Asymptotic Solutions Of Scientific Interest

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1 Introduction

The exact mathematical solution of many scientific problems is not feasible, and even when an exact solution can be found it may be difficult to interpret or inconvenient for numerical evaluation. An asymptotic solution may provide the insight we need however, and often enhances or complements any numerical computation that we might make. Thus the solution f sought may depend on some small parameter ϵ say, such that

$$f(\epsilon) \sim g_0(\epsilon), \qquad \epsilon \to 0$$

(equivalent to $\lim_{\epsilon\to 0} |f(\epsilon)/g_0(\epsilon)| = 1$) defines the solution behaviour by reference to a known function g_0. We say that "f is asymptotic to g_0" or "g_0 is an asymptotic approximation for f". Sometimes we can be more precise, and write f as an "asymptotic expansion"

$$f(\epsilon) \sim g_0(\epsilon) + g_1(\epsilon) + g_2(\epsilon) + \dots, \qquad \epsilon \to 0$$

where $\{g_i(\epsilon)\}$ is a sequence of known functions such that $\lim_{\epsilon\to 0} |g_{j+1}(\epsilon)/g_j(\epsilon)| = 0 \; \forall j$. However, the function f usually depends on another scalar or vector variable x (say) independent of ϵ, so an asymptotic expansion may be invalid in certain regions – i.e. not *uniformly valid* (for all x). Such singular behaviour often arises when the domain of x is unbounded, or when the order or type of differential equation in f changes at the limit $\epsilon = 0$. There are various mathematical techniques to cope with singular behaviour, which can be an important feature to recognise and interpret in the mathematical sciences.

Another useful area of asymptotic analysis allows us to approximate analytically difficult integrals. Over 200 years ago Laplace evaluated the integral

$$F(\lambda) = \int_\alpha^\beta \exp[-\lambda f(t)] g(t) dt \qquad (\lambda > 0)$$

in the large parameter limit ($\lambda \to \infty$), when the known functions f and g are assumed sufficiently smooth. Thus if f has an absolute minimum at $t_0 \in (\alpha, \beta)$, we have

$$F(\lambda) \sim g(t_0) \exp[-\lambda f(t_0)] \sqrt{\frac{2\pi}{\lambda f''(t_0)}}, \qquad \lambda \to \infty. \tag{1}$$

When λ is purely imaginary, the analogous "method of stationary phase" produces a similar result. The "method of steepest descent" or "saddle point method", referring to a maximum or saddle point in the integrand, represents a generalisation to the complex plane originally due to Riemann. The essential idea is to deform the path of integration to avoid rapid oscillations of the integrand in the large parameter limit.

2 Viscous Flow

One of the earliest applications of asymptotic analysis arose in hydrodynamics. The velocity field for incompressible flow satisfies $\nabla \cdot \mathbf{v} = 0$, so in a spherical coordinate system Stokes (1843) introduced a "stream function" $\psi(r, \theta)$ such that the related velocity components are $v_r = \psi_\theta/(r^2 \sin\theta)$ and $v_\theta = -\psi_r/(r \sin\theta)$, for axisymmetric incompressible flow (where the motion is the same in any plane through a common line). The corresponding Navier-Stokes equation for steady viscous flow is

$$D^4\psi = \frac{R}{r^2 \sin\theta}(\psi_\theta \frac{\partial}{\partial r} - \psi_r \frac{\partial}{\partial \theta} + 2\cot\theta \psi_r - 2\frac{\psi_\theta}{r})D^2\psi, \tag{2}$$

where R denotes the dimensionless Reynolds number and the differential operator

$$D^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial \theta}(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}).$$

The corresponding boundary conditions to be satisfied for flow past a sphere of unit radius are $\psi(1, \theta) = \psi_r(1, \theta) = 0$ and $\psi(r, \theta) \to \frac{1}{2}r^2 \sin^2\theta$ as $r \to \infty$.

A straightforward perturbation expansion for "slow" flow (i.e. in the limit of *small* Reynolds number $R \to 0$) is of form $\psi \sim \psi_0 + R\psi_1 + \ldots$. Thus on substituting into equation (2), at zeroth order we obtain the solution originally found by Stokes (1843), which is uniformly valid; but when first-order terms (of order R) are included, the result

$$\psi \sim \frac{1}{4}(2r^2 - 3r + \frac{1}{r})\sin^2\theta + R[-\frac{3}{32}(2r^2 - 3r + 1 - \frac{1}{r} + \frac{1}{r^2})\sin^2\theta \cos\theta + \psi_{1c}] \tag{3}$$

is not valid at large radial distances ($r > R^{-1}$) from the sphere. In particular, the boundary condition as $r \to \infty$ is not satisfied (Whitehead 1889).

We may derive another asymptotic solution valid for large distances by first scaling the independent variable. Thus on introducing $\rho = Rr$, the differential equation becomes

$$D^4\psi = \frac{R^2}{\rho^2 \sin\theta}(\psi_\theta \frac{\partial}{\partial \rho} - \psi_\rho \frac{\partial}{\partial \theta} + 2\cot\theta \psi_\rho - 2\frac{\psi_\theta}{\rho})D^2\psi, \tag{4}$$

where

$$D^2 = \frac{\partial^2}{\partial \rho^2} + \frac{\sin\theta}{\rho^2}\frac{\partial}{\partial \theta}(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}).$$

The uniform validity of the earlier zeroth order approximation (Stokes' solution) allows us to infer that

$$\psi \sim \frac{1}{2}\frac{\rho^2}{R^2}\sin^2\theta + \frac{1}{R}g_1(\rho, \theta) + \ldots$$

as $R \to 0$, which on substituting into (4) produces "Oseen's equation"

$$(D^2 - \cos\theta \frac{\partial}{\partial \rho} + \frac{\sin\theta}{\rho}\frac{\partial}{\partial\theta}) D^2 g_1 = 0, \tag{5}$$

which Oseen (1910) derived using physical arguments. Following Goldstein (1929), in equation (5) we may set $D^2 g_1 = \phi \exp(\frac{1}{2}\rho\cos\theta)$ so that $(D^2 - \frac{1}{4})\phi = 0$, to obtain (as $R \to 0$)

$$\psi \sim \frac{1}{2R^2}\rho^2 \sin^2\theta - \frac{3}{2R}(1 + \cos\theta)\{1 - \exp[-\frac{1}{2}\rho(1 - \cos\theta)]\}.$$

Rewriting this result in terms of r and expanding the exponential, we have

$$\psi \sim \frac{1}{2}r^2 \sin^2\theta - \frac{3}{4}r\sin^2\theta + \frac{3}{16}r^2 R \sin^2\theta(1 - \cos\theta) + \ldots \tag{6}$$

Thus the "inner" asymptotic solution (3) and the "outer" asymptotic solution (6) match for intermediate values $1 \ll r \ll R^{-1}$ provided $\psi_{1c} \sim \frac{3}{16}r^2 \sin^2\theta$, so the uniformly valid first-order expansion replacing them is

$$\psi \sim \frac{1}{4}(2r^2 - 3r + \frac{1}{r})\sin^2\theta + \frac{3}{32}R[(2r^2 - 3r + \frac{1}{r})\sin^2\theta - (2r^2 - 3r + 1 - \frac{1}{r} + \frac{1}{r^2})\sin^2\theta\cos\theta].$$

Perhaps the best known case where a governing differential equation reduces to an equation of lower order is the solution of the Navier-Stokes equation for steady flow past an obstacle in the limit of *large* Reynolds number ($R \to \infty$). In the context of otherwise uniform plane flow, we use a stream function $\psi(x, y)$ in Cartesian coordinates to get

$$(\psi_y \frac{\partial}{\partial x} - \psi_x \frac{\partial}{\partial y} - \frac{1}{R}\nabla^2)\nabla^2\psi = 0, \tag{7}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. This form of the Navier-Stokes equation is to be solved subject to $\psi(x, F(x)) = 0$ and $F'(x)\psi_y(x, F(x)) + \psi_x(x, F(x)) = 0$ at the surface $y = F(x)$ of any obstacle. In this case, an asymptotic expansion of form $\psi \sim \psi_0 + R^{-1}\psi_1 + \ldots$. ($R \to \infty$) produces a third-order (rather than a fourth-order) differential equation to zeroth order: namely the equation

$$(\psi_{0y}\frac{\partial}{\partial x} - \psi_{0x}\frac{\partial}{\partial y})\nabla^2\psi_0 = 0,$$

which describes idealised *inviscid* flow. In the neighbourhood of the obstacle (i.e. in the inner region often called the "boundary layer") where the term $R^{-1}\nabla^4\psi$ is not small, Prandtl (1905) essentially adopted the original equation (7) to avoid this reduction of order, so that the viscous no-slip boundary condition $\psi(x, F(x)) = 0$ may be satisfied.

3 Seepage Flow to a Drain

Although the traditional rice paddy in Asia exploits the natural wet conditions well, drainage methods may be important for both urban and rural land. In particular, rows of buried drainage pipes may control the level of the local water table, or even drain ponded water. We can demonstrate the "method of matched asymptotic expansions" in this context.

Let us consider a horizontal row of hollow circular cylindrical pipes made of some non-porous material, buried in the soil and equally spaced. Water seeps through the soil to small gaps between the pipes and hence drains away. We assume the soil is homogeneous so that Darcy's law is valid, and there exists an harmonic potential ϕ for the seepage velocity (see for example, Muskat 1937). Although the water table (or the soil surface) tends to be almost horizontal, if it is taken to be a plane the governing Laplace equation does not separate, so we avoid this difficulty by assuming it to be a circular cylinder of radius b coaxial with the drain of radius a. We shall see that the outer geometry is unimportant, for the flow is largely determined by the gaps. Thus in cylindrical polar coordinates, we have an axisymmetric boundary value problem in dimensionless independent variables:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial \phi}{\partial r}) + \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad (\lambda_1 < r < \lambda_2, \quad |z| < 1)$$

subject to $\phi(\lambda_1, z) = V_o$ for $|z| < \epsilon$ but $\phi_r(\lambda_1, z) = 0$ for $\epsilon < |z| < 1$, $\phi(\lambda_2, z) = 0$, and $\phi_2(r, \pm 1) = 0$. If h denotes a characteristic horizontal length, the geometry is defined by $\lambda_1 = a/h$, $\lambda_2 = b/h$, and the dimensionless small gap width is $z = d/h = \epsilon \ll 1$.

The solution ϕ must be an even function of z, and it may be written (Sneyd & Hosking 1976):

$$\phi(r, z) = \int_{-\epsilon}^{\epsilon} K(r, z, \alpha) f(\alpha) d\alpha, \tag{8}$$

where the kernel function is

$$K(r, z, \alpha) = \frac{1}{2}\lambda_1 \ln(\lambda_2/r) + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{T_n(r)}{n} \cos(n\pi z)\cos(n\pi\alpha) \tag{9}$$

with

$$T_n(r) = [K_0(n\pi r)I_0(n\pi\lambda_2) - I_0(n\pi r)K_0(n\pi\lambda_2)]/[K_1(n\pi\lambda_1)I_0(n\pi\lambda_2) - I_1(n\pi\lambda_1)K_0(n\pi\lambda_2)]$$

an appropriate combination of modified Bessel functions. Note that the range of integration is restricted to across the gap where the flux function $f(z) = -\phi_r(\lambda_1, z)$ is non-zero, and this function is defined by the associated Fredholm integral equation

$$\int_{-\epsilon}^{\epsilon} K(\lambda_1, z, \alpha) f(\alpha) d\alpha = V_0 \qquad (|z| < \epsilon).$$

We are interested in how the ratio of the flux through the drain to the applied velocity potential V_0 depends upon the geometry, so we may "normalise" the flux function such that $\int_{-\epsilon}^{\epsilon} f(\alpha)d\alpha = 1$. An asymptotic solution for $\epsilon \ll 1$ can now be found by the method of matched asymptotic expansions.

On introducing the scaled variable $\xi = \alpha/\epsilon$ and writing $F(\xi) = \epsilon f(\epsilon\xi)$, from equation (8) we have

$$\phi(r, z) = \int_{-1}^{1} K(r, z, \epsilon\xi) F(\xi) d\xi,$$

so expanding as a power series in ϵ we formally obtain the outer expansion valid far away from the gap: namely

$$\phi_{outer} \sim K(r, z, 0) + O(\epsilon^2),$$

since $\int_{-1}^{1} F(\xi)d\xi = \int_{-\epsilon}^{\epsilon} f(\alpha)d\alpha = 1$ and $\int_{-1}^{1} \xi F(\xi)d\xi = 0$. The infinite series (9) for $K(r,z,0)$ diverges at the point $r = \lambda_1$, $z = 0$ in the centre of the gap between the pipes. For $n \gg 1$ we have

$$T_n(r) \sim \frac{K_0(n\pi r)}{K_1(n\pi\lambda_1)} \sim \frac{\lambda_1}{r} e^{-n\pi(r-\lambda_1)}[1 - \frac{1}{8n\pi}(\frac{1}{r} + \frac{3}{\lambda_1}) + O(n^{-2})],$$

on invoking well known asymptotic expansions of the modified Bessel functions for large arguments. Thus if we write

$$R_1(r) = T_1(r) - \sqrt{\frac{\lambda_1}{r}} \exp[-\pi(r - \lambda_1)]$$

and

$$R_n(r) = T_n(r) - \sqrt{\frac{\lambda_1}{r}} e^{-n\pi(r-\lambda_1)}[1 - \frac{1}{8(n-1)\pi}(\frac{1}{r} + \frac{3}{\lambda_1})] \qquad for \quad n \geq 2,$$

we have an explicit expression for the outer expansion: namely

$$\phi_{outer} \sim \frac{1}{2}\lambda_1 \ln(\lambda_2/r) + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} R_n(r)\cos(n\pi z) - \frac{1}{\pi}\sqrt{\frac{\lambda_1}{r}} \ln|1 - x|$$

$$-\frac{1}{8\pi^2}\sqrt{\frac{\lambda_1}{r}}(\frac{1}{r} + \frac{3}{\lambda_1}) Re\{x + (1-x)\ln(1-x)\} \tag{10}$$

where $x = \exp[\pi(iz + \lambda_1 - r)]$, on using the formulas $\sum_{n=1}^{\infty} x^n/n = -\ln(1-x)$ and $\sum_{n=2}^{\infty} x^n/(n(n-1)) = x + (1-x)\ln(1-x)$. The singular behaviour near the gap (as $r \to \lambda_1$ and $z \to 0$) is confined to the last two terms in this expression. To obtain the inner expansion valid in the region near the gap, we introduce the scaled independent variables $X = (r - \lambda_1)/\epsilon$, $Z = z/\epsilon$ to rewrite the fundamental Laplace equation

$$\frac{\partial^2\phi}{\partial X^2} + \frac{\epsilon}{\lambda_1 + \epsilon X}\frac{\partial\phi}{\partial X} + \frac{\partial^2\phi}{\partial Z^2} = 0.$$

Thus assuming $\phi = \phi_0 + \epsilon\phi_1 + \ldots$, we have to solve (to zeroth order)

$$\frac{\partial^2\phi_0}{\partial X^2} + \frac{\partial^2\phi_0}{\partial Z^2} = 0$$

subject to the inner boundary conditions $\phi_0(0,Z) = V_0$ for $|Z| < 1$ and $\phi_{0X}(0,Z) = 0$ for $|Z| > 1$. (The outer boundary conditions on the surfaces $r = \lambda_2$ and $z = 1$ are typically ignored in the construction of the inner expansion.) It is convenient to use elliptic cylindrical coordinates (u,v) defined by $X = \sinh u \sin v$, $Z = \cosh u \cos v$ to obtain the inner expansion

$$\phi_{inner} \sim V_0 - \pi^{-1}u + O(\epsilon). \tag{11}$$

We now match over intermediate distances from the gap by expressing the outer expansion (10) in terms of the inner variables X and Z for comparison with the inner expansion (11) rewritten in terms of the outer variables r and z (this is sometimes called a "matching principle"). Thus $x = \exp[\pi\epsilon(iZ - X)] = 1 + \pi\epsilon(iZ - X) + O(\epsilon^2)$, so near the gap we find (noting two terms in $\epsilon \ln \epsilon$ cancel)

$$\phi_{outer} \sim \frac{1}{2}\lambda_1 \ln(\lambda_2/\lambda_1) + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}R_n(\lambda_1) - \frac{1}{\pi}\ln(\pi\epsilon\sqrt{X^2+Z^2}) - \frac{1}{2\pi^2\lambda_1} + O(\epsilon). \qquad (10')$$

For the inner expansion we have $(r - \lambda_1)/\epsilon \sim e^u \sin v$ and $z/\epsilon \sim e^u \cos v$, so that in terms of the outer variables

$$\phi_{inner} \sim V_0 - \frac{1}{\pi}\ln\left(\frac{2}{\epsilon}\sqrt{(r-\lambda_1)^2 + z^2}\right) + O(\epsilon). \qquad (11')$$

The inner and outer expansions therefore match over intermediate distances provided we identify

$$V_0 = \frac{1}{2}\lambda_1 \ln(\lambda_2/\lambda_1) + \frac{1}{\pi}\ln(\frac{2}{\pi\epsilon}) - \frac{1}{2\pi^2\lambda_1} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}R_n(\lambda_1).$$

The main interest in this problem is the so-called relative flow (Kirkham 1950) – i.e.

$$R = \frac{\lambda_1 \log(\lambda_2/\lambda_1)}{2V_0},$$

the ratio of the calculated flow to that for a corresponding completely open drain (when $\epsilon = 1$). Thus we have (for $\epsilon \ll 1$)

$$R = \{1 + \frac{2}{\pi\lambda_1 \ln(\lambda_2/\lambda_1)}[\ln(\frac{2}{\pi\epsilon}) - \frac{1}{2\pi\lambda_1} + \sum_{n=1}^{\infty}\frac{1}{n}R_n(\lambda_1)]\}^{-1},$$

which proves to be a remarkably accurate formula (Sneyd & Hosking 1976). In the limit $\epsilon \to 0$ we have $R \to \pi\lambda_1 \ln(\lambda_2/\lambda_1)/[2\ln(2/\pi\epsilon)]$, when the so-called "total flow rate"

$$Q_\epsilon = \frac{2\pi^2 a V_0}{\ln(2/\pi\epsilon)} = \frac{\pi V_0}{\ln(2/\pi\epsilon)} \times (gap\ circumference\ = 2\pi a)$$

is governed only by the gap parameter ϵ for a given applied potential V_0.

4 Non-Ideal Plasma Instabilities

The instability of magnetic confinement schemes has been a major problem in controlled thermonuclear fusion research for over thirty years. There are laboratory configurations designed to avoid ideal magnetohydrodynamic (MHD) instability, but residual non-ideal modes can be quite destructive. Their important feature is the relaxation of the ideal constraint that any plasma fluid element must move with the magnetic field, so consequently the system may shed additional potential energy. From the mathematician's viewpoint, the ideal theory is again a singular limit corresponding to reduction of order. The interesting seminal paper by Furth, Killeen & Rosenbluth (1963) identified three types of resistive instabilities ("resistive-g", "tearing" and "rippling" modes), associated with relaxation of the ideal constraint in thin boundary layers due to small but non-zero resistivity. The present author's first research paper showed that inclusion of the so-called Hall effect similarly introduces new instabilities, corresponding to a differential equation of higher order than in the ideal MHD limit (Hosking 1965). There is now a very extensive literative that treats much more complex geometric configurations and other non-ideal plasma properties such as magneto-viscosity, but there is no space to discuss these intriguing matters further here.

5 Moving Loads over Continuously–Supported Plates

The asymptotic evaluation of integrals has often arisen in the mathematical theory for the response of a continuously-supported beam or plate due to a moving load. The beam or plate may represent a rail or road surface, an airport runway or a floating ice sheet in a cold region. The moving load might be a conventional vehicle, a landing aeroplane or a hovercraft. Moving loads on ice plates is the subject of a recent monograph by Squire, Hosking, Kerr & Langhorne (1996), where it is emphasised that the deflexion can be much greater when the load is moving than when it is stationary. Hovercraft exploiting this phenomenon are used as ice breakers on the St. Lawrence Seaway in Canada, but the safe operation of transport systems of course usually means avoiding any fracture.

The accepted mathematical model involves a thin plate equation for the vertical deflexion $\eta(x, y, t)$: namely

$$D\nabla^4\eta + \rho h\frac{\partial^2\eta}{\partial t^2} = p - f(x, y, t)$$

or its visco-elastic counterpart (Squire *et al.* 1996). Here $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ as before, D is the effective flexural rigidity of the plate, ρ and h are the plate density and thickness, p is the pressure exerted by the underlying foundation on the plate, and $f(x, y, t)$ is the downward forcing function on the plate due to the moving load. For example, in the context of a floating ice sheet when the underlying foundation is water of density ρ_w, from Bernoulli's equation for incompressible irrotational flow we have

$$p = -\rho_w(\frac{\partial\phi}{\partial t})_{z=0} - \rho g\eta,$$

where ϕ denotes the velocity potentical and g the gravitational acceleration.

Using a Fourier transform

$$\eta(k_1, k_2, \omega) = (2\pi)^{-\frac{3}{2}}\int\eta(x, y, t)e^{i(k_1x+k_2y-\omega t)}dxdydt,$$

we immediately obtain the deflexion due to a load moving in the x-direction with uniform speed V: namely

$$\eta(X, y) = -\frac{1}{2\pi}\int\frac{\hat{F}(k_1, k_2)}{B(k_1, k_2)}e^{-i[k_1X+k_2y]}dk_1dk_2, \qquad (12)$$

where $\hat{F}(k_1, k_2)$ is the Fourier transform of the loading function $f(x, y, t) = F(x - Vt, y) = F(X, y)$, and $B(k_1, k_2) = Dk^4 + \rho g - \rho h'V^2{k_1}^2 - (\rho V^2{k_1}^2/k)\coth(kH)$ with wavenumber $k = \sqrt{k_1^2 + k_2^2}$ and $h' = \rho h/\rho_w$. Note that $B(k_1, k_2) = 0$ is just the dispersion relation with ω replaced by Vk_1 as required for a steady wave pattern, since the component of the load velocity normal to any wave crest (Vk_1/k) must equal the crest phase speed (ω/k).

Generally, one would not expect to evaluate the integral in equation (12) exactly, but since it is of considerable interest to predict the deflexion some distance away from a localised (point) load we can exploit asymptotic methods. After first integrating with respect to k_1 by contour integration, we seek asymptotic approximations for the integral

$$\eta(X,y) = i\int_{C_k} \frac{\hat{F}(k_1,k_2)}{B_{k_1}(k_1,k_2)} e^{-i(k_1X+k_2y)} dk_2, \tag{13}$$

arising due to the pole at $k_1(k_2)$ associated with the point on the *wavenumber curve* C_k defined in the real (k_1,k_2)-plane by $B(k_1,k_2)=0$ (see Lighthill 1978).

For most directions away from the load, we deform the path of integration in (13) to pass through saddle points in the complex plane, where we have truncated Taylor representations of form $k_1(k_2) = k_0 + k_1''(k_{20})(k-k_{20})^2/2$ and get stationary phase approximations analogous to the Laplace integral asymptotic approximation (1): namely

$$\eta(X,y) \sim -F(k_{10},k_{20})[(\partial B/\partial n)_0]^{-1}\sqrt{\frac{2\pi}{\kappa_0 r}}e^{-i(k_1X+k_2y+\Theta)},$$

where $r=\sqrt{X^2+y^2}\to\infty$, $\kappa_0\equiv|k_1''(k_{20})|$ denotes the small curvature on C_k at $P_0(k_{10},k_{20})$, $\partial/\partial n$ denotes differentiation normal to C_k in the sense of increasing ω, and Θ is a phase factor equalling $\pi/4$ if C_k is convex to the n-direction at P_0 and π otherwise (Davys, Hosking & Sneyd 1985). There are variations in the neighbourhood of a caustic associated with a point of inflexion on C_k (where $k_1'(k_2)=k_1''(k_2)=0$), and in the neighbourhood of a "supercaustic" corresponding to when two caustics just merge (where $k_1'(k_2)=k_1''(k_2)=k_1'''(k_2)=0$). All of this enables us to ascertain preferred directions of energy propagation corresponding to the caustics (or supercaustics), and to produce wave patterns that are notably dependent on the load speed (Davys *et al.* 1985; see also Squire *et al.* 1996).

More recently, research attention has shifted to the time-dependent response due to an impulsive-started load, to further examine the marked resonant response at so-called critical load speed. In this case, we consider the Fourier double integral (Nugroho, private communication 1996):

$$\eta(X,y,t) = -\frac{P_0}{8\pi^2\rho}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\tanh(kH)}{c(k)}e^{-i(k_1X+k_2z)}\{\frac{1-e^{i\Psi_1 t}}{\Psi_1}+\frac{1-e^{-i\Psi_2 t}}{\Psi_2}\}dk_1dk_2 \tag{14}$$

where $\Psi_1 = kc - k_1V$ and $\Psi_2 = kc + k_1V$ ($k=\sqrt{k_1^2+k_2^2}$), the coordinates ($X = x - Vt, y$) are again centred at the point load that instantaneously moves with constant speed V, and $t>0$ denotes the time. We can also derive a corresponding integral for the time-dependent deflexion under the centre of a uniform circular load of radius R (Wang & Hosking 1996):

$$\eta(t) = -\frac{P_0}{\pi\rho R}\int_0^{\infty}\frac{\tanh(kH)J_1(kR)}{c(k)}\int_0^t \sin(kcs)J_0(kVs)dsdk. \tag{15}$$

The asymptotic evaluation of the integrals in equation (14) or (15) as $t\to\infty$ predicts that at all but one load speed a steady state is approached, more quickly than the earlier one-dimensional (line load) time-dependent theory suggested (Schulkes & Sneyd 1988, Squire *et al.* 1996); but that at the traditional critical load speed $V = c_{min}$ (the minimum wave phase speed) there is a resonant response, where the deflexion grows continuously as $\ln t$.

6 Summary

Asymptotic analysis can often provide valuable insight in the mathematical sciences. Singular mathematical behaviour, which often arises when a solution domain is infinite or when the order or type of a governing equation changes in some limit, is commonly associated with the mathematical description of important scientific phenomena. Thus when an asymptotic expansion is not uniformly valid, a more complete description may be constructed by judicious scaling, or by the matching of more than one asymptotic expansion. Another useful area of asymptotic analysis is the approximation of integrals, when some parameter is large or small. Several applications have been presented in this paper.

Recently, asymptotic expansions that include small transcendental terms have become the subject of considerable research, variously referred to as "asymptotics beyond all orders" or "superasymptotics" and "hyperasymptotics". These mathematical developments relate to the "Stokes phenomenon", where there is an abrupt change in the coefficients of otherwise exponentially small terms in compound asymptotic expansions, across certain rays in the complex plane (see for example, Paris and Wood 1995). This is an example of feedback into modern analysis from the work of an outstanding scientist, who (in the words of G H Hardy) "was primarily a mathematical physicist" but "also a most acute pure mathematician". And no doubt there will be further applications of this new asymptotics involving transcendental terms, and indeed other developments in areas such as the asymptotic evaluation of multiple integrals, in the splendid tradition of creative interaction between "pure" and "applied" mathematics.

[This paper is a modified and up-dated version of Report Number 9 of the then Department of Mathematics (now the Department of Mathematics & Statistics) at James Cook University of North Queensland, written by the author in June 1988.]

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