

COMPUTATIONAL IMPLEMENTATIONS OF THE DORODNITSYN  
BOUNDARY LAYER FORMULATION

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1. INTRODUCTION

Conventionally the boundary layer governing equations are discretised by treating the velocity components,  $u$  and  $v$ , as the dependent variables and the coordinates,  $x$  and  $y$  in two dimensions, as the independent variables. However there are many advantages in adopting a Dorodnitsyn boundary layer formulation which uses a non-dimensional normal velocity gradient as the dependent variable and  $x$  and  $u$  as the independent variables.

An immediate computational advantage is that an infinite domain in the  $y$  direction is replaced by a finite domain in  $u$ ;  $u$  is scaled to vary between zero and unity in traversing the boundary layer. The scaling of  $u$  means that the grid automatically captures the boundary layer growth in the downstream direction. In  $(x,y)$  space periodic readjustment of the boundary layer grid at the downstream stations is computationally expensive.

In the Dorodnitsyn formulation it is convenient to specify a uniform grid in the  $u$  direction. For the finite element Dorodnitsyn formulation this permits a higher accuracy to be achieved. In contrast in physical space a non-uniform grid is invariably required which implies, for the finite difference or finite element method, a larger truncation error than if a uniform grid is used. The use of a uniform grid in  $u$ -space provides high resolution in physical space adjacent to the wall. This is particularly important for turbulent boundary layers.

For two-dimensional flows the Dorodnitsyn formulation offers the additional advantage of avoiding the explicit appearance of the normal velocity component,  $v$ . Although it can be recovered if required. Consequently only one equation is solved with the Dorodnitsyn formulation.

By choosing the non-dimensional velocity gradient as the dependent variable the shear stress is computed accurately. This is particularly important in determining the skin friction behaviour.

## 2. DORODNITSYN BOUNDARY LAYER FORMULATION

In this section the Dorodnitsyn formulation of the equations governing two-dimensional, turbulent boundary layer flow with blowing or suction in the normal direction, is described.

For two-dimensional turbulent boundary layer flow the governing equations can be written in the following form,

$$\partial u / \partial x + \partial v / \partial y = 0 \quad (1)$$

$$u \partial u / \partial x + v \partial u / \partial y = u_e \frac{du_e}{dx} + \{1/Re\} \partial \{ (1 + \nu_T / \nu) \partial u / \partial y \} / \partial y \quad (2)$$

In eq. (2) the expression  $\rho \nu_T \partial u / \partial y$  has replaced the Reynolds stress,  $-\overline{\rho u'v'}$ , where  $\nu_T$  is the eddy viscosity. The equation system, (1) and (2) is parabolic in character and requires initial conditions

$$u(x_0, y) = u_i(y) \quad \text{and} \quad v(x_0, y) = v_i(y) \quad (3)$$

and boundary conditions,

$$u(x, 0) = 0 \quad , \quad v(x, 0) = v_w \quad \text{and} \quad u(x, \infty) = u_e(x) \quad (4)$$

where  $v_w$  is the prescribed normal velocity at the wall.

In eqs. (2) and (4),  $u_e(x)$  is the known velocity distribution at the outer edge of the boundary layer. In eqs. (1) to (4)  $u$ ,  $u_e$  and  $v$  have been nondimensionalised with a reference velocity  $U_\infty$  and  $x$  and  $y$  have been nondimensionalised with a reference length  $L$ . Consequently the Reynolds number,  $Re = U_\infty L / \nu$ .

The following variables are introduced,

$$\xi = \int_0^x u_e(x') dx' \quad , \quad \eta = Re^{\frac{1}{2}} u_e y$$

$$u' = u / u_e \quad , \quad v' = Re^{\frac{1}{2}} v / u_e \quad \text{and} \quad w = v' + \eta u' \{ \partial u_e / \partial \xi \} / u_e \quad .$$

Then eqs. (1) and (4) take the form,

$$\frac{\partial u'}{\partial \xi} + \frac{\partial w}{\partial \eta} = 0 \quad (5)$$

$$u' \frac{\partial u'}{\partial \xi} + w \frac{\partial u'}{\partial \eta} = \frac{1}{u_e} \frac{\partial u_e}{\partial \xi} + \frac{\partial}{\partial \eta} \left\{ (1 + \nu_T / \nu) \frac{\partial u'}{\partial \eta} \right\} \quad , \quad (6)$$

with auxiliary conditions,

$$u' = 0 \quad , \quad w = Re^{\frac{1}{2}} v_w / u_e \quad \text{at} \quad \eta = 0 \quad \text{and} \quad u' = 1 \quad \text{at} \quad \eta = \infty \quad . \quad (7)$$

Equations (5) and (6) are combined in the following way,

$$f_k \times \text{equation (5)} + (df_k / du') \times \text{equation (6)} = 0 \quad , \quad (8)$$

where  $f_k(u')$  is a general test function. Evaluation of eq. (8), after dropping the superscript  $'$ , gives

$$\frac{\partial}{\partial \xi} (u f_k) + \frac{\partial}{\partial \eta} (w f_k) = \frac{1}{u_e} \frac{\partial u_e}{\partial \xi} \frac{\partial f_k}{\partial u} (1-u^2) + \frac{\partial f_k}{\partial u} \frac{\partial}{\partial \eta} \left\{ (1+v_T/v) \frac{\partial u}{\partial \eta} \right\}. \quad (9)$$

An integration with respect to  $\eta$  is made,

$$\frac{\partial}{\partial \xi} \int_0^\infty u f_k d\eta + \left[ w f_k \right]_0^\infty = \frac{1}{u_e} \frac{du_e}{d\xi} \int_0^\infty \frac{df_k}{du} (1-u^2) d\eta + \int_0^\infty \frac{df_k}{du} \frac{\partial}{\partial \eta} \left\{ (1 + \frac{v_T}{v}) \frac{\partial u}{\partial \eta} \right\} d\eta. \quad (10)$$

The function,  $f_k$ , is chosen so that  $f_k(\infty) = 0$ . The second stage of the Dorodnitsyn formulation changes the variable of integration from  $\eta$  to  $u$  and introduces new dependent variables,

$$T = 1/\bar{H} = \partial u / \partial \eta. \quad (11)$$

As a result eq. (10) becomes

$$\frac{\partial}{\partial \xi} \int_0^1 u f_k \bar{H} du - R_e^{1/2} F f_k(0) = \frac{1}{u_e} \frac{du_e}{d\xi} \int_0^1 \frac{df_k}{du} (1-u^2) \bar{H} du + \int_0^1 \frac{df_k}{du} \frac{\partial}{\partial u} \left\{ (1+v_T/v) T \right\} du, \quad (12)$$

where the wall-blowing parameter,  $F = v_w/u_e$ .

Equation (12) is the Dorodnitsyn turbulent boundary layer formulation with a known normal velocity,  $v_w$ , at the wall. In the original Dorodnitsyn method, polynomial trial solutions for  $\bar{H}$  and  $T$  are utilised. The original Dorodnitsyn formulation (the *Method of Integral Relations*) is effective as long as the number of unknown coefficients in the trial solution is small (say  $N = 2$  to  $4$ ). The Method of Integral Relations is described by Holt (1977).

### 3. DORODNITSYN SPECTRAL FORMULATION

In this formulation the polynomial trial functions and the test function  $f_k$ , in eq. (12) are replaced by related *orthonormal* functions,  $g_k(u)$ . The orthonormal functions are constructed as follows,

$$g_k(u) = \sum_{r=1}^k e_{rk} (1-u)^r \quad (13)$$

where the coefficients  $e_{rk}$  are evaluated via the Gram-Schmidt orthonormalisation process (Isaacson and Keller, 1966) so that

$$\int_0^1 g_j(u) g_k(u) w(u) du = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \quad (14)$$

The appropriate form of  $w(u)$  will be indicated below.

The trial solution is

$$\textcircled{H} = \frac{1}{(1-u)} \left[ b_0 + \sum_{j=1}^{N-1} b_j g_j(u) \right]. \quad (15)$$

With  $g_k$  replacing  $f_k$  in eq. (12), and substitution of eq. (15), the following is obtained

$$\frac{d}{d\xi} \int_0^1 \left[ b_0 + \sum_{j=1}^{N-1} b_j g_j(u) \right] g_k(u) \left\{ u/(1-u) \right\} du = C_k \quad (16)$$

where

$$C_k = \text{Re} \frac{1}{2} F g_k(0) + \frac{1}{u} \frac{du}{d\xi} e \int_0^1 \frac{dg_k}{du} (1-u^2) \textcircled{H} du + \int_0^1 \frac{dg_k}{du} \frac{\partial}{\partial u} \left\{ \frac{(1+v_T/v)}{\textcircled{H}} \right\} du. \quad (17)$$

A comparison of eqs. (14) and (16) indicates that the choice,  $w(u) = u/(1-u)$ , permits a *significant* simplification of eq. (16). That is eq. (16) becomes

$$db_k/d\xi = C_k - C_N v_k/v_N, \quad k = 1, \dots, N-1 \quad (18)$$

and 
$$db_0/d\xi = C_N/v_N, \quad \text{when } k = N. \quad (19)$$

where  $v_k$  etc. can be evaluated, once and for all, as

$$v_k = \int_0^1 g_k(u) u/(1-u) du. \quad (20)$$

The spectral formulation is implemented by numerically integrating eqs. (18) and (19) in the  $\xi$  direction. The variable-step, variable-order predictor-corrector method due to Gear (1971) is particularly suitable for this purpose.

Accurate solutions using the Dorodnitsyn spectral formulation are obtained with, typically,  $N = 4$  to  $6$  in eq. (15). The Dorodnitsyn spectral boundary layer formulation has been applied to incompressible (Fletcher and Holt, 1975) and compressible (Fletcher and Holt, 1976) laminar flows and to incompressible (Yeung and Yang, 1981; Fletcher and Fleet, 1984b) and compressible (Fleet and Fletcher, 1983) turbulent flows.

#### DORODNITSYN FINITE ELEMENT FORMULATION

Trial solutions are introduced for  $\textcircled{H}$  and  $(1+v_T/v)T$ , in eq. (12), in the following way,

$$\textcircled{H} = \sum_{j=1}^M N_j(u)/(1-u) \theta_j(\xi) \quad (21)$$

and

$$(1+v_T/v)T = \sum_{j=1}^M (1-u)N_j(u) (1+v_T/v) \tau_j(\xi) \quad (22)$$

In eqs. (21) and (22) the factor  $(1-u)$  has been introduced to ensure that  $\theta$  and  $T$  have the correct behaviour at the edge of the boundary layer. The terms  $N_j(u)$  are one-dimensional shape functions, typically linear or quadratic (Fletcher, 1984).

In eq. (22) the trial solution has been introduced for the group of terms,  $(1+v_T/v)T$ . This is an example of the *group finite element formulation* (Fletcher, 1983) and is partly responsible for the very economical execution of the current algorithm.

The test function,  $f_k(u)$ , is given the following form,

$$f_k(u) = (1-u)N_k(u), \quad (23)$$

which ensures that  $f_k(u) = 0$  at the outer edge of the boundary layer,  $u=1$ .

The substitution of eqs. (21) to (23) into eq. (12) indicates that a modified Galerkin method is produced. Evaluation of the various integrals produces the following system of ordinary differential equations for the nodal values,  $\theta_j$  and  $\tau_j$ ,

$$\begin{aligned} \sum_{j=1}^M CC_{kj} \, d\theta_j/d\xi = \operatorname{Re}^{\frac{1}{2}} u_e F \delta_{1k} + \left\{ \frac{1}{u_e} \frac{du_e}{d\xi} \right\} \sum_{j=1}^M EF_{kj} \theta_j \\ + u_e \sum_{j=1}^M AA_{kj} (1+v_T/v) \tau_j \end{aligned} \quad (24)$$

where  $\delta_{1k} = 1$  if  $k = 1$   
 $= 0$  if  $k \neq 1$ .

The various coefficients in eq. (24) are given by

$$CC_{kj} = \int_0^1 N_k N_j u \, du, \quad EF_{kj} = \int_0^1 N_j \left\{ \frac{dN_k}{du} (1-u) - N_k \right\} (1+u) \, du \quad (25)$$

$$\text{and} \quad AA_{kj} = \int_0^1 \left( \frac{dN_j}{du} (1-u) - N_j \right) \left( \frac{dN_k}{du} (1-u) - N_k \right) \, du.$$

The system of equation (24), has a very compact form due to *simultaneously* prescribing trial solutions for  $\theta_j$  and  $\tau_j$ . However this feature prevents eq. (11) being satisfied except at the nodes, where  $\theta_j = 1/\tau_j$ , or in the limit  $M \rightarrow \infty$ .

An efficient non-iterative, implicit marching algorithm for the system of equations,(25), is described by Fletcher and Fleet (1984a). This paper also provides numerical convergence results for the Dorodnitsyn finite element formulation.

A comparison with finite difference solutions (STAN 5) obtained for turbulent boundary layers in different pressure gradients is made by Fletcher and Fleet (1984b). STAN5 (Reynolds, 1976) is typical of the more efficient finite difference boundary layers formulations. Solutions obtained with the present method, DOROD-FEM, demonstrate comparable accuracy to those produced by STAN5. However DOROD-FEM is about *ten times more economical* than STAN5.

The superior economy comes partly from the ability of the Dorodnitsyn formulation to obtain accurate solutions with fewer points across the boundary layer, and partly from the economical implementation of the finite element method.

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