

One problem which we immediately confront in the θ-summable case is the convergence of integrals of the form

$$\int_1^\infty t^s \tau(X \exp(-tD^2)) dt \tag{4.6}$$

and

$$\int_1^\infty t^{-1/2} \tau(D \exp(-tD^2)) dt. \tag{4.7}$$

where X is a bounded self adjoint operator in $\mathcal{N}$. Mathai showed [M0] that if D has trivial kernel (4.7) converges without any special assumptions on the large time behaviour of $\tau(\exp(-tD^2))$ however the same cannot be said for (4.6) which arises in the θ-summable case when we try to show that the spectral flow is the integral of a one form. Any assumption which guarantees that (4.6) converges must be stable under perturbations of the form $D' = D + A$ where A is bounded self adjoint with D and D' having zero kernel. However it is easy to show by example that this cannot hold. Thus in the type II case a new strategy is required. (Note that the decay of $\tau(\exp(-tD^2))$ for large t is the object of study in [ES] and [GS] and leads to the definition of the Novikov-Shubin invariants.)

There are many other formidable technical issues as well but a discussion would take us too far afield. The interested reader is referred to the forthcoming preprint [CP1] for the details of proof of the results announced here. Work on the θ summable case is on-going.

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[1] Department of Pure Mathematics University of Adelaide, Adelaide, Australia, [2] Department of Mathematics University of Victoria, Victoria, Canada