ON THE BLOW-UP BEHAVIOR OF SOLUTIONS OF SCALAR CURVATURE EQUATION AND ITS APPLICATION

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1. Introduction

In this expository article, I want to report the recent joint work with Chiun-Chuen Chen. Condider positive smooth solutions of the scalar curvature equation

$$\Delta u + K(x)u^{\frac{n+2}{n-2}} = 0 \quad \text{in } \Omega \subseteq \mathbf{R}^n, \tag{1}$$

where Δ is the Laplace operator, $K(x)$ is a positive $\mathbf{C}^1$ function and $n \geq 3$. *Throughout the paper, we always assume that $K(x)$ is bounded between two positive constants.* One of the motivations in studying equation (1) arises from the problem of prescribing scalar curvature in conformal geometry. Let (M, g_0) be a n-dimensional Riemannian manifold and $K(x)$ be a given smooth function on M, we would like to find a metric g conformal to g_0 such that K is the scalar curvature of g. Set $g = u^{\frac{4}{n-2}} g_0$ for some positive function u, then the problem above is equivalent to finding positive smooth solutions of

$$\frac{n-1}{4(n-2)}\Delta_0 u - k_0 u + K(x)u^{\frac{n+2}{n-2}} = 0 \text{ in } M, \tag{2}$$

where Δ_0 denotes the Beltrami-Laplace operator of (M, g_0) and $k_0(x)$ is the scalar curvature of g_0. When (M, g_0) is the n-dimensional Euchidean space $\mathbf{R}^n$, then we have $k_0 \equiv 0$ and equation (2) reduces to (1) after an appropriate scaling.

For the case $K(x) \equiv$ a positive constant, say $K(x) \equiv n(n-2)$, and $\Omega \equiv \mathbf{R}^n$, all slutions of equation (1) can be completely classified.

Theorem 1.1. (Caffarelli-Gidas-Spruck) *Any positive smooth solution u of*

$$\Delta u + n(n-2)u^{\frac{n+2}{n-2}} = 0 \quad \textit{in } \mathbf{R}^n$$

must satisfy

$$u(x) = (\frac{\lambda}{1+\lambda^2|x-x_0|^2})^{\frac{n-2}{2}}$$

for some $\lambda > 0$ *and* $x_0 \in \mathbf{R}^n$.

It is not difficult to see that

(i) The total energy, which is defined by

$$\begin{aligned}\int_{\mathbf{R}^n} |\nabla u|^2 &= n(n-2)\int_{\mathbf{R}^n} u^{\frac{2n}{n-2}}dx \\ &= [n(n-2)]^{1-\frac{n}{2}} S_n^{\frac{n}{2}},\end{aligned}$$

is independent of λ. Here S_n is the Sobolev best constant. And the energy is concentrated in a small neigh of x_0 (say $x_0 = 0$), i.e., for any $\delta > 0$

$$\int_{|x|\geq\delta} u^{\frac{2n}{n-2}}(x)dx = O(\lambda^{-\frac{n-2}{2}})$$

as $\lambda \longrightarrow +\infty$.

(ii) Denote $M = \max_{\mathbf{R}^n} u = \lambda^{\frac{n-2}{2}}$. Then

$$u(x) \leq M^{-1}|x|^{2-n},$$

i.e.,

$$\min_{|x|\geq\delta} u = O(M^{-1}).$$

(iii) Let $w(r) \equiv u(r)r^{\frac{n-2}{2}} = (\frac{\lambda r}{1+\lambda^2 r^2})^{\frac{n-2}{2}}$. Then $w(r)$ has a unique critical point in $r > 0$, i,e, the maximum point $r = \lambda^{-1}$. (the property (iii) was first observed by R. Schoen. It is an important notion concerning the below-up behavior.)

Obviously, the difficulty for studying equation (1) comes from the concentration phenomenon mentioned above. Of course, it is of great interest to study the blow-up behavior of solution of (1) when $K(x)$ is not a constant function. (or even $K(x) \equiv$ a constant, but solutions u is not defined in the whole space $\mathbf{R}^n$.) In the following sections, we will discuss the blow up behavior and see what is the property of K affecting the blow-up behavior of a sequence of solutions of (1). Before going into the next section, we would like to point out that a Harnack-type inequality holds for solutions of equation (1) with a constant $K(x)$.

Theorem 1.2. *There exists a constant $c > 0$ such that for any solution u of*

$$\Delta u + u^{\frac{n+2}{n-2}} = 0 \quad in\ |x| \le 2R,$$

the inequality,

$$(\max_{|x|\le R} u)(\min_{|x|\le 2R} u) \le \frac{c}{R^{n-2}}$$

holds.

Theorem 1.2 was proved in [CLn1], where a more geneal nonlinear term was considered.

2. Simple blow-ups

Let u_i be a sequence of solutions of equation (1). *A point x_0 is called a blow up point if there exists a sequence of x_i such that $x_0 = \lim_{i\longrightarrow+\infty} x_i$ and $\overline{\lim}_{i\longrightarrow+\infty} u_i(x_i) = +\infty$.* Following R. Schoen, *a blow-up point x_0 is called isolated if there exists a local maximum x_i of u_i such that*

$$u_i(x_i + x) \le c|x|^{-\frac{n-2}{2}} \quad for\ |x| \le \delta_0, \tag{3}$$

where both constants c and δ_0 are independent of i. Note that if x_0 is an isolated blow up point, then u_i is uniformly bounded in any compact set of $B_{\delta_0}(x_0)\backslash\{x_0\}$. Thus we let $M_i = \max_{|x-x_0|\le\delta_0} u_i(x) = u_i(x_i)$. Obviously, $x_i \longrightarrow x_0$ as $i \longrightarrow +\infty$. *The blow-up point x_0 is called simple* if

$$u_i(x_i + x) \le cM_i^{-1}|x|^{2-n}. \tag{4}$$

Another notion of the simple blow up is defined originally by R. Schoen in the following. (See [L1]). Let

$$w_i(r) = \bar{u}_i(r)r^{\frac{n-2}{2}}, \tag{5}$$

where $\bar{u}_i(r) = \fint_{|x|=r} u$ is the average of u over the sphere $|x| = r$ (for the simplicity of notations, we assume $x_0 = 0$). Then we have

Proposition 2.1. *Let u_i be a sequence of solutions of equation (1). Assume that 0 is an isolated blow-up point of u_i. Then 0 is a simple isolated blow-up point if and only if there exists $r_0 > 0$ such that $w_i(r)$ has a unique critical point in $(0, r_0)$.*

Proof. The sufficient part was proved by Y. Y. Li, [L1]. We will give a proof for this part which is different from the one in [L1]. For the proof of Proposition 2.1, we need the following lemma, which can be derived by integrating the differential inequality hold for w. For a proof of Lemma 2.2 below, we refer the reader to [CLn3].

Lemma 2.2. *Let $w(r)$ be defined as in (5) and $r = e^t$. (The index i is omitted for the simplicity.) Then*

(i) Suppose that w is nonincreasing in (t_0, t_1) and t_1 is a local minimum of w, then

$$\frac{2}{n-2}\log\frac{w(t_0)}{w(t_1)} \leq t_1 - t_0 \leq \frac{2}{n-2}\log\frac{w(t_0)}{w(t_1)} + C. \tag{6}$$

(ii) Suppose that w is nondecreasing in (t_1, t_2) and t_1 is a local minimum of w. Then

$$\frac{2}{n-2}\log\frac{w(t_2)}{w(t_1)} \leq t_2 - t_1 \leq \frac{2}{n-2}\log\frac{w(t_2)}{w(t_1)} + C, \tag{7}$$

where C are a constant depending on n onely.

Return now to the proof of Proposition 2.1.

First, we assume that 0 is a simple blow up point. Let $T_i < t_i$ denote the first local maximum point and the first local minimum point respectively. Suppose the conclusion of Proposition 2.1 does not hold, i.e., $\lim_{i\longrightarrow+\infty} t_i = -\infty$. By a simple argument of scaling, we have

$$T_i = -\frac{n-2}{2}\log M_i + O(1), \quad \text{and} \tag{8}$$

$$\lim_{i\longrightarrow+\infty} w_i(t_i) = 0. \tag{9}$$

By (9), we always can find $t_i^* > t_i$ such that $w_i(t)$ is increasing in $[t_i, t_i^*]$ and $t_i^* - t_i \longrightarrow +\infty$ as $i \longrightarrow +\infty$. By (6), (7) and (8), we have

$$\begin{aligned} \bar{u}_i(r_i^*) &\geq c_1\bar{u}_i(r_i) \geq c_2 M_i^{-1} r_i^{2-n} \\ &= c_2\left(\frac{r_i^*}{r_i}\right)^{n-2} M_i^{-1} r_i^{*2-n}, \end{aligned} \tag{10}$$

where $r_i^* = e^{t_i^*}$ and $r_i = e^{t_i}$. Since $\lim_{i \longrightarrow +\infty} \frac{r_i^*}{r_i} = +\infty$, applying the Harnack inequality, (10) yields a contradiction to (4).

The necessary part follows immediatly from the second inequality of (6) and (8).

Q.E.D.

To state our first result, we assume that for any critical point x_0 of K, there exists a neighborhood U of x_0 such that one of the following conditions is satisfied:

(K1) For $x \in U$, we have

$$c_1|x|^{\alpha-1} \leq |\bigtriangledown K(x)| \leq c_2|x|^{\alpha-1}$$

for some constant $\alpha \geq n-2$.

(K2) For $x \in U$ we have

$$|\bigtriangledown^k K(x)| \leq c|\bigtriangledown K(x)|^{\frac{\alpha-k}{\alpha-1}},$$

where $2 \leq k \leq \alpha = n-2$.

Theorem 2.2. *Assume that (i) $K \in C^1$ for $n = 3$, (ii) For $n \geq 4$, at any critical point of K, either (K1) or (K2) is satisfied. Suppose that u is a positive solution of*

$$\Delta u + K(x)u^{\frac{n+2}{n-2}} = 0 \quad in\ B_1. \tag{11}$$

Then for any $r \in (0, \frac{1}{2})$, we have

$$(\max_{B_r} u)(\min_{B_{2r}} u) \leq c\, r^{2-n}. \tag{12}$$

Furthemore, if u_i is a sequence of solutions of (12), then any blow up point is a simple blow up point.

When u is a global solution defined on S^n, then Theorem 2.2 was proved by Chang-Gursky-Yang for $n = 3$, Schoen-Zheng, for $n = 3, 4$ and Y.Y. Li for $n \geq 4$. In [CLn2], the authors proved Theorem 2.2 via the method of moving planes. For the details, we refer the reader to [CLn2]. An immediate consequence of Theorem 2.2 is that any blow up point must be a critical point of K.

3. Main Theroems

In this section, we always assume $K \in C^1(\bar{B}_1)$ and satisfies the following conditions:
(K3) For any $\varepsilon > 0$, there exists $c(\varepsilon) > 0$ such that $c(\varepsilon) \leq |\bigtriangledown K(x)| \leq c_1$ for $|x| \geq \varepsilon$ where c_1 is a positive constant independent of i and ε.

(K4) The origin is a critical point of K and $K(x) = K(0) + Q(x) + R(x)$ in a neighborhood of 0 where $Q(x)$ is a C^1 homogeneous function of order $\alpha > 1$ satisfying

$$c_1|x|^{\alpha-1} \leq |\bigtriangledown Q(x)| \leq c_2|x|^{\alpha-1},$$

and both $R(x)|x|^{-\alpha}$ and $|\bigtriangledown R(x)||x|^{1-\alpha}$ tend to zero as $|x| \longrightarrow 0$.

Let U_0 be the positive solution of

$$\Delta U_0 + K(0)U_0^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbf{R}^n. \tag{13}$$

Then, Q in (K4) satisfies

$$\text{[Q]} \qquad \begin{pmatrix} \int_{\mathbf{R}^n} \bigtriangledown Q(\xi + y)U_0^{\frac{2n}{n-2}}(y)dy \\ \int_{\mathbf{R}^n} Q(\xi + y)U_0^{\frac{2n}{n-2}}(y)dy \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{for all } \xi \in \mathbf{R}^n.$$

The first result in this section is

Theorem 3.1. *Suppose $\{u_i\}$ is a sequence of positive solutions of (11). Assume (K3) and (K4) with $1 < \alpha < n-2$. If Q satisfies*

$$\int_{\mathbf{R}^n} Q(\xi + y)U_0^{\frac{2u}{n-2}}(y)dy > 0$$

whenever $\int_{\mathbf{R}^n} \bigtriangledown Q(\xi + y)U_0^{\frac{2n}{n-2}}(y)dy = 0$. *Then u_i is uniformly bounded in $\bar{B}_{\frac{1}{2}}$.*

Remark 3.2 If $\alpha \geq n-2$, then Theorem 3.1 does not hold in general. For a counter example, please see [LL].

Thoerem 3.3. *Assume (K3) and (K4) hold. Suppose 0 is a blow-up point of a sequence of solutions of (11). Then 0 is an isolated blow up point. Furthemore, the inequality*

$$u_i(x)|x|^{\frac{n-2}{2}} \leq C \tag{14}$$

holds for $|x| \leq \frac{1}{2}$.

Let $u_i(x_i) = \max_{\bar{B}_{\frac{1}{2}}} u_i$. Then, by (14), we have

$$\xi = \lim_{i \longrightarrow +\infty} M_i^{\frac{2}{n-2}} x_i, \tag{15}$$

In the proof of Theorem 3.3, ξ satisfies

$$\int_{\mathbf{R}^n} \nabla Q(\xi + y) U_0^{\frac{2n}{n-2}}(y) dy = 0, \quad \text{and} \tag{16}$$

$$\int_{\mathbf{R}} \nabla Q(\xi + y) U_0^{\frac{2n}{n-2}}(y) dy \leq 0, \tag{17}$$

where U_0 is the solution of (13).

By assuming [Q], we have more precise description of $u_i(x)$ near its blow up point.

Theorem 3.4. *Suppose (K3), (K4) and [Q] with* $\frac{n-2}{2} \leq \alpha < n-2$ *are satisfied. Assume 0 is a blow up point of a sequence of solutions* u_i. *Let* $M_i = \max_{B_{\frac{1}{2}}} u_i$, *and* $m_i = \min_{\bar{B}_{\frac{1}{2}}} u_i$. *Then there exists a constant* $c > 0$ *such taht*

$$u_i(x + x_i) \leq c M_i^{-1} |x|^{2-n} \quad \textit{for } |x| \leq M_i^{-\beta}, \tag{18}$$

where $\beta = \frac{2}{n-2}(1 - \frac{\alpha}{n-2})$.

$$u_i(x + x_i) \sim M_i^{1 - \frac{2\alpha}{n-2}} \quad \textit{for } |x| \geq M_i^{-\beta}. \tag{19}$$

In particular,

$$\begin{cases} \lim_{i \longrightarrow +\infty} m_i = 0 & \textit{if } \alpha > \frac{n-2}{2}, \\ m_i \sim 1 & \textit{if } \alpha = \frac{n-2}{2}. \end{cases}$$

Furthemore, we have

$$\lim_{i \longrightarrow +\infty} \int_{B_1} K(x) u_i^{\frac{2n}{n-2}} dx = S_n^{\frac{n}{2}} \quad \textit{if } \alpha > \frac{n-2}{2},$$

and

$$\lim_{i \longrightarrow +\infty} \int_{B_r} K(x) u_i^{\frac{2n}{n-2}} dx = S_n^{\frac{n}{2}}(1 + o(1)) \quad \textit{if } \alpha = \frac{n-2}{2},$$

where $K(0) = n(n-2)$ *is assumed.*

For $\alpha < \dfrac{n-2}{2}$, we have

Theorem 3.5. *Suppose the assumption of Theorem 3.4 holds except that* α *satisfies* $1 < \alpha < \dfrac{n-2}{2}$. *Then*

$$\lim_{i \longrightarrow +\infty} \int_{\bar{B}_{\frac{1}{2}}} u_i^{\frac{2n}{n-2}}(x)dx = +\infty.$$

Furthemore, there exists a subsequence of u_i *(still denoted by* u_i*) such that* u_i *converges to a singular solution* u *of (11) with 0 as a nonrremovable singularity. The conformal metric* $ds^2 = u^{\frac{4}{n-2}}|dx|^2$ *is complete in* $\bar{B}_{\frac{1}{2}} \backslash \{0\}$. *If we assume 0 is the only zero of*

$$\int_{\mathbf{R}^n} \nabla Q(\xi + y) U_0^{\frac{2n}{n-2}}(y)dy = 0.$$

Then $u(x) = \bar{u}(|x|)(1 + o(1))$ *as* $|x| \longrightarrow 0$.

For the proofs of Theorem 3.1 $\sim$ 3.5, we refer the reader to [CLn3]. As an application, we have

Theorem 3.6. *Let* $K(x)$ *be a Morse function on* S^5, *and satisfy* $\Delta K(P) \neq 0$ *for any critical point* P *of* K. *Then there exists a constant* $C > 0$ *such that for any conformal metric* $g = u^{\frac{4}{3}} g_0$ *with* $K(x)$ *as the scalar curvature, we have*

$$C^{-1} \leq u(x) \leq C \quad \text{for } x \in S^5,$$

Let d *denote the Leray-Schauder degree among all solutions. Then* $d = 0$.

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