Local Error Growth and Predictability of Ensemble Perturbations in Chaotic Dynamical Systems

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Abstract. In ensemble predictions, particularly in numerical weather forecasts, the true initial errors are usually not known, and can be optimally represented by an ensemble of perturbations. Thus the growth rate and predictability of an ensemble of perturbations have direct impact on the quality and predictability of an ensemble forecast. In this paper, local metric entropy (LME) is introduced and used as a measure of local error growth of an ensemble of perturbations in chaotic dynamical systems. The predictability time scale of a dynamical system during a given period of time can also be estimated with higher accuracy by using the LME. It is shown that LME, at any time during the evolution of a dynamical system, can be calculated as the sum of all the positive local Lyapunov exponents. LME does not depend upon the amplitudes nor the configurations of initial perturbations, it depends on the positive local LEs which are intrinsic properties of dynamical systems.
LME is adopted as a measure of local error growth to examine the dependence of an ensemble of perturbations to the dynamics of the system. In analysing local error growth rates, LME is compared with the local Lyapunov exponents, normal modes, optimal modes which are all commonly used in meteorological applictions. The correlations between LME, locally largest Lyapunov exponent, the first local Lyapunov exponent, growth rates of the first normal mode and first optimal perturbation are studied. When LME is used to estimate the predictability time scale of a system over a specified time period, it is found that the time scale defined by the LME is closer to the autocorrelation times for some variables than the commonly used Lyapunov time and Kolmogorov - Sinai time in the two dynamical systems we have tested.

1.   Introduction

It is now known that both local and finite-time Lyapunov exponents (LEs) are more directly related to predictability and error growth during certain periods than global Lyapunov exponents [Farrell & Ioannou, 1996; Vastano & Moser, 1991; Toth & Kalnay, 1993; Buizza & Palmer, 1995], since the most relevant and interesting timescales for most of our applications such as weather or climate predictions are from $t = 0.0$ to a finite-time, not infinite time. The global LEs are the long-time averaged growth rates and have almost no correlation with the local growth rates.

The local LEs describe the local growth rates, while finite-time LEs describe the fluctuations of growth rates for certain periods of time. However, when local LEs are used, usually only the first local LE and its associated Lyapunov vector (LV) is followed. For instance, the points on a trajectory with maximum instability are identified with the

maxima of a time-series of the first local LEs. This is not accurate in general, since the local growth rate of the first LV is not always the largest, however its long-time averaged growth rate is the largest. The largest local growth rate cannot be represented solely by the first LE. Different perturbation directions must be considered, and this is especially true for systems with many degrees of freedom.

In error growth and predictability studies, normal modes and optimal modes are also used to describe the local error growth and structure. Normal modes are the eigenvectors of the Jacobian matrix calculated at a particular point on the trjectory [Frederiksen & Bell, 1990; Farrell & Ioannou, 1996; Wei & Frederiksen, 1998]. When the basic state is time-independent, the leading normal modes associated with the largest real parts of eigenvalues will dominant the perturbation. Optimal modes are the eigen-modes of a symmetric matrix that is composed of the Jacobian and the transpose of the Jacobian [Frederiksen & Bell, 1990; Farrell & Ioannou, 1996]. Optimal modes provide the optimal local growth rates for a general dynamical system. However, in analysing the predictability time of a dynamical system, usually the first growth rate is used to characterize the predictability time scale. This may not be accurate in reality, since any true initial perturbations which are not known initially can not be necessarily represented by these largest growing modes.

In quantifying the predictability time scale of a dynamical system during certain period of time, Lyapunov time and Kolmogorov-Sinai time (K-S time), which are defined as the inverses of the first global LE and the Kolmogorov-Sinai entropy (K-S entropy) respectively, have been used most often to estimate predictability times. Again both Lyapunov time and K-S time are not accurate, since the first global LE does not include the contributions from the other LVs such as 2nd, 3rd ... LVs, and thus the Lyapunov predictability time is only expected to be valid for dynamical systems of low order. The K-S entropy only includes contributions from LEs whose long-time averages are positive; it ignores LEs whose long-time averages are negative but whose short-time contributions may be positive. These neglected contributions may be significant for large systems.

The predictabilities of nonlinear systems are severely limited due to their chaotic nature and difficulties of locating the true initial errors. For instance, weather forecasts still can not go beyond several days by single integrations of the most sophisticated numerical prediction models on powerful supercomputers. Ensemble predictions have already shown superiority to the single integrations of models in some major meteorology centers [Toth & Kalnay, 1993; Buizza & Palmer, 1995]. It is likely that all the daily weather forecasts by traditional single integrations will be replaced with ensemble forecasts in the near future. In fact one of the difficult tasks in numerical weather predictions is how to choose the initial perturbations which can optimally represent all the possible true errors. In reality these true errors, which are usually not known in advance, are best simulated by an ensemble of perturbations usually with equal probability. Thus for dynamical systems with some unknown true errors, the local error growth rates and the predictability times during a period of time are best estimated by analysing the local error growth rate and predictability time of an ensemble of perturbations.

The primary motivation for this study is to examine questions concerning the estimation of predictability time and measuring local instability and error growth of a dynamical system within chaotic dynamical theory framework. These include the following: (a) How to estimate the local error growth rate and predictability time of an ensemble of pertur-

bations during a specified period of time with higher accuracy. (b) Do the first local LE, the growth rates of first normal mode and first optimal mode provide good description of local error growth of a dynamical system? (c) If we assume that the true errors in practical dynamical systems such as numerical weather prediction models are represented by an ensemble of errors, then to what extent the true error growth rate can be described by the first local LE, the largest local LE, the growth rates of first normal mode and first optimal mode? To what extent the predictability time of a true error can be estimated using the first LE, K-S entropy, first normal mode and first optimal mode?

In this paper we use what we may call *local metric entropy* (LME) to estimate the predictabilty time scale for a dynamical system. We use the inverse of the average of the LME over a time interval as an estimate of characteristic predictability time. The time scale defined through LME is compared with Lyapunov time and K-S time for two dynamical systems originated from the atmospheric research. Since the predictability of a dynamical system is limited by all the contributions from the different expanding directions and the LME describes the local growth rate of information creation due to perturbation growth in all expanding directions, it is found that the time scale defined through LME is indeed of higher accuracy than Lyapunov time and K-S time for these two systems we have studied. The time scales defined by the growth rates of first normal mode, first optimal mode, first singular vector and first finite-time normal mode may not be accurate.

The local error growth rate and instability of a dynamical system might be better described by the LME as a function of time, the maximum instability points could be identified with higher accuracy. The cause of the instability can then be examined by analysing the basic state trajectory and the perturbations at the time when the LME is maximum. We also study the correlations between LME, locally largest LE which is not necessarily the first LE, growth rates of the first normal mode and first optimal mode.

Before the local metric entropy is presented in Sec. 3., we briefly discuss local Lyapunov exponents and K-S entropy in Sec. 2.. Numerical experiments of the Lorenz [1963] system and a simple barotropic model will be carried out in Sec. 4.. Finally, our discussion is summarized in Sec. 5..

2.   Local Lyapunov Exponents and K-S Entropy

Let $\mathbf{X}$ be an n-dimensional state vector in the phase space of a system described by the nonlinear evolution equation

$$\frac{d\mathbf{X}}{dt} = \mathbf{H}(\mathbf{X}), \tag{2.1}$$

where $\mathbf{H}$ is a nonlinear operator. Suppose that $\mathbf{x}$ is a small perturbation to the state vector $\mathbf{X}$. Then, for sufficiently short time, its evolution can be described by the linearized equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}. \tag{2.2}$$

Here $\mathbf{A} = d\mathbf{H}/d\mathbf{X}|_{\mathbf{X}(t)}$ is the tangent linear operator evaluated on the nonlinear trajectory $\mathbf{X}(t)$. Eq. (2.2) can be written in the integral form

$$\mathbf{x}(t) = \mathbf{G}(t, t_0)\mathbf{x}(t_0). \tag{2.3}$$

The operator $\mathbf{G}(t, t_0)$ is called the forward tangent propagator which maps perturbations along the nonlinear trajectory from an initial time t_0 to a future time t. For the application to weather prediction, if $\mathbf{x}(t_0)$ is the typical error in the initial condition for a weather forecast, then (2.2) and (2.3) should hold for about 2 - 3 days.

Generally, if we follow any random initial perturbation $\mathbf{x}(t_0)$ for the nonlinear system (2.1) using (2.2) or (2.3), this perturbation will be attracted to the fastest growing direction and gives rise to the largest LE such that $\lim_{t \to \infty}(1/(t - t_0)) \ln(||\mathbf{x}(t)||/||\mathbf{x}(t_0)||) = \lambda_1$, where $||\cdot||$ is the Euclidean norm. Thus $\mathbf{x}(t_0)$ will grow as $||\mathbf{x}(t)|| \approx ||\mathbf{x}(t_0)|| \exp \lambda_1(t - t_0)$ [Benetin *et al.*, 1980]. Consequently, if $||\mathbf{x}(t_0)|| = \delta_0$ and one accepts δ_{max} as the maximum tolerance error of the system, then the system is predictable up to time $T \sim (1/\lambda_1) \ln(\delta_{max}/\delta_0)$.

If one chooses the maximum tolerance δ_{max} as the e times of the initial error δ_0, the predictability time T_{λ_1} is just proportional to the inverse of the maximum LE [Lorenz, 1996]. That is, $T_{\lambda_1} \equiv (1/\lambda_1)$, where T_{λ_1} is often called Lyapunov time [Dellago & Posch, 1997].

Since we know that the most relevant and interesting timescales for weather or climate development are from $t = 0$ to a finite-time, it is necessary to define a finite-time and instantaneous quantity in order to describe the fluctuations of growth rate. We define finite-time Lyapunov exponents as $\lambda_i(t, \tau) = (1/\tau) \ln(||\mathbf{x}_i(t + \tau)||/||\mathbf{x}_i(t)||)$, where $\mathbf{x}_i(t) \in F_i(t)$, $F_i(t)$ are a set of disjoint subspaces defined in Eckmann & Ruelle [1985] and $\tau = L\Delta t$ (L is a positive integer, Δt is an infinitesimal integration time step). Clearly $\lambda_i(t, \tau)$ depend on τ, t and position of the trajectory $\mathbf{X}(t)$; they measure the average perturbation growth over a given interval τ. The global LEs are recovered by taking the limit $\tau \to \infty$. When $L = 1$, we call them *local Lyapunov exponents*, which are denoted by $\lambda_i(t)$. Thus

$$\lambda_i(t) = \frac{1}{\Delta t} \ln \frac{||\mathbf{x}_i(t + \Delta t)||}{||\mathbf{x}_i(t)||}. \tag{2.4}$$

Since both $F_i(t)$ and $\lambda_i(t)$ are local properties of the dynamical system (2.1), and they give the directions of perturbations and their growth rates in those directions respectively, they can be used to identify the local instabilities. It should be mentioned that these perturbations are generally different from the eigenfunctions of the local Jacobian at time t.

The algorithm we are going to use for computing the $\lambda_i(t)$ and $F_i(t)$ is based on the standard method [Benetin *et al.*, 1980; Shimada & Nagashima, 1979]. The method consists of evolving a set of initial orthonormal vectors, chosen at random in the tangent space $T^n(t)$ at $\mathbf{X}(t)$ by integrating the equations for both basic state flow (2.1) and the perturbations (2.2) or (2.3). The method determines the growth rates $\lambda_i(t)$ and the associated $F_i(t)$. The $F_i(t)$ components of a perturbation will grow or shrink exponentially with $\lambda_i(t)$.

The set of orthonormal vectors obtained by the standard method characterize the local directions of stretching or contraction of any perturbation and is identical to the set obtained by orthogonalizing the F_i starting from F_1 [Vastano & Moser, 1991]. We call this orthonormal set of vectors generated through the standard method *Lyapunov vectors*. The LVs span the same spaces as $F_i(t)$, and therefore their dimensions are the same, but $F_i(t)$ have the advantage that the exponential rates of growth or decrease of

any perturbation along $F_i(t)$ are given by $\lambda_i(t)$ when $t \to$ both $-\infty$ and $+\infty$ [Eckmann & Ruelle, 1985].

Note that the standard method for calculating the global LEs is also a numerical integration method. This means that t is not strictly continuous, but that a small finite time step Δt is used in the integrations. Let $t = k\Delta t$, and $k = 0, 1, 2, \ldots, N - 1$, then . (2.4) can be written as $\lambda_i(k\Delta t) = (1/\Delta t)\ln(||\mathbf{x}_i((k+1)\Delta t)||/||\mathbf{x}_i(k\Delta t)||)$ and the global LEs are calculated as

$$\lambda_i = \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\frac{1}{\Delta t}\ln\frac{||\mathbf{x}_i((k+1)\Delta t)||}{||\mathbf{x}_i(k\Delta t)||} = \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\lambda_i(k\Delta t). \qquad (2.5)$$

Then,

$$\begin{aligned}
\sum_{i=1}^{n}\lambda_i &= \sum_{i=1}^{n}\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\lambda_i(k\Delta t) = \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\sum_{i=1}^{n}\lambda_i(k\Delta t) \\
&= \lim_{t\to\infty}\frac{1}{t}\int_0^t \nabla\cdot\mathbf{H}(\tau)d\tau = \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}[\nabla\cdot\mathbf{H}(k\Delta t)], \qquad (2.6)
\end{aligned}$$

which shows that $\sum_{i=1}^{n}\lambda_i(k\Delta t) = \nabla\cdot\mathbf{H}(k\Delta t)$. Thus at any time t on the trajectory $\mathbf{X}(t)$ we have $\sum_{i=1}^{n}\lambda_i(t) = \nabla\cdot\mathbf{H}(t)$. This relation is confirmed in our computations, and in turn it verifies our numerical algorithms.

A system with sensitive dependence on initial conditions produces information in the sense that two different initial conditions, which are indistinguishable to a certain experimental precision, will evolve into distinguishable states in a finite time, because the difference between the two conditions grows exponentially fast [Eckmann & Ruelle, 1985]. Kolmogorov and Sinai applied the concept of metric entropy introduced by Shannon in his information theory to dynamical systems and were able to prove that this metric entropy is a topological invariant. This metric entropy describes the mean rate of creation of information, also known as measure-theoretic entropy. It is now called *Kolmogorov-Sinai entropy* [Eckmann & Ruelle, 1985].

Pesin [1977] was able to show that, under some conditions, a link exists between the K-S entropy of a domain V of the phase space and the LEs. He showed that $KS = \int_V \sum_{i=1}^{m}\lambda_i(\mathbf{X})\rho(\mathbf{X})d\mathbf{X}$, where m is the largest Lyapunov index number such that $\lambda_m > 0$ and $\rho(\mathbf{X})$ is the invariant natural measure, i.e. the probability density. In many cases, the λ_i are independent of $\mathbf{X}$, so that this equation can be simplified to $KS = \sum_{i=1}^{m}\lambda_i$. We will use this to calculate KS in this paper. If the probability density ρ is continuous in all directions F_i associated with positive λ_i, one has at least $KS \le \sum_{i=1}^{m}\lambda_i(\mathbf{X})$ [Eckmann & Ruelle, 1985; Argyris et al.,1994].

The K-S entropy has turned out to be an extremely useful quantity in nonlinear dynamics. It is invariant under changes of coordinates. It is often used to define chaos: a dynamical system is defined to be chaotic if it possesses a positive KS. Generally speaking, in higher dimensions the K-S entropy rather than the first LE characterizes the creation of information [Schuster, 1988; Argyris et al.,1994].

One of the most important applications of the K-S entropy is that it determines the average time over which the state of a chaotic system can be predicted [Schuster, 1988; Argyris et al.,1994]. It has been suggested that the predictability time for a dynamical

system, especially for a higher dimensional system, is $T \sim (1/KS) \ln(\delta_{max}/\delta_0)$ (Chapter 5 of Schuster [1988]). By analogy with T_{λ_1}, one can define a time scale associated with the KS as $T_{KS} \equiv (1/KS)$. T_{KS} is often called the Kolmogorov-Sinai time (see Dellago & Posch, 1997 and references therein). The arguments leading to the above equations are similar to those resulting in T_{λ_1}, but replacing λ_1 with KS. We shall calculate both T_{λ_1} and T_{KS} and compare them with T_p, a time scale based on the Local Metric Entropy.

Despite its usefulness, K-S entropy is of little help when studying local and finite-time error growth, which are of primary concern for weather forecasting as discussed in Sec. 2.. What we need is an indicator that is able to capture the essential feature of local error growth in a dynamical system and identify the local information creation rate which is closely related to local error growth and predictability.

As we mentiond in Sec. 1., in numerical weather forecasts the true initial errors are usually not known, and they can be optimally represented by an ensemble of perturbations with each member having equal probability in most cases. Hence qualifying the local growth rate and measuring the predictability time scale of an ensemble of perturbations of a dynamical system are particularly important in numerical weather forecasts. We provide such a tool in the next section.

3. Local Metric Entropy

At any time t, we consider an infinitesimal volume of ensemble points around $\mathbf{X}(t)$. All these points also represent an ensemble of perturbations with respect to the reference point $\mathbf{X}(t)$. When all the points including the reference point $\mathbf{X}(t)$ in this small volume $\Theta(t)$ move for the next time step according to (2.1), the perturbations are simultaneously propagated by the linear propagator $\mathbf{G}$, and this process results in the deformation of this small volume $\Theta(t)$. For a fixed initial time, the sum of all the global LEs is obtained by the volume growth rate when taking the limit of $t \rightarrow \infty$. [Benetin *et al.*, 1980; Shimada & Nagashima, 1979; Argyris *et al.*, 1994; Rasband, 1990].

If we follow the infinitesimal volume of this ensemble of perturbations, according to Shannon's information theory(see Schuster [1988] pp.110-113 and the Appendix F, and Hilborn [1994]), the information needed to describe the state of this ensemble of perturbations within $\Theta(t)$ is proportional to $S(t)$, where $S(t) = - \sum_{i \in \Theta(t)} P_i \ln P_i$, and P_i is the probability of the ith perturbation, where we define $P \ln P \equiv 0$ if $P = 0$. S is called the Shannon entropy and S is always positive.

Since we are interested in the change of local information of this ensemble of perturbations within $\Theta(t)$, we must consider the information change from t to $t + \Delta t$. Let $ME(t)$ denote the rate of this information change, then

$$ME(t) = \frac{dS}{dt} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} (S(t + \Delta t) - S(t)). \qquad (3.1)$$

This is our mathematical definition of $ME(t)$ describing the rate of local information change of an infinitesimal volume which contains an ensemble of perturbations. We call this *local metric entropy* due to its relationship with the K-S entropy.

Simliar definitions to (3.1) are given for a general dynamical system in Sec. 9.6 of Hilborn [1994], but the right-hand side of (3.1) is not studied further and only used as the first step to introduce the K-S entropy in this book. In this study we apply these definitions to an infinitesimal volume $\Theta(t)$ around $\mathbf{X}(t)$ at any time t, in particular we

will calculate $ME(t)$ defined by (3.1). We should note that $ME(t)$ is a local quantity, it doesn't cover the whole phase space, where "local" is meant both temporally and spatially. Due to the deformation nature of $\Theta(t)$ during the evolution, $S(t)$ is a function of time.

Following Hilborn [1994], if we average $ME(t)$ over the whole trajectories or over the entire attractor in a dissipative system, we have the K-S entropy, i.e.

$$KS = \lim_{N\to\infty} \lim_{\Delta t\to 0} \frac{1}{N\Delta t} \sum_{k=0}^{N-1} [S((k+1)\Delta t) - S(k\Delta t)] = \lim_{N\to\infty} \lim_{\Delta t\to 0} \frac{1}{N\Delta t} [S(N\Delta t) - S(0)],$$

where $t = k\Delta t$, and $k = 0, 1, 2, \ldots, N-1$. It should be pointed out that the information gain, or loss, is produced by the expanding directions only, and this process cannot be restored by contracting directions which do not generate new information. Hence, in studying the information change of the small volume of ensemble perturbations, we are only concerned with the expanding directions.

From Sec. 2. we know that at any time t, there are in general some positive local LEs which are associated with the expanding directions in the tangent space $T^n(t)$. Suppose that there are m positive local LEs at t, i.e. $\lambda_{i_1}(t), \lambda_{i_2}(t), \ldots, \lambda_{i_m}(t) > 0$, where $i_j = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Let $\mathbf{x}_{i_1}(t) \in F_{i_1}(t), \mathbf{x}_{i_2}(t) \in F_{i_2}(t), \ldots, \mathbf{x}_{i_m}(t) \in F_{i_m}(t)$ be the corresponding perturbation vectors which represent all the different expanding directions. Each of these positive λ_{i_j} corresponds to a local expansion of small areas along the direction of $\mathbf{x}_{i_j}$. In the vicinity of $\mathbf{X}(t)$, a piece of manifold in the subspace spanned by these perturbation vectors can be called *local unstable manifold* of $\mathbf{X}(t)$, and similarly a piece of manifold in the subspace spanned by the perturbation vectors with negative local LEs can be called *local stable manifold* of $\mathbf{X}(t)$ [Eckmann & Ruelle, 1985].

The volume of the m-dimensional parallelepiped on the unstable manifold at t is given by $V_m(t) = ||\mathbf{x}_{i_1}(t) \wedge \mathbf{x}_{i_2}(t) \wedge \ldots \wedge \mathbf{x}_{i_m}(t)||$, where the wedge operator denotes the exterior product of vectors; it is a generalization of the ordinary cross product of vectors in three dimensional space. Due to the expanding nature of the unstable manifold, after one time step $V_m(t)$ will become $V_m(t + \Delta t) = ||\mathbf{x}_{i_1}(t + \Delta t) \wedge \mathbf{x}_{i_2}(t + \Delta t) \wedge, \ldots, \wedge \mathbf{x}_{i_m}(t + \Delta t)||$. From (2.4) one has $\mathbf{x}_{i_1}(t + \Delta t) = \mathbf{x}_{i_1}(t) e^{\lambda_{i_1}\Delta t}, \mathbf{x}_{i_2}(t + \Delta t) = \mathbf{x}_{i_2}(t) e^{\lambda_{i_2}\Delta t}, \ldots, \mathbf{x}_{i_m}(t + \Delta t) = \mathbf{x}_{i_m}(t) e^{\lambda_{i_m}\Delta t}$. Therefore we have $V_m(t + \Delta t) = V_m(t) e^{\Delta t \sum_{j=1}^{m} \lambda_{i_j}(t)}$.

Note that, since $\mathbf{x}_i(t)$ $(\in F_i(t))$ are all very small perturbations in the tangent space at $\mathbf{X}(t)$, it is not difficult to see that $V_m(t)$ is also very small. One can think of this volume expansion process as follows: an m-parallelepiped of points whose volume is $V_m(t)$ at t evolves into another stretched m-parallelepiped of points with volume $V_p(t + \Delta t)$ at $t + \Delta t$.

At initial time, the m-parallelepiped of points contains an ensemble of initial conditions around $\mathbf{X}(t_0)$. The distances between all the points and the reference point $\mathbf{X}(t_0)$ consist of an ensemble of perturbations. At any time t, there is always an ensemble of expanding perturbations (say m perturbations) which are represented by $\mathbf{x}_{i_1}(t), \mathbf{x}_{i_2}(t), \ldots, \mathbf{x}_{i_m}(t)$ in the tangent space. The volume of this m-parallelepiped of expanding perturbations changes from time to time under the linear action of the propagator $\mathbf{G}$.

For a classical dynamical system, the number of states is generally proportional to the volume in phase space [Rasband, 1990]. We recall that in ensemble numerical weather predictions, all the initial errors are usually assumed to have equal probabilities to optimally represent the true unknown errors [Buizza & Palmer, 1995]. Here we assume

that the m-parallelepiped has $Q(t)$ states and each is equally probable. This assumption should be well satistified by the m-dimensional parallelepiped. One has $P_i = 1/Q$ and $S(t) = \ln Q(t)$, where $Q(t) = \alpha V_m(t)$ and α is a real constant. From (3.1), we have

$$ME(t) = \frac{1}{\Delta t} \ln \frac{Q(t + \Delta t)}{Q(t)} = \frac{1}{\Delta t} \ln \frac{V_m(t + \Delta t)}{V_m(t)} = \sum_{j=1}^{m} \lambda_{i_j}(t). \tag{3.2}$$

where $\lambda_{i_j}(t) > 0$. We notice that the local information creation rate is equal to the local volume expanding rate in the local unstable space under our assumption. It is assumed that at any time t, each member of the ensemble perturbations in the m-dimensional parallelepiped is equally probable. In fact this is not a strong restraint on most classical dynamical systems. Because the m-parallelepiped is composed of an ensemble of expanding perturbations, the local volume growth rate at any time t itself is fascinating and important in ensemble predictability study.

We remember that the standard method [Benetin *et al.*, 1980; Shimada & Nagashima, 1979] for calculating the global LEs was actually developed from considering the volume growth rate of a k-dimensional parallelepiped, here k is an integer and $k \leq n$. For a fixed initial time t_0, the k-th order LE which is the sum of the first k global LEs is obtained by taking the limit of $t \to \infty$. This has become a standard way of introducing the so-called standard method in textbooks [e.g. Argyris *et al.*, 1994; Rasband, 1990]. A similar consideration to this has been taken here in deriving the LME.

Equation (3.2) offers an easy way of calculating the local information creation rate for a dynamical system. We should emphasize again that $\sum_{j=1}^{m} \lambda_{i_j}(t)$ itself is very interesting to study, no matter if the assumption is valid or not. We will use this to calculate the LME in this paper.

$ME(t)$ is more useful than KS since it quantifies the local sustainable ensemble perturbation growth rate. It includes all the contributions from all the expanding perturbations at any time. The dependence of local ensemble perturbations on the dynamics of the flow fields can be studied by calculating LME. From eq. (3.2), one can see that LME doesn't depend upon the amplitudes nor the configurations of initial perturbations, it depends on the positive local LEs which are the intrinsic properties of dynamical systems. Thus LME directly represents the local instability properties of the flows. We adopt LME as a measure of the error growth in this paper.

It is expected that the maxima of ensemble instability can be identified by studying the time series of $ME(t)$. In the mean time $ME(t)$ can also be used to quantify the predictability time for a dynamical system during a given period of time. We define Local Metric Entropy time by $T_p \equiv (1/\langle ME(t) \rangle)$, where $\langle \cdot \rangle$ is the average over a time interval. This time interval can be extended to arbitrary time scales of interest. We will compare T_p with T_{λ_1} and T_{KS} in our numerical experiments.

4. Numerical Results

In this section, we carry out numerical experiments with two simple models, both of which originate from atmospheric dynamics, and test the utility of $ME(t)$. The independent variables in these two models are non-dimensional and, for time integrations, a fourth-order Runge-Kutta scheme is used with a time step 0.001 units in all the numerical computations.

4.1. The Lorenz system The first model is the well-studied Lorenz system[Lorenz 1963] with only three degrees of freedom. This system has been studied largely as a good example of a simple nonlinear dynamical system exhibiting chaotic behaviour. The relevant differential equations are $dx/dt = -\Sigma x + \Sigma y$; $dy/dt = Rx - y - xz$; $dz/dt = xy - bz$. We choose the parameters as follows: $\Sigma = 16.0$, $b = 4.0$ and $R = 45.92$. From almost any initial point in the phase space the trajectory will converge to the familiar butterfly strange attractor. Any small perturbation vector at any point along a time-varying trajectory is described by the tangent linear system.

We start the integration for 19990 steps from the initial condition $(10.0, 0.0, 30.0)$ which is chosen at random. This is to allow the trajectory converge onto the attractor. Once the trajectory is on the attractor, we then simultaneously integrate both the nonlinear system and the associated tangent linear system for another 5×10^6 steps. This will ensure that the trajectory is on a well established attractor. All our numerical experiments discussed below for the Lorenz system are based on the last 2000 steps segment, i.e. from $t = 4998 \times 10^3$ to 5×10^6.

Fig. 1a shows the first local LE, $\lambda_1(t)$, the LME, $ME(t)$, the real part of the largest eigenvalue, $\mu_1(t)$, of the tangent linear matrix $\mathbf{A}(t)$ which appears in (2.2), the first optimal local growth rate, $\sigma_1(t)$, and the K-S entropy, KS. The instantaneous optimal growth rate is defined as the largest eigenvalues of $\frac{1}{2}(\mathbf{A} + \mathbf{A}^\dagger)$ [Frederiksen & Bell 1990; Farrell & Ioannou, 1996]. The $Re[\mu_i(t)]$ can be understood as the growth rates along the normal modes. It is interesting to compare $Re[\mu_1(t)]$ with $\sigma_1(t)$, $\lambda_1(t)$ and $ME(t)$, although the normal mode theory applies when the basic flow is steady, i.e. $Re[\mu_i]$ can be used to quantify the perturbation growth when $\mathbf{A}$ is independent of time.

Displayed in Fig. 1b are the 2nd and 3rd LEs, both global and local. One can easily see that the periods during which $\lambda_1(t) < 0$, but $\lambda_2(t) > 0$, do exist in this short time interval. This means that $\lambda_1(t)$ fails to identify the error growth property during some time intervals. Both $\lambda_3(t)$ and λ_3 are always negative, which shows that the perturbation is a decaying structure along the 3rd LV direction. It is this feature that makes the strange attractor possible. One can also see that the optimal growth rate bounds nearly all the instantaneous growth rates when these growth rates are positive during this selected time segment. Both $\sigma_1(t)$ and $Re[\mu_1(t)]$ suffer from the similar problem as $\lambda_1(t)$, i.e. there are some times when $\sigma_1(t) < 0$ and $Re[\mu_1(t)] < 0$ while as $\lambda_2(t) > 0$. We conclude that neither $\lambda_1(t)$ nor $\sigma_1(t)$ and $Re[\mu_1(t)]$, can be used to quantify the local perturbation growth during all time periods in this simple dynamical system.

Fig. 2a shows the linear correlations of the $ME(t)$ with $\lambda_i(t)$ where $i = 1, 2, 3$, $Re[\mu_1(t)]$ and $\sigma_1(t)$. A near zero coefficient between $ME(t)$ and $\lambda_2(t)$ indicates that they are nearly uncorrelated. We expect that $\lambda_1(t)$ makes a large contribution to $ME(t)$ since the correlation coefficient between them is close to 1. In contrast, $\lambda_2(t)$, which oscillates around zero and has small magnitude, contributes little to the $ME(t)$. We understand that $\lambda_1(t) + \lambda_2(t) + \lambda_3(t) = -21.0$ for the Lorenz system. Hence, one would expect a highly negative relationship (close to -1) between $ME(t)$ and $\lambda_3(t)$, i.e. an increase in $ME(t)$ is accompanied by a decrease in $\lambda_3(t)$. This is also shown in Fig. 2a. The correlations of $ME(t)$ with both $Re[\mu_1(t)]$ and $\sigma_1(t)$ are also close to 1.

We have seen from Fig. 2a that $\lambda_1(t)$ dominates the $ME(t)$. This is not surprising, since the total number of degrees of freedom of the system is only three. It is also the reason why $\lambda_1(t)$ is able to quantify the local error growth to some extent. However, this
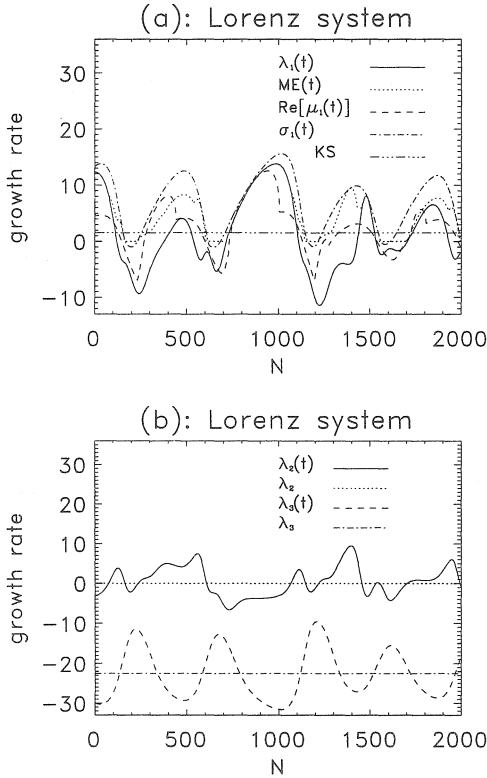
(a): Lorenz system

Fig. 1. Numerical results from the Lorenz system during the last 2000 steps. (a) The time evolutions of $\lambda_1(t)$, $ME(t)$, $Re\{\mu_1(t)\}$, $\sigma_1(t)$ and K-S entropy KS. (b) The 2nd and 3rd global and local LEs.
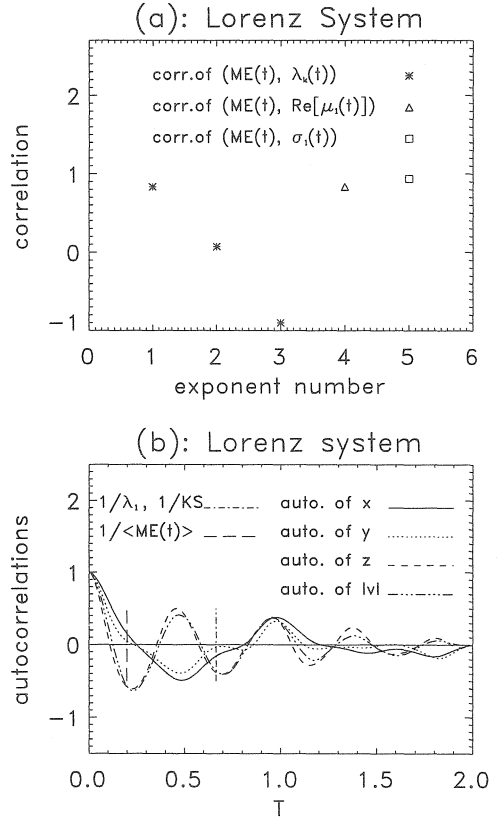
Fig. 2. (a) Linear correlations of $ME(t)$ with $\lambda_k(t)$, here $k = 1, 2, 3$, $Re\{\mu_1(t)\}$ and $\sigma_1(t)$. (b) The time series of autocorrelation functions of x, y, z and $|\mathbf{v}|$ with the delay time T. Also shown are $T_{\lambda_1} = (1/\lambda_1)$ and $T_p = (1/\langle ME(t)\rangle)$.

is not accurate for large dynamical systems as we will see in our next numerical example. In general the larger the systems are, the greater the difference between $ME(t)$ and $\lambda_1(t)$ would be, thus the smaller the correlations between them would be.

We argue that for large dynamical systems, $ME(t)$ should be used to quantify the local perturbation growth. This is particularly true in numerical weather forecasts where the true initial errors are not known. A common and realistic practice is to represent the unknown true errors by an ensemble of perturbations. Hence, to study the growth rate of an unknown error in a dynamical system, it is best to analyze the error growth rate of an ensemble of perturbations. With $ME(t)$, we don't need to integrate all the ensemble members, because $ME(t)$ describes the local error growth rate of an ensemble of perturbations.

Next, we examine the extent to which the future state of a chaotic dynamical system can be predicted from an initially chosen condition. A traditional measure of the loss

of information during prediction has been the autocorrelation time of a variable of a dynamical system. For a chaotic system the autocorrelation of a variable that is sensitive to initial conditions rapidly falls to zero as the delay time T is increased. Beyond a certain time T_c the variable is uncorrelated with itself and the system loses the memory of its previous states [Argyris et $al.$,1994]. We define this autocorrelation time T_c as the first zero of the autocorrelation function. Fig. 2b shows the autocorrelation functions of x, y, z and the Euclidean norm of the phase velocity $|\mathbf{v}|$ with the delay time T. Also shown are $T_{\lambda_1} = (1/\lambda_1)$ and $T_p = (1/\langle ME(t)\rangle)$. The Lyapunov time is the same as the K-S time since $KS = \lambda_1$ for this system. It is evident that T_p is much closer to all the autocorrelation times of x, y, z and $|\mathbf{v}|$ than T_{λ_1} or T_{KS}, thus supporting our argument that T_p is a useful measure of predictability time.

4.2.   Simplified baroptropic model The second dynamical system that we consider consists of severe truncation of the barotropic vorticity equation describing two-dimensional flow on the sphere. We use the spectral method and represent our system by 27 independent variables without using any fast Fourier transform and the popular spectral transform method[Bourke, 1972].

Barotropic model is the earliest developed and simplest model for weather prediction [Bourke, 1972]. For flow on a sphere, the barotropic vorticity equation is given by $\frac{\partial \xi}{\partial t} = -J(\psi, \xi + 2\Omega\mu) - \eta\nabla^2\xi$, where $J(\psi, \xi) = \frac{\partial \psi}{\partial \lambda}\frac{\partial \xi}{\partial \mu} - \frac{\partial \psi}{\partial \mu}\frac{\partial \xi}{\partial \lambda}$, also $\xi = \nabla^2\psi$ is the vorticity, ψ is the streamfunction, t is time, λ is longitude, μ is sine of latitude, Ω is the earth's angular velocity and η is the coefficient of eddy viscosity.

We expand the streamfunction and vorticity in spherical harmonics, for example

$$\psi(\lambda, \mu, t) = \sum_{m,l} \psi_{ml}(t) P_l^m(\mu) \exp(im\lambda),$$

where $P_l^m(\mu)$ are orthonormalised Legendre functions, m is the zonal wave number and l is the total wave number. The prognostic spectral equations may then be expressed in terms of $Re(\psi_{ml})$ and $Im(\psi_{ml})$. With

$$\mathbf{X} = (\ldots, Re(\psi_{ml}), \ldots, Im(\psi_{ml}), \ldots)^T$$

denoting the column of real and imaginary parts of ψ_{ml}, the spectral equations can be written formally in the form of (2.1). For simplicity, the spectral equation is truncated to have only 27 independent variables. Details of the model can be found in Wei [1996] and Wei & Frederiksen [1998].

As with the Lorenz model, we do not propose to use this simple model to simulate any real atmospheric phenomena, as it is far too simple compared with current standard numerical prediction models. The main reason for using this model is that it has many more degrees of freedom than the Lorenz model. Thus the difference between the LME, first LE and K-S entropy can be better demonstrated. Secondly, it is simple to use and computationally very cheap.

First let us look at the inviscid situation, in this case the system is conservative. The dynamics in different regions of the phase space could be very different [Benetin et $al.$, 1980]. Fig. 3 shows the numerical results from Run A whose statistical and ergodic properties were studied in Wei [1996]. The integration is from the same initial condition as that in Wei [1996] and carried out for 95500 steps. Displayed in Fig. 3a are $ME(t)$, the first

three local LEs ($\lambda_1(t)$, $\lambda_2(t)$ and $\lambda_3(t)$) and local exponential growth rate corresponding to the most unstable Lyapunov vector $\lambda_m(t) = max[\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)]$.
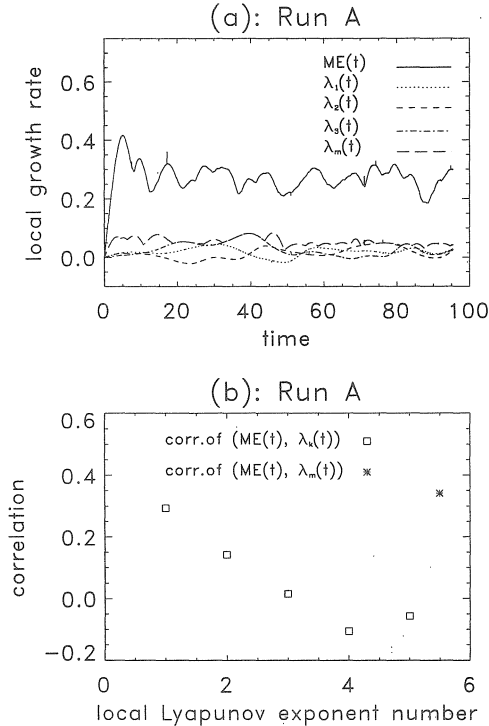

(a): Run A

(b): Run A

It is obvious that $\lambda_m(t)$ is very different from $\lambda_1(t)$. This shows that $\lambda_1(t)$ is far from being the maximum growth rate. $ME(t)$ is different from both $\lambda_m(t)$ and $\lambda_1(t)$, the correlations of $ME(t)$ with them are very small, 0.35 and 0.3 respectively, as displayed in Fig. 3b. Unlike the Lorenz system, where $ME(t)$ is dominated by $\lambda_1(t)$ because of the very small number of degrees of freedom, there is almost no linear relationship between $ME(t)$ and $\lambda_1(t)$ or $\lambda_m(t)$ here. Also shown in Fig. 3b are the correlations between $ME(t)$ and the other dominant local LEs $\lambda_k(t)$ with $k = 1, 2, 3, 4, 5$. All of these correlations are quite small in this system.

Similar numerical analyses from Runs B and C, which were studied along with Run A in Wei[1996] have been considered, but with an integration interval of 10^4 steps. The results are not shown here, the above conclusions for Run A also hold for these other two runs. As discussed above, we argue that the LME is a useful measure of the time scale of predictability for a dynamical system. Next, we examine this for different values of viscosity for Runs A, B and C. After initial nonlinear integrations for 10^4 steps, both nonlinear and perturbation equations are then integrated simultaneously for another 10^5 steps for Runs B and C and 10^6 steps for Run A, which converges more slowly than Runs B and C.

Fig. 3. Numerical results from Run A. (a) The time evolutions of $ME(t)$, the top three numbered local LEs ($\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)$) and local exponential growth rate corresponding to the most unstable Lyapunov vector $\lambda_m(t) = max\{\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)\}$. (b) Correlations of $ME(t)$ with $\lambda_k(t)$, where $k = 1, 2, 3, 4, 5$ and $\lambda_m(t)$.

The numerical results displayed in Fig. 4 are based on the last 10^4 steps for each of these three runs. As in Fig. 2b, we compare T_p, T_{λ_1} and T_{KS} with the autocorrelation time of the Euclidean norm of phase position $|\mathbf{d}| = \sqrt{\sum_{k=1}^{n} |X_k(t)|^2}$. Again T_p is much closer to the autocorrelation time T_c compared with T_{λ_1} and T_{KS} for all values of viscosity that we considered in Run A. This is also true for Run B. For Run C, all the predictability time scales are relatively close to T_c with the T_{λ_1} even closer.

However, we should mention that the autocorrelation time estimates the rate of information loss based on the evolution of just one variable. The autocorrelation times for different variables may be different as demonstrated in Fig. 2b for the Lorenz system.

Nevertheless, our numerical results do indicate that T_p is reasonably close to the auto-correlation time T_c of $|\mathbf{d}|$ for all the tested cases. Both Lyapunov time T_{λ_1} and the K-S time T_{KS} are very different from T_c in Runs A and B. Runs A, B and C start from very different regions in the phase space. The first LE and K-S entropy in each run are in fact the average values for the period we have integrated.

5. Conclusions and Remarks

We have presented theoretical arguments and numerical experiments in this paper supporting the usefulness of the Local Metric Entropy. The utility of LME has been demonstrated in comparisons with Lyapunov exponents (both global and local) and K-S entropy in studies of error growth. In particular, we have demonstrated that the LME employed in this paper is very useful in describing the local error growth rate of an ensemble of perturbations and estimating the predictability time of dynamical systems during given period of time. The local growth rate of an ensemble of perturbations with equal probability is determined by the sum of positive local LEs which are the intrinsic properties of a dynamical system, it doesn't depend upon the amplitudes nor the configurations of initial perturbations. It is evident that LME directly represents the local instability properties of the flows. The above conclusions make LME particularly important and useful in ensemble weather forecasts.

The method is by no means limited to the dynamical systems derived from atmospheric dynamics, although these models were used here to illustrate our ideas. The LME is expected to be potentially useful in dealing with chaotic phenomena in a wide range of applications. When applied to more general complex dynamical systems, the method can improve our understanding of the detailed mechanism of true local error growth and the related structures. At the same time, the characteristic predictability time limit during a specified period of time can be estimated with better accuracy.

The LME is easier to compute than the LEs and the K-S entropy. It should be applicable to systems in which $KS = \sum_{\lambda_i > 0} \lambda_i$ doesn't hold, e.g. one of the notable restrictions for this relation is that the LEs are the same for almost all initial conditions, i.e. independent of position in phase space. Without this simple restriction, the K-S entropy can not be found easily.

We notice that when LME is maximum, the instability properties have not been shown significantly different compared with the instability properties when $\lambda_1(t)$ is maximum in these two simple dynamical systems. The advantages of LME may be better demonstrated in larger systems. Applications to more complex dynammical systems are being carried out. In fact we are planing to use the method proposed in this paper to atmospheric numerical models to study atmospheric and climate phenomena such as blocking and El Niño-Southern Oscillation in the future.

In the other application aspects, we expect that the local LVs associated with the largest local error growth rates would be good perturbations for ensemble predictions which will be widely used at the meteorology centers around the world. These vectors are somehow similar to the "breeding" vectors which have been used at NCEP (National Centers for Enviromental Prediction, USA) [Toth & Kalnay, 1993]. The breeding vectors are actually the superpositions of the leading LVs sampled at different periods of time, but they are not the vectors associated with the largest local LEs. This is demonstrated clearly in our two experiements.

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The importance of ensemble predictions has been realized due to the difficulty of finding the true intial errors and the chaotic nature of atmosphere and climate systems. The future numerical weather forecasts will be based on ensemble predictions. One of the critical challenges to the scientists in meteorology community is how to choose the best possible initial enemble perturbations which can represent the true errors as accurate as possible. Since these true errors are not known, they are assumed to be represented by the fastest growing combinations of possible analysis errors. At ECMWF (European Centre for Medium-Range Weather Forecasts), singular vectors have been used as initial errors for the ensemble prediction system [Buizza & Palmer, 1995].

Fig. 4. Comparison of time scales based on different measures as functions of viscosity for (a) Run A; (b) Run B; (c) Run C.

Since operational ensemble forecasts began at ECMWF and NCEP, there has been controversy in the meteorology literature about the suitability of their different methods for generating the ensemble initial perturbations [Toth & Kalnay, 1993; Buizza & Palmer, 1995; Farrell & Ioannou, 1996]. Are the initial ensemble perturbations best represented by the breeding vectors, fastest growing singular vectors or the LVs associated with the largest local growth rates? There is still no definite answer to this question yet. It would be interesting to compare the local error growth rate and the predictability time scale defined by LME with the skills of an ensemble prediction systems based on breeding vectors, singular vectors, finite-time normal modes and Monte Carlo method that was first studied by Leith [1974] in meteorology. In a Monte Carlo ensemble forecast, the large number of initial perturbations are usually chosen at random with equal probability in order to locate the true unknown errors. All the members of the ensemble will be integrated, this is very expensive. By using the LME, we don't need to carry out this large number of integrations.
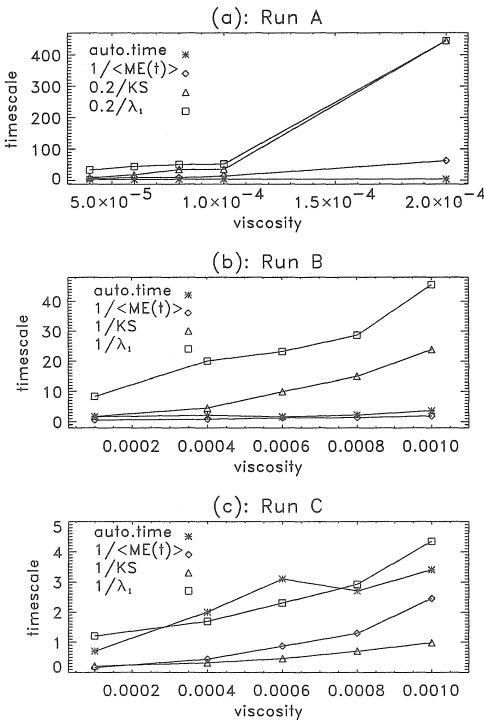
Acknowledgments

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