Balancing of Diffusion Partial Differential Equation

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Abstract

This paper concerns with the balancing theory for the system governed by diffusion partial differential equation, which is refereed here as infinite dimensional system or distributed-parameter system. Based on the eigenvalue-eigenfunction structure of Laplacian differential operator, the approximate controllability and initial observability are constructed. In order to perform the balanced realization for infinite dimensional system, a brief review on finite dimensional balancing is presented, and more intuitive meaning of balanced realization is then obtained. After defining the Hankel operator of the infinite dimensional system, we compute the Hankel singular value and construct the energy balancing. And then, the model reduction problem is discussed. In the sequel, numerical simulation of the balanced model reduction for one-dimensional heat equation is conducted.

Keywords: Hankel singular value, balancing theory, diffusion system, infinite dimensional system, distributed parameter system.

I. Introduction

In this paper, we consider the diffusion PDE as following

$$\frac{\partial \phi}{\partial t} = A\phi + Bu, \quad \phi(0,\xi) = \phi_0(\xi)$$
$$y = C\phi, \quad \phi(t,0) = u_l(t), \phi(t,l) = u_r(t)$$

which can be related to three types of control problems:

1) Boundary control problem, the control forces are applied through the functions $u_l(t), u_r(t)$.
2) Point control problems, the control forces are applied at discrete points, say ξ_a, i.e., the operator B is $\sum \delta(\xi - \xi_a)$.
3) Distributed control problem, the operator B is a distributed function of location where the control forces are acting on e.g. $H(\xi - \xi_1) - H(\xi - \xi_2)$.

On the other hand, the output operator C can also be divided into two types: point sensor and distributed sensor.

In many engineering application dynamic system is described by complex models which are difficult to analyze and to control. Reduction of the order of the model (e.g. the minimum number of state to describe the model, also known as McMillan degree) may overcome these difficulties, but it is quite possible that the model reduction incurs a significant loss of accuracy. In practical application, model approximation is often based on trail and error methods, which is obviously not the best form from an analysis and control point of view. Formalization of model approximation of finite-dimensional system has been studied by a number of authors. Glover[1] investigated the optimality of model approximations in the Hankel-norm, and gave a formal characterization of all optimal Hankel-norm approximations. Moore[2] introduces the balancing for stable minimal linear system. The balancing method offers a tool to measure the contribution of the different state components to the past input and future output energy of the system, which are the measures of controllability and observability. This analysis yields a methodology for model reduction problem.

Since the introduction of balancing method, balancing theory for stable linear finite-dimensional system has been explored in several directions. Balancing as a model reduction technique has been formalized by Glover[1], and Enns[5], who obtained an upper bound for the error in the Hankel- and L^∞-norm, respectively. Open-loop balancing method for unstable linear systems is further developed by Meyer[6], Obser et al[7]. Van der Schaft et al[8] discussed the balancing method for mechanical system. The closed-loop balancing method was due to Jonckheere et al[9], Opdenacker et al[10] and Mustafa et al[11]. Scherpen[12] extended the balancing method to nonlinear system. All these results are restricted to finite dimensional system.

The controllability and observability for infinite dimensional system are more complicated than those for finite dimensional systems. Curtain and Pritchard[13] has overcome many technical difficulties to the study of the controllability and observability for an infinite-dimensional system. Bensoussan et al[14,15] has discussed the controllability and observability with different approach.

In this paper, the eigenvalue and eigenfunction structure is used to analyze the infinite-dimensional system, which is very similar to modal analysis used in the analysis of structural vibration. The state-space representation for infinite-dimensional system is then constructed and the conditions for controllability and observability can be evidently obtained. The balancing method is conducted as same as that for finite-dimensional system. In the sequel, the model-reduction problem for parabolic partial differential equation is performed.

II. Controllability and Observability of Infinite Dimensional System

Let $\Omega=[0,l]$ and $H=L^2(\Omega)$. Consider the following parameter distribution system:

$$\frac{\partial\phi}{\partial t}=A\phi+Bu,\quad \phi(0,\xi)=\phi_0(\xi) \tag{1.a}$$

$$y=C\phi \tag{1.b}$$

where

1) $\phi\in V$, with $V=C([0,T];L^2(\Omega))$
2) $A:D(A)\to H$ is a partial differential operator on Hilbert space H associated with boundary conditions $\phi(t,0)=u_l(t),\phi(t,l)=u_r(t)$. The domain of operator A is $D(A)=\{\phi\in H|A\phi \text{ exists}\}$. Simple examples of the operator A are $A=\frac{\partial^2}{\partial\xi^2}$, $A=\frac{\partial}{\partial\xi}\left(a(\xi)\frac{\partial^2}{\partial\xi^2}\right)$, $A=\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)$.
3) $B:\mathrm{R}^m\to H$ with

$$Bu=\begin{bmatrix}b_1(\xi), & \cdots, & b_m(\xi)\end{bmatrix}\begin{bmatrix}u_1(t)\\ \vdots\\ u_m(t)\end{bmatrix}=b_1(\xi)u_1(t)+\cdots+b_m(\xi)u_m(t) \tag{2}$$

4) $C:H\to\mathrm{R}^p$ with

$$C\phi=\begin{bmatrix}\langle c_1,\phi(t,\cdot)\rangle_H\\ \vdots\\ \langle c_p,\phi(t,\cdot)\rangle_H\end{bmatrix} \tag{3}$$

The inner product on Hilbert space H is defined as $\langle f,g\rangle_H=\int_0^l f^*(\xi)g(\xi)d\xi$, and $\overline{\langle f,g\rangle_H}=\langle g,f\rangle_H$. The subscript H may be omitted without making confusion if it denotes the space $L^2(\Omega)$ on in the following sections.

2.1 Eigenfunctions and adjoint operator of A.

In most engineering application, the eigenfunctions $\{\phi_i(\xi)\}_{i=1}^{\infty}$ of the operator A can form the bases for space H, and this operator is called a regular spectral operator. The adjoint operator of A is recognized as A^* with eigenfunctions $\{\psi_i(\xi)\}_{i=1}^{\infty}$ and satisfies

$$\langle A^*x,y\rangle=\langle x,Ay\rangle,\quad \forall x,y\in H \tag{4}$$

The eigenfunctions ϕ_i and ψ_i are defined as

$$A\phi_i(\xi)=\lambda_i\phi_i(\xi),\quad A^*\psi_j(\xi)=\lambda_j^*\psi_j(\xi) \tag{5}$$

then

$$\langle\psi_i,\phi_j\rangle=\delta_{ij} \tag{6}$$

i.e. after normalization $\{\phi_i(\xi)\}_{i=1}^{\infty}$ and $\{\psi_j(\xi)\}_{j=1}^{\infty}$ are biorthonormal to each other. and they also form two bases for Hilbert space H. Hence the solution of Eq.(1), ϕ, can be represented by

$$\phi(t,\xi)=\sum_{i=1}^{\infty}x_i(t)\phi_i(\xi) \tag{7}$$

where $x_i(t)=<\psi_i,\phi(t,\cdot)>$ are the coefficients of the series expansion of $\phi(t,\xi)$ with respect to the basis $\{\phi_i(\xi)\}_{i=1}^{\infty}$. After some algebraic operations, the following lemma is deduced:

Lemma 1:

(1) $A\phi(t,\xi)=\sum_{i=1}^{\infty}\lambda_i <\psi_i,\phi(t,\cdot)> \phi_i(\xi)$ (8)

(2) $e^{tA}\phi(t,\xi)=\sum_{i=1}^{\infty}e^{\lambda_i t} <\psi_i,\phi(t,\cdot)> \phi_i(\xi)$ (9)

(3) $\phi(t,\xi)=\sum_{i=1}^{\infty}<\phi_i,\phi(t,\cdot)> \psi_i(\xi)$ (10)

(4) $A^*\phi(t,\xi)=\sum_{i=1}^{\infty}\lambda_i^* <\phi_i,\phi(t,\cdot)> \psi_i(\xi)$ (11)

(5) $e^{tA^*}\phi(t,\xi)=\sum_{i=1}^{\infty}e^{\lambda_i^* t} <\phi_i,\phi(t,\cdot)> \psi_i(\xi)$ (12)

2.2 State-space representation

Since the solution $\phi(t,\xi)$ can be expressed respect to the basis $\{\phi_i(\xi)\}_{i=1}^{\infty}$, we can compute the following terms:

(1) $\dfrac{\partial\phi(t,\xi)}{\partial t}$:

$$\frac{\partial\phi(t,\xi)}{\partial t}=[\phi_1(\xi),\quad \phi_2(\xi),\quad \cdots]\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \vdots\end{bmatrix} \tag{13}$$

(2) Bu: Since the term $b_i(\xi)$ can be expressed as the series expansion of the basis $\{\phi_i(\xi)\}_{i=1}^{\infty}$, for instance:

$$b_1(\xi)=\sum_{i=1}^{\infty}<\psi_i,b_1>\phi_i(\xi)=\sum_{i=1}^{\infty}\phi_i(\xi)b_{i1},\quad b_{i1}=<\psi_i,b_1> \tag{14}$$

thus, Bu can be rewritten as

$$Bu=[\phi_1(\xi)\quad \phi_2(\xi)\quad \cdots]\begin{bmatrix}B_1\\ B_2\\ \vdots\end{bmatrix}\begin{bmatrix}u_1(t)\\ \vdots\\ u_m(t)\end{bmatrix}=[\phi_1(\xi)\quad \phi_2(\xi)\quad \cdots][B]u \tag{15}$$

where

$$B_i=[b_{i1}\quad b_{i2}\quad \cdots\quad b_{im}]=[\langle\psi_i(\xi),b_1(\xi)\rangle\quad \langle\psi_i(\xi),b_2(\xi)\rangle\quad \cdots\quad \langle\psi_i(\xi),b_m(\xi)\rangle] \tag{16}$$

(3) $y=C\phi$: Similarly, $y=C\phi$ can be expanded as following. Since

$$<c_1,\phi(t,\cdot)>=<c_1,\sum_{i=1}^{\infty}x_i(t)\phi_i>=[<c_1,\phi_1>\quad <c_1,\phi_2>\quad \cdots]\begin{bmatrix}x_1(t)\\ x_2(t)\\ \vdots\end{bmatrix} \tag{17}$$

then

$$y=C\phi=[C_1\quad C_2\quad \cdots]\begin{bmatrix}x_1\\ x_2\\ \vdots\end{bmatrix}=[C]\begin{bmatrix}x_1\\ x_2\\ \vdots\end{bmatrix} \tag{18}$$

where $C_i=[\langle c_1,\phi_i\rangle\quad \langle c_2,\phi_i\rangle\quad \cdots\quad \langle c_p,\phi_i\rangle]^T$

(4) $A\phi(t,\xi)$:

$$A\phi(t,\xi)=\sum_{i=1}^{\infty}x_i(t)A\phi_i(\xi)=[\phi_1(\xi)\quad \phi_2(\xi)\quad \cdots]\begin{bmatrix}\lambda_1 & 0 & \cdots\\ 0 & \lambda_2 & \\ \vdots & & \ddots\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ \vdots\end{bmatrix} \tag{19}$$

Substitution of above terms into Eq.(1), comparing the coefficients of eigenfunctions $\{\phi_i(\xi)\}_{i=1}^{\infty}$ leads to the state-space representation of Eq.(1)

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \\ \vdots & & \ddots \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ \vdots \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} C_1 & C_2 & \cdots \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \end{bmatrix} \tag{20}$$

Eq.(20) contains infinity many states which are also decoupled. This is why we call this type of system as infinite dimensional system. Thus if the value of B_i is zero, the corresponding state $x_i(t)$ is uncontrollable since the control input $u(t)$ can't affect the state $x_i(t)$. But B_i is a row vector, hence the controllability condition of Eq.(20) is

$$\text{Rank}(B_i)=1, \quad i=1,2,\ldots$$

Similarly, if C_i has zero value, the measured output signal $y(t)$ contains no information of the state $x_i(t)$, that is, the state $x_i(t)$ could not be observable from variable $y(t)$. Thus, the condition for each state to be observable is $C_i \neq 0$, $i=1,2,\ldots$. Because C_i is a column vector, the observability condition of Eq.(20) is

$$\text{Rank}(C_i)=1, \ i=1,2,\ldots$$

Therefore, the following theorem can be obtained:

Theorem 1 *Given a diffusion partial differential equation*

$$\frac{\partial \phi}{\partial t} = A\phi + Bu$$

$$y = C\phi$$

the eigenfunctions of the partial differential operator A and A^* are $\{\phi_i(\xi)\}_{i=1}^{\infty}$ *and* $\{\psi_i(\xi)\}_{i=1}^{\infty}$ *respectively, then*

1) *this system is approximately controllable if and only if*

$$\text{Rank}\left(\left[<\psi_i, b_1> \quad <\psi_i, b_2> \quad \cdots \quad <\psi_i, b_m>\right]\right) = 1, \quad i = 1,2,\ldots$$

2) *this system is initially observable if and only if*

$$\text{Rank}\left(\left[<c_1, \phi_i> \quad <c_2, \phi_i> \quad \cdots \quad <c_p, \phi_i>\right]^T\right) = 1, \quad i = 1,2,\ldots$$

Remarks:

1. It can be easily verified that the controllability condition in Theorem 1 is equivalent to Theorem 3.11 or Proposition 3.13 in Ref. 13 and the observability condition is equivalent to Theorem 3.25 in Ref. 13.
2. If we choose $b_i = \psi_i$, then the system is fully decoupled which means that we can control each state $x_i(t)$ by the corresponding input vector element $u_i(t)$. This is also referred as modal control in the study of vibration control of flexible structure.
3. If the value of b_i are orthogonal to certain eigenfunction ψ_j of the adjoint operator A^*, then the corresponding state x_j is uncontrollable.
4. Similar results for c_i's as b_i's in Remark 2 and 3 can be obtained if the observability of state x is concerned.

III. Balancing of Finite Dimensional System

Let G be a stable system with differential equation of the form:

$$G: \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) & (21.a) \\ y(t) = Cx(t) & (21.b) \end{cases}$$

where $A \in R^{n \times n}, B \in R^{n \times m}, C \in R^{p \times n}$ are continuous real matrix value functions. It is often convenient to assume that the system is relaxed in the infinitely remote pass, *i.e.*, $\lim_{t \to -\infty} x(t) = 0$. The triple of matrices (A,B,C) is called a realization of the system. Assume this system is both controllable and observable, then the pair (A,B) is controllable and (C,A) is observable and the triple (A,B,C) is called minimal realization.

3.1 Controllability function and observability function

Define the controllability function $L_c(x_0)$ and observability function $L_o(x_0)$ of the linear system (21) as:

$$L_c(x_0) \equiv \min_{\substack{u \in L^2(-\infty,0) \\ x(-\infty)=0,\ x(0)=x_0}} \frac{1}{2}\int_{-\infty}^{0} \|u(t)\|^2 dt \tag{22a}$$

$$L_o(x_0) \equiv \frac{1}{2}\int_{0}^{\infty} \|y(t)\|^2 dt, \quad x(0) = x_0, \quad u(t) = 0, \quad 0 \le t \le \infty \tag{22b}$$

Note that $L_c(x_0)$ is the minimum control input energy which is used to drive the system state $x(t)$ from rest $(x(-\infty) = 0)$ to the current state $(x(0)=x_0)$ and the function $L_o(x_0)$ is the free output response energy of the system which is released from state $x(0)=x_0$ without any control input $(u(t)=0, t \geq 0)$. It is very easy to verify that

$$L_c(x_0)=\frac{1}{2}x_0^T P^{-1}x_0,\quad L_o(x_0)=\frac{1}{2}x_0^T Q x_0 \tag{23}$$

where

$$P=\int_0^\infty e^{tA}BB^Te^{tA^T}dt,\qquad Q=\int_0^\infty e^{tA^T}C^TCe^{tA}dt \tag{24}$$

P and Q are called the system controllability gramian and system observability gramian, respectively, which can also be obtained as solutions of the following Lyapunov equations:

$$AP+PA^T+BB^T=0,\qquad A^TQ+QA+C^TC=0 \tag{25}$$

For stable A, these solutions are positive. The controllability gramian P and observability gramian Q are the convenient form of the joint characteristics of a system's controllability and observability. Let σ be the Hankel singular value of this system, it can be proved that

$$PQx_0=\sigma^2x_0 \tag{26}$$

i.e. the square ot the Hankel singular value σ^2 is the eigenvalue of the matrix PQ. Since the matrix PQ has n positive eigenvalues, we can denote their square roots by a diagonal matrix Σ

$$\Sigma=\begin{bmatrix}\sigma_1 & & & 0\\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_n\end{bmatrix}$$

where $\sigma_1,\sigma_2,\ldots,\sigma_n$ are ordered with the decreasing magnitude.

3.2 Why balanced realization?

The system triple (A,B,C) is balanced, if its controllability and observability gramians are equal and diagonal [2] i.e. $P=Q=\Sigma$. We say that this realization is an ordered balanced realization if the elements in Σ satisfying $\sigma_1>\sigma_2>\cdots>\sigma_n>0$. Let the current state $x(0)$ of this balanced system be partitioned as

$$x_0=[x_{01}\quad x_{02}\quad\cdots\quad x_{0n}]^T$$

it follows directly from Eq.(23) that the controllability and observability functions are written as

$$L_c(x_0)=\frac{1}{2}\left[\frac{1}{\sigma_1}x_{01}^2+\frac{1}{\sigma_2}x_{02}^2+\cdots+\frac{1}{\sigma_n}x_{0n}^2\right],\quad L_o(x_0)=\frac{1}{2}\left[\sigma_1x_{01}^2+\sigma_2x_{02}^2+\cdots+\sigma_nx_{0n}^2\right] \tag{27}$$

The more intuitively physical meaning of the controllability and observability functions can be explained as follows. $L_c(x_0)$ is the input energy used to drive the system state which can be related to the actuator power, and $L_o(x_0)$ is the output response energy which depends on the sensitivity of the sensor measurement. If an order balanced system (A,B,C) has large Hankel singular values σ_1, σ_2, …, σ_n, then it means that

1) $L_c(x_0)$ is small, *i.e.* only moderate actuator powers are necessary to drive the system from rest to state x_0.
2) The related $L_o(x_0)$ is also large, *i.e.* the very accurate and sensitive sensors are not critically required.

On the other hand, for an ordered balanced system with some small Hankel singular values, say σ_k, …, σ_n, the magnitude $\sum_{i=k}^{n}x_{0i}^2/\sigma_i$ is large as the major contribution term in $L_c(x_0)$, so this system can only be driven to current state with large input actuator power. Similarly, the magnitude $\sum_{i=k}^{n}\sigma_ix_{0i}^2$ is small compared with other terms in $L_o(x_0)$, so only very accurate and sensitive measurement system can be used to detect these signals. Hence if $\sigma_k>>\sigma_{k+1}$, the state components x_{k+1} to x_n are insignificant from this energy point of view and may be removed to reduce the number of state components of the model without making significant drawback in system performance. Therefore the Hankel singular values can not only be used to judge the joint characterization of controllability and observability of the system but also indicate that the balanced realization can serve as a good technique to fulfill system model reduction.

3.3 The implementation of balanced realization

Assume that (A,B,C) is minimal but not balanced realization of system G, given by Eqs.(21.a) (21.b), i.e. (A,B) is controllable and (C,A) is observable, its controllability gramian P and observability gramian can't satisfy Eq.(38). In order to find the balanced realization of system G, we have to perform the coordinate transformation. Let T be an invertable matrix such that the new state $\bar{x}$ is related to the original state x by

$$x=T\bar{x}$$

then Eqs.(21.a)(21.b) become as

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \\ y = \bar{C}\bar{x} \tag{28}$$

where

$$\bar{A} = T^{-1}AT,\ \bar{B} = T^{-1}B,\ \bar{C} = CT \tag{29}$$

$(\bar{A},\bar{B},\bar{C})$ is the new representation. Is it possible for us to find similarity transform T such that $(\bar{A},\bar{B},\bar{C})$ is a balanced realization? That is, the solutions of the following two Lyaponov equations:

$$\bar{A}\bar{P} + \bar{P}\bar{A}^T + \bar{B}\bar{B}^T = 0,\qquad \bar{A}^T\bar{Q} + \bar{Q}\bar{A} + \bar{C}^T\bar{C} = 0 \tag{30}$$

must satisfy

$$\bar{P} = \bar{Q} = \Sigma \tag{31}$$

It has been proved that this similarity transform exists and given by

$$T = Q^{-\frac{1}{2}}V\Sigma^{\frac{1}{2}} \tag{32}$$

Therefore, the triple $(\bar{A},\bar{B},\bar{C})$ in Eq.(29) which is obtained from Eq.(30) is the state-space representation of the balanced realization of system G.

3.4 Balanced model truncation

Suppose the ordered Hankel singular values $\sigma_1,\sigma_2,\cdots,\sigma_n$ of the system G satisfy the condition $\sigma_k >> \sigma_{k+1}$ for some k, then we can partition the states $\bar{x}$ in balanced realization corresponding to k as

$$\bar{x}(t) = \begin{bmatrix} \bar{x}_a \\ \bar{x}_b \end{bmatrix} = \begin{bmatrix} [\bar{x}_1 & \cdots & \bar{x}_k]^T \\ [\bar{x}_{k+1} & \cdots & \bar{x}_n]^T \end{bmatrix} \tag{33}$$

The k-vector $\bar{x}_a$ contains the components to be retained, while the $(n-k)$-vector $\bar{x}_b$ contains the components to be discarded. Now partition the matrices $\bar{A}$, $\bar{B}$ and $\bar{C}$ conformably with $\bar{x}$ to obtain

$$\bar{A} = \begin{bmatrix} \bar{A}_{aa} & \bar{A}_{ab} \\ \bar{A}_{ba} & \bar{A}_{bb} \end{bmatrix},\quad \bar{B} = \begin{bmatrix} \bar{B}_a \\ \bar{B}_b \end{bmatrix},\quad \bar{C} = \begin{bmatrix} \bar{C}_a & \bar{C}_b \end{bmatrix} \tag{34}$$

By omitting the states and dynamics associated with $\bar{x}_b$, we obtain the lower-order system

$$\bar{x}_a = \bar{A}_{aa}\bar{x}_a + \bar{B}_a u \\ y = \bar{C}_a\bar{x}_a \tag{35}$$

The k^{th}-order truncation of the balanced realization $(\bar{A},\bar{B},\bar{C})$ is given by $(\bar{A}_{aa},\bar{B}_a,\bar{C}_a)$. The diagonal matrix Σ is also partitioned as

$$\Sigma = \begin{bmatrix} \Sigma_a & 0 \\ 0 & \Sigma_b \end{bmatrix} \tag{36}$$

More intuitive meaning about the balanced model truncation under the condition $\sigma_k >> \sigma_{k+1}$ can be understood as follows. Let σ_k be 10 times larger than σ_{k+1}, we split the controllability gramian into

$$L_c(\bar{x}) = L_c(\bar{x}_a) + L_c(\bar{x}_b) \tag{37}$$

where

$$L_c(\bar{x}_a) = \frac{1}{2}\left[\frac{1}{\sigma_1}\bar{x}_1^2 + \cdots + \frac{1}{\sigma_k}\bar{x}_k^2\right],\quad L_c(\bar{x}_b) = \frac{1}{2}\left[\frac{1}{\sigma_{k+1}}\bar{x}_{k+1}^2 + \cdots + \frac{1}{\sigma_n}\bar{x}_n^2\right]$$

and since

$$L_c(\bar{x}_a) \le \frac{1}{2}\frac{1}{\sigma_k}\left[\bar{x}_1^2 + \cdots + \bar{x}_k^2\right],\quad L_c(\bar{x}_b) \ge \frac{1}{2}\frac{1}{\sigma_{k+1}}\left[\bar{x}_{k+1}^2 + \cdots + \bar{x}_n^2\right] \tag{38}$$

it follows that 10 times larger control input energy is spent to drive the states $(\bar{x}_{k+1},\cdots,\bar{x}_n)$ than to drive states $(\bar{x}_1,\cdots,\bar{x}_k)$. Similarly, the sensor to detect the output energy contributed from states $(\bar{x}_{k+1},\cdots,\bar{x}_n)$ must be 10 times more sensitive than from states $(\bar{x}_1,\cdots,\bar{x}_k)$. Hence, we may say that states $(\bar{x}_{k+1},\cdots,\bar{x}_n)$ is relatively less controllable and observable than states $(\bar{x}_1,\cdots,\bar{x}_k)$.

Remarks:

1. The above observation leads to conclude that the states $(\bar{x}_{k+1},\cdots,\bar{x}_n)$ are relatively **less** controllable and observable than the states $(\bar{x}_1,\cdots,\bar{x}_k)$ instead of that states $(\bar{x}_{k+1},\cdots,\bar{x}_n)$ is uncontrollable and/or

unobservable.

2. The Hankel singular value σ_i is recognized as the indicator to measure the relative controllability and observability which is not the same as the application of rank conditions

$$\text{rank}\begin{bmatrix}B & AB & \cdots & A^{n-1}B\end{bmatrix}=n,\ \text{rank}\begin{bmatrix}C & CA & \cdots & CA^{n-1}\end{bmatrix}^T=n$$

to judge the overall system's controllability and observability. We can use σ_i not only to indicate the distance of the state $\bar{x}_i$ to the uncontrollable region but also to denote the distance of same state to the unobservable region. Thus the value of σ_i can be considered as the controllability margin and the observability margin of measuring the relative information about state $\bar{x}_i$. This is similar to the stability analysis in finite-dimensional system in which Nyquist criterion provides the same information on the absolute stability of a control system as does the Routh-Hurwitz criterion and phase margin and gain margin give qualitative indication on the relative stability of a closed-loop system as well.

3. There are infinite many different types of state representation of system G, and each representation has a set of state variable $x(t)=\begin{bmatrix}x_1 & x_2 & \cdots & x_n\end{bmatrix}^T$ to realize the system information. We can compute the controllability margin and observability margin associated with state x_i. In general, these two margins are not the same but they do have the same value when the state-space representation is obtained through the balanced realization. If we put these two margins on two sides of weights and measures, they are in the state of equilibrium.

IV. Balanced Realization of Infinite Dimensional System

4.1 Hankel operator of infinite dimensional system

Consider the same distributed-parameter system governed by Eq.(1), we can define the Hankel operator corresponding to G as in the case of finite-dimensional system, that is

$$\Gamma_G : L_2(-\infty,0)\to L_2(0,\infty)$$

$$u \mapsto y \equiv (\Gamma_G u)(t) = Ce^{tA}\phi_0$$

where

$$\phi_0(\xi)=\int_{-\infty}^{0} e^{-\tau A}Bu(\tau)d\tau \tag{39}$$

and e^{tA} is a strongly continuous semigroup with the differential operator A as its infinitesimal generator which can be constructed by

$$e^{tA}\equiv I+tA+\frac{t^2}{2!}A^2+\frac{t^3}{3!}A^3+\cdots$$

The Hankel operator can be rewritten as

$$(\Gamma_G u)(t)=\int_{-\infty}^{0} Ce^{(t-\tau)A}Bu(\tau)d\tau,\quad \text{for } t\geq 0$$

The adjoint of Γ_G is then given by

$$\Gamma_G^* : L^2[0,\infty)\to L^2(-\infty,0]$$

$$y \mapsto u \equiv (\Gamma_G^* y)(t)=\int_0^{\infty} B^* e^{(\tau-t)A^*}C^* y(\tau)d\tau$$

where A^*, B^* and C^* are the adjoint operators of the operators A, B and C, respectively. Let $\sigma^2 \neq 0$ be an eigenvalue of $\Gamma_G^*\Gamma_G$ and nonzero $u\in L^2(-\infty,0]$ be a corresponding eigenvector i.e.

$$\Gamma_G^*\Gamma_G u=\sigma^2 u \tag{40}$$

Define

$$v:=\frac{1}{\sigma}\Gamma_G u\in L^2[0,\infty).$$

Then (u,v) satisfy

$$\Gamma_G u=\sigma v,\quad \Gamma_G^* v=\sigma u$$

This pair of vectors (u,v) are called a Schimdt pair of Γ_G and σ is called the Hankel singular value. It is evident that Γ_G has infinite many singular values due to the existence of infinite many eigenvalues associated with the differential operator A. The Hankel singular values and the corresponding Schimdt pairs can not be easily constructed by the integral operator theory. And we will develop the realization technique in which the matrices with rank infinite are used to represent these pairs and then calculate these singular values.

4.2 Balanced Realization

Let $\phi_0(\xi)$ be expressed in terms of the bases $\{\phi_i(\xi)\}_{i=1}^{\infty}$,

$$\phi_0(\xi)=\sum_{i=1}^{\infty} v_i\phi_i(\xi)=[\phi_1 \quad \phi_2 \quad \cdots]V \tag{41}$$

The Hankel singular values can be computed with the aids of Lemma 1 and are the eigenvalues of an infinite matrix PQ, i.e.

$$PQV=\sigma^2 V \tag{42}$$

And

$$P=\int_0^{\infty}[e^{\lambda_i\tau}][B][B]^*[e^{\bar{\lambda}_i\tau}]d\tau,\quad Q=\int_0^{\infty}[e^{\bar{\lambda}_i\tau}][C]^*[C][e^{\lambda_i\tau}]d\tau \tag{43}$$

where the matrices $[B],[C]$ are given in Eqs.(16)(17). Assume for stable system, $Re(\lambda_i)<0$, direct integration leads to

$$P=\left[\frac{B_iB_j^*}{Re(\lambda_i+\bar{\lambda}_j)}\right]_{i,j=1}^{\infty},\quad Q=\left[\frac{C_j^*C_i}{Re(\bar{\lambda}_i+\lambda_j)}\right]_{i,j=1}^{\infty} \tag{44}$$

From engeneering point of view, the computation of Hankel singular values can be solved by Eqs.(42)-(44) with large number of terms in the matrix PQ instead of using infinite many terms. Now let the ordered Hankel singular values are organized as

$$\Sigma=\begin{bmatrix}\sigma_1 & & \\ & \sigma_2 & \\ & & \ddots\end{bmatrix}$$

and T be an invertable matrix such that the new state $\bar{x}=[\bar{x}_1 \quad \bar{x}_2 \quad \cdots]$ is related to the original state $x=[x_1 \quad x_2 \quad \cdots]$ by

$$x=T\bar{x}$$

where

$$T=Q^{-1/2}V\Sigma^{1/2} \tag{45}$$

with the matrix V satisfying

$$Q^{1/2}PQ^{1/2}V=V\Sigma^2$$

Hence the balanced realization $(\bar{A},\bar{B},\bar{C})$ of this system is

$$\bar{A}=\Sigma^{-1/2}V^TQ^{1/2}AQ^{1/2}V\Sigma^{1/2},\ \bar{B}=\Sigma^{-1/2}V^TQ^{1/2}B,\ \bar{C}=CQ^{1/2}V\Sigma^{1/2} \tag{46}$$

Therefore, the corresponding balanced state-space representation of Eq.(1) is

$$\frac{d}{dt}\begin{bmatrix}\bar{x}_1(t)\\ \bar{x}_2(t)\\ \vdots\end{bmatrix}=\bar{A}\begin{bmatrix}\bar{x}_1(t)\\ \bar{x}_2(t)\\ \vdots\end{bmatrix}+B,\quad y=\bar{C}\begin{bmatrix}\bar{x}_1(t)\\ \bar{x}_2(t)\\ \vdots\end{bmatrix} \tag{47}$$

4.3 Balanced model truncation

Suppose the ordered Hankel singular values $\sigma_1,\sigma_2,\cdots$ of the system G satisfy the condition $\sigma_n >> \sigma_{n+1}$ for some n, then we can partition the states $\bar{x}$ in balanced realization corresponding to n as

$$\bar{x}=[\bar{x}_1(t)\ \cdots\bar{x}_n(t)\,|\,\bar{x}_{n+1}(t)\cdots]^T=[\bar{x}_a(t)\,|\,\bar{x}_b(t)]$$

The n-vector $\bar{x}_a$ contains the components to be retained, while the vector $\bar{x}_b$ contains the components to be discarded. Now partition the matrices $\bar{A}$, $\bar{B}$ and $\bar{C}$ conformably with $\bar{x}$ to obtain

$$\bar{A}=\begin{bmatrix}\bar{A}_{aa} & \bar{A}_{ab}\\ \bar{A}_{ba} & \bar{A}_{bb}\end{bmatrix},\quad \bar{B}=\begin{bmatrix}\bar{B}_a\\ \bar{B}_b\end{bmatrix},\quad \bar{C}=[\bar{C}_a \quad \bar{C}_b] \tag{48}$$

with $\bar{A}_{aa}\in R^{n\times n}, \bar{B}_a\in R^{n\times m}, \bar{C}_a\in R^{p\times n}$. By omitting the states and dynamics associated with $\bar{x}_b$, we obtain the lower-order system

$$\begin{aligned}\dot{\bar{x}}_a&=\bar{A}_{aa}\bar{x}_a+\bar{B}_a u\\ y&=\bar{C}_a\bar{x}_a\end{aligned} \tag{49}$$

The n^{th}-order truncation of the balanced realization $(\bar{A},\bar{B},\bar{C})$ is given by $(\bar{A}_{aa},\bar{B}_a,\bar{C}_a)$ which is an approximated system for the orginal infinite dimensional system.

V. Illustration Example

Consider initial boundary value problem for temperature in a homogenous, isotropic rod with insulated sides as shown in Fig. 1:

$$\frac{\partial\phi(t,\xi)}{\partial t}=\frac{1}{\pi^2}\frac{\partial^2\phi(t,\xi)}{\partial\xi^2}+b(\xi)u(t) \quad (50.a)$$

$$y(t)=\int_0^1 c(\xi)\phi(t,\xi)d\xi \quad (50.b)$$

The associated boundary condition and initial condition are

$$\frac{\partial\phi(t,0)}{\partial\xi}=0,\qquad \frac{\partial\phi(t,1)}{\partial\xi}=0,\qquad \phi(0,\xi)=\phi_0(\xi) \quad (50.c)$$

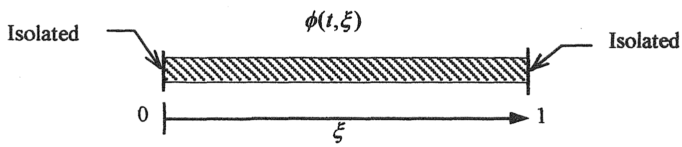

Figure 1. Temperature distribution in a homogenous isotropic rod with insulated sides

It is easy verified that the cooresponding differential operator A is self-adjoint, i.e. $A^*=A$. And then the eigenvalues and eigenfunctions of A and A^* are equal. It is noted that the eigenvalues are $\lambda_n=\lambda_n^*=-n^2$ and the eigenfunctions are $\phi_n=\psi_n=\sqrt{2}\cos(n\pi\xi)$ and $\phi_0=\psi_0=1$.

Suppose that the point control and point observation are used. The actuator is placed at ξ_a i.e. $b(\xi)=\delta(\xi-\xi_a)$ and the sensor is placed at ξ_s i.e. $c(\xi)=\delta(\xi-\xi_s)$.

Hence, the diffusion P.D.E. state-space realization as follows.

$$\frac{d}{dt}\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\\\vdots\end{bmatrix}=\begin{bmatrix}-1 & & & 0\\ & -4 & & \\ & & -9 & \\ 0 & & & \ddots\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\\\vdots\end{bmatrix}+\sqrt{2}\begin{bmatrix}\cos(\pi\xi_a)\\\cos(2\pi\xi_a)\\\cos(3\pi\xi_a)\\\vdots\end{bmatrix}u(t) \quad (51.a)$$

$$y(t)=\sqrt{2}\begin{bmatrix}\cos(\pi\xi_s) & \cos(2\pi\xi_s) & \cos(3\pi\xi_s) & \cdots\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\\\vdots\end{bmatrix} \quad (51.b)$$

The conditions of controllability and observability of this system are $\cos(n\pi\xi_a)\neq 0$ and $\cos(n\pi\xi_s)\neq 0$, $n=1,2,\cdots$, respectively. These conditions are equivalent to

$$\xi_a,\xi_s\notin\left\{\xi\in[0,1]\,\middle|\,\xi=\frac{2m-1}{2n},n,m\in N\right\}$$

This result means that for single direction heat conduction problem , if only one energy source is located at specified position and only one sensor is used to measure the temperature at certain location, the control of the whole range of temperature cannot be achived if the sensor and /or actuator are placed at $\xi=2m-1/2n$.

The corresponding controllability gramian P and observability gramian Q are:

$$P=2\begin{bmatrix}\frac{\cos^2(\pi\xi_a)}{1^2+1^2} & \frac{\cos(\pi\xi_a)\cos(2\pi\xi_a)}{1^2+2^2} & \frac{\cos(\pi\xi_a)\cos(3\pi\xi_a)}{1^2+3^2} & \cdots\\ \frac{\cos(2\pi\xi_a)\cos(\pi\xi_a)}{2^2+1^2} & \frac{\cos(2\pi\xi_a)\cos(2\pi\xi_a)}{2^2+2^2} & \frac{\cos(2\pi\xi_a)\cos(3\pi\xi_a)}{2^2+3^2} & \cdots\\ \vdots & \vdots & & \ddots\end{bmatrix} \quad (52a)$$

$$Q = 2\begin{bmatrix} \frac{\cos^2(\pi\xi_s)}{1^2+1^2} & \frac{\cos(\pi\xi_s)\cos(2\pi\xi_s)}{1^2+2^2} & \frac{\cos(\pi\xi_s)\cos(3\pi\xi_s)}{1^2+3^2} & \cdots \\ \frac{\cos(2\pi\xi_s)\cos(\pi\xi_s)}{2^2+1^2} & \frac{\cos(2\pi\xi_s)\cos(2\pi\xi_s)}{2^2+2^2} & \frac{\cos(2\pi\xi_s)\cos(3\pi\xi_s)}{2^2+3^2} & \cdots \\ \vdots & \vdots & & \ddots \end{bmatrix} \tag{52b}$$

Since the magnitude of the elements of P, Q is decreasing by $1/n^2$ as the element position (i,j) increasing, the most effective terms of eigenvalues of PQ are first few term. Thus we can take the first finite $l \times l$ terms, say P_l and Q_l, to approximate P and Q. Let $PQu = \sigma^2 u$ and $P_l Q_l u_l = \sigma_l^2 u_l$, we have $\lim_{l\to\infty} P_l = P$, $\lim_{l\to\infty} Q_l = Q$,. In general, we may take $l = 10$ or $l = 50$ to meet the requirement of accuracy.

The balanced realization of this system can be solved using the approximated controllability gramian P_l and obsevability gramain Q_l as stated in previous section. Hence, the model reduction can also be performed.

VI. Conclusion

This paper deals with an analysis of diffusion partial differential equation and offers a easy tool for infinite-dimensional system model reduction. Intead of using operator theory to compute the Hankel singular values, we develop an simple algorithm to compute these values through a matrix equation with infinite many components. In order to solve this problem in engineering sense, we can only use the first few terms of this equation without loss of significance. At the meanwhile, the balancing for finite dimensional system is briefly reviewed and more intuitive meaning has been discussed. As stated in our paper, we believe there may be some new design criterion of control system may be incurred in terms of controllability margin and observability margin.

References

[1] Glover, K., "All Optimal Hankel-norm approximations of linear multivariable systems and there L^∞-error bounds," *Int. J. of Contr.*, Vol. 39, 1984, pp. 1115-1193.

[2] Moore, B. C., "Principal Component Analysis in Linear Systems: Controllability, Observability and Model Reduction," *IEEE Transaction of Automatic Control*, AC-26, 1981, pp. 17-32.

[3] Meirovitch, L., *Methods of Analytical Dynamics*, McGraw-Hill, New York, 1970.

[4] Merirovitch, L., *Dynamics and Control of Structures*, John Wiley & Sons, Singapore, 1992.

[5] Enns, D. F., "Model Reduction with Balanced Realizations: An Error Bound and A Frequency weighted Generalization," *Proceedings of the 23rd Conference on Decision and Control*, IEEE, New York, 1984., pp. 127-132.

[6] Meyer, D. G., "Fractional Balanced Reduction - Model Reduction via a Fractional Representation," *IEEE Transaction of Automatic Control*, AC-26, 1990, pp. 1341-1345.

[7] Ober, R. and McFarlane, D., "Balanced Canonical Forms for Minimal Systems: A Normalized Comprime Factor Approach," *Linear Algebra and its Applications*, 1989, pp. 23-64.

[8] Van der Schaft, A. J. and Oeloff, L. E., "Model Reduction of Linear Conservative Mechanical Systems," *IEEE Transaction of Automatic Control*, AC-35, 1990, pp. 729-733.

[9] Jonckheere, E. A. and Silverman, L. M., "Singular Value Analysis of Deformable System," *Journal of Circuit System and Signal Processing*, Vol. 1, 1982, pp. 447-470.

[10] Opdernacker, Ph. and Jonckheere, E. A., "LQG Balancing and Reduced LQG Compensation of Symmetric Passive Systems," *International Journal of Control*, Vol. 41, 1985, 73-109.

[11] Mustafa, D. and Glover, K., "Controller Reduction by H_∞-balanced truncation", *IEEE Transaction of Automatic Control*, AC-36, 1991, pp. 668-682.

[12] Scherpen , J. M.A., *Balancing for Nonlinear Systems*, Doctoral dissertation, University of Twente, The Netherlands, 1994.

[13] Curtain, R. F., and Pritchard, A. J., *Infinite Dimensional Linear System Theory*, Lecture Notes in Control and Information Sciences 8, Springer Verlag, Berlin, 1978.

[14] Bensoussan, A., Da Prato, G., Delfour, M. C., and Mitter, S. K., *Representation and Control of Infinite Dimensional Systems*, Vol. I, Systems & Control: Foundations & Applications, Birkhauser, Boston, 1992.

[15] Bensoussan, A., Da Prato, G., Delfour, M. C., and Mitter, S. K., *Representation and Control of Infinite Dimensional Systems*, Vol. II, Systems & Control: Foundations & Applications, Birkhauser, Boston, 1993.