SINC METHODS OF APPROXIMATE SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

Frank Stenger

1. INTRODUCTION AND SUMMARY

In the author's experience, nearly all solutions of PDE (partial differential equations) encountered in applications are piecewise analytic in each variable. Except in the case of inverse problems, we can predict a priori the regions of analyticity of the solutions of PDE. The solutions of linear PDE are analytic whenever the coefficients of the PDE are analytic, although singularities may also occur on the boundary of the region.

In this paper we derive two families of methods for solving second order PDE. Each of these families is based on the Whittaker cardinal function, or sinc function expansion of a function f defined on the real line $\mathbb{R}$. This expansion takes the form

$$C(f,h) = \sum_{k\in\mathbb{Z}} f(kh)S(k,h) \tag{1.1}$$

where $h > 0$, $\mathbb{Z} = \{0,\pm1,\pm2,...\}$, and where the sinc function $S(k,h)$ is defined by

$$S(k,h)\circ(x) = \frac{\sin\{\frac{\pi}{h}(x-kh)\}}{\frac{\pi}{h}(x-kh)} , \quad k \in \mathbb{Z} . \tag{1.2}$$

While formulas for approximating derivatives of function defined on $\mathbb{R}$ are immediately obtainable from (1.1), we obtain formulas over other intervals (or contours) Γ via the use of a function ϕ, where ϕ is a one-to-one transformation of Γ onto $\mathbb{R}$. Using N^p points in p dimensions, the resulting approximate solutions converge at the rate $O(N^{3p/2}\exp(-\gamma N^{\frac{1}{2}}))$ where the constant $\gamma > 0$ is the largest possible in the case when the $\{Lip_\alpha\} \cap \{analytic\}$ class housing the solution is explicitly known, i.e., in that case there does not exist a method which converges at a faster rate [1].

The paper is organized as follows. In Sec. 2 we briefly review the relevant properties of the cardinal function (1.1). These formulas are then extended in Sec. 3, to the derivation of two classes of methods for reducing PDE to the solution of a system of algebraic equations. One of these, which we refer to as formulas of type I, has been derived elsewhere [5], while the other, the class of formulas of type II, is new. Both classes of formulas have the same order of convergence. The type I formulas have been tested on model problems, while the type II formulas have not. The two classes of formulas are considerably different except in the case when $\Gamma = \mathbb{R}$, although at this stage the advantage of one type over the other are not clear. Both types lead to full matrices, as opposed to sparse matrices for the case of finite difference and finite element methods, although the rate of convergence of the methods of this paper is considerably faster than that of finite difference and finite element methods. In Sec. 4 we consider some special regions, and we show that within these regions the formulas of type I and II converge at the same rate. In Sec. 5 we describe some function spaces for solving PDE in $\mathbb{R}^n$. In Sec. 6 we illustrate the solution of some ODE and PDE boundary value problems via methods of this paper.

2. SOME PROPERTIES OF THE CARDINAL FUNCTION

The results of this section, taken from [6], consist of certain identities of the Whittaker cardinal function which we shall use to derive the formulas of Sec. 3.

Definition 2.1: Let $h > 0$, and let $B(h)$ denote the family of all functions f that are analytic in the entire complex plane $\mathbb{C}$, such that

$$|f(z)| \leq C e^{\pi|z|/h}, \qquad (2.1)$$

and such that $f \in L^2(\mathbb{R})$. Let $C(f,h)$ and $S(k,h)$ be defined by (1.1) and (1.2) respectively, and let us set

$$\delta_{jk}^{(n)} = \left(\frac{d}{dx}\right)^n S(j,1)(x)\Big|_{x=k}.$$

In particular, we have

$$\begin{cases} \delta_{jh}^{(0)} = \begin{cases} 1 & \text{if } j=k \\ 0 & \text{if } j \neq h \end{cases} \\ \delta_{jh}^{(1)} = \begin{cases} 0 & \text{if } f=k \\ \dfrac{(-1)^{k-j}}{k-j} & \text{if } j \neq h \end{cases} \\ \delta_{jk}^{(2)} = \begin{cases} -\pi^2/3 & \text{if } j=k \\ \dfrac{-2(-1)^{k-j}}{(k-j)^2} & \text{if } j \neq h \end{cases} \end{cases} \qquad (2.3)$$

Theorem 2.2: Let $f \in B(h)$. Then

(a) $$f(z) = C(f,h,z) \quad \text{for all} \quad z \in \mathbb{C} ; \tag{2.4}$$

(b) $$\int_{\mathbb{R}} f(x)\,dx = h \sum_{k \in \mathbb{Z}} f(kh) ; \tag{2.5}$$

(c) $$\int_{\mathbb{R}} f(t) S(k,h) \circ (t)\,dt = hf(kh) , \quad k \in \mathbb{Z} ; \tag{2.6}$$

(d) $$f' \in B(h) ; \tag{2.7}$$

3. FORMULAS FOR DISCRETIZING DIFFERENTIAL EQUATIONS

The definitions, notations and results of this section are important to the rest of the paper. Two families of formulas are derived for a general contour Γ . These two families reduce to a single family for the case when $\Gamma = \mathbb{R}$.

3.1 The Domain $\mathcal{D}_d$ and Approximating Functions on $\mathbb{R}$.

Definition 3.1: Let $\mathbb{R}$ denote the real line, $\mathbb{R} = (-\infty,\infty)$, let $\mathbb{C} = \{z=x+iy : x \in \mathbb{R} , y \in \mathbb{R}\}$, and let $\mathbb{Z} = \{k : k=0,\pm1,\pm2,\ldots\}$. Let d and h be positive numbers and set

$$\mathcal{D}_d = \{z \in \mathbb{C} : |\operatorname{Im} z| < d\} \tag{3.1}$$

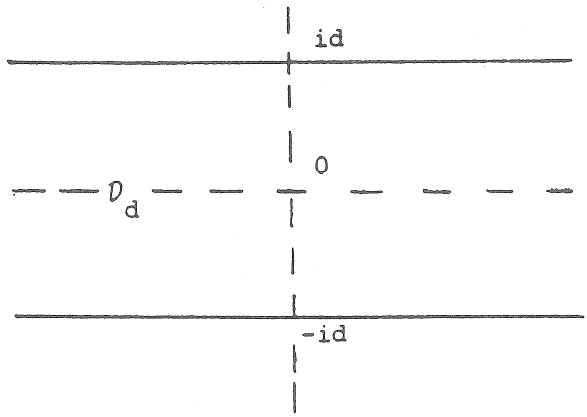

Fig. 3.1.
The Region $\mathcal{D}_d$ of Eq. (3.1).

3.2 The More General Domain $\mathcal{D}$ and Approximation on Γ.

Definition 3.2: Let $\mathcal{D}$ be a simply connected domain in the complex plane $\mathbb{C}$, and denote by $\partial\mathcal{D}$ the boundary of $\mathcal{D}$. Let a and b $(b\neq a)$ be boundary points of $\mathcal{D}$, and let ϕ be a conformal map of $\mathcal{D}$ onto $\mathcal{D}_d$, such that $\phi(a) = -\infty$, $\phi(b) = \infty$. Let $\psi = \phi^{-1}$ denote the inverse map, and set

$$\Gamma = \{\psi(x) : x \in \mathbb{R}\} ; \qquad (3.2)$$

$$z_k = \psi(kh) , k \in \mathbb{Z} . \qquad (3.3)$$

Let $B(\mathcal{D})$ denote the family of all functions F that are analytic in $\mathcal{D}$, and such that

$$N(F,\mathcal{D}) \equiv \int_{\partial\mathcal{D}} |F(z)dz| \equiv \inf_{C\to\partial\mathcal{D}, C\subset\mathcal{D}} \int_C |F(z)dz| < \infty . \qquad (3.4)$$

3.3 Formulas of Type I.

Theorem 3.3 [4]: Let $F \in B(\mathcal{D})$, and let $S(k,h)$ be defined as in (1.2). Then

(a) For all $x \in \Gamma$

$$\frac{F(x)}{\phi'(x)} - \sum_{k \in \mathbb{Z}} \frac{F(z_k)}{\phi'(z_k)} S(k,h) \circ \phi(x)$$

$$= \frac{\sin[\pi\phi(x)/h]}{2\pi i} \int_{\partial\mathcal{D}} \frac{F(z)\,dz}{[\phi(z)-\phi(x)]\sin[\pi\phi(z)/h]} \; ; \tag{3.5}$$

(b)

$$\int_\Gamma F(x)\,dx - h \sum_{k \in \mathbb{Z}} \frac{F(z_k)}{\phi'(z_k)}$$

$$= \frac{i}{2} \int_{\partial\mathcal{D}} \frac{F(z)\exp[(i\pi\phi(z)/h)\operatorname{sgn}\operatorname{Im}\phi(z)]\,dz}{\sin[\pi\phi(z)/h]} \; ; \tag{3.6}$$

(c)

$$\int_\Gamma F(x) S(k,h) \circ \phi(x)\,dx - \frac{hF(z_k)}{\phi'(z_k)}$$

$$= \frac{(-1)^k ih}{2\pi} \int_{\partial\mathcal{D}} \frac{F(z)\exp[(i\pi\phi(z)/h)\operatorname{sgn}\operatorname{Im}\phi(z)]}{\phi(z)-kh}\,dz \; . \tag{3.7}$$

Moreover, if the left hand sides of (3.5), (3.6) and (3.7) are denoted by $\eta_1(x)$, η_2 and η_3 respectively, we have

$$\left\{ \begin{aligned} |\eta_1(x)| &\le \frac{N(F,\mathcal{D})}{2\pi d \sinh(\pi d/h)} , \quad x \in \Gamma \; ; \\ |\eta_2| &\le \frac{e^{-\pi d/h} N(F,\mathcal{D})}{2\sinh(\pi d/h)} \; ; \\ |\eta_3| &\le \frac{h}{2\pi d} e^{-\pi d/h} N(F,\mathcal{D}) \; . \end{aligned} \right. \tag{3.8}$$

Assumption 3.4: In addition to $F \in B(\mathcal{D})$, let us assume that for all $x \in \Gamma$,

$$\frac{|F(x)|}{\phi'(x)} \leq C\, e^{-\alpha|\phi(x)|} \tag{3.9}$$

where C and α are positive numbers.

Theorem 3.5 [6]: Let $\delta_{i,N}$ denote the left hand side of Eq. (2.6+i), $i = 1,2,3$, for the case when the infinite sums $\Sigma_{k \in \mathbb{Z}}$ are replaced by finite sums, $\Sigma_{k=-N}^{N}$. Let $F \in B(\mathcal{D})$ satisfy (3.9) on Γ. If h is selected by the formula

$$h = \left(\frac{\pi d}{\alpha N}\right)^{1/2} \tag{3.10}$$

then there exist constants C_i which are independent of N such that

$$|\delta_{i,N}| \leq C_i N^{1/2} \exp\{-(\pi d \alpha N)^{1/2}\}, \quad i = 1,2,3 . \tag{3.11}$$

If h is selected by the formula

$$h = (2\pi d/\alpha N)^{1/2} \tag{3.12}$$

then there exists a constant C_2 such that

$$|\delta_{2,N}| \leq C_2 \exp\{-(2\pi d \alpha N)^{1/2}\} . \tag{3.13}$$

If h is selected by the formula

$$h = \gamma/N^{1/2} \tag{3.14}$$

where γ is a positive constant, then there exist positive constants C' and δ such that

$$|\delta_{i,N}| \leq C'\, e^{-\delta N^{1/2}}, \quad i = 1,2,3 . \tag{3.15}$$

We remark that the bounds (3.11) and (3.15) apply uniformly for all $x \in \Gamma$, in the case when $\eta_{i,N}$ depends on x.

Theorem 3.6 [5]: (a) If $\phi' F \in B(\mathcal{D})$, then

$$\left| \int_\Gamma F'(x) S(k,h)\circ\phi(x)\,dx + \sum_{j\in\mathbb{Z}} \delta_{kj}^{(1)} F(z_j) \right| \tag{3.16}$$

$$\leq \left\{ \frac{1}{2d\ \tanh(\pi d/h)} + \frac{h}{2\pi d^2} \right\} e^{-\pi d/h} N(\phi' F, \mathcal{D}) \ .$$

(b) If $\phi' F \in B(\mathcal{D})$, $F(x)/[\phi(x)\phi'(x)] \to 0$ as $x \to a$ and as $x \to b$ along Γ, then

$$\left| \int_\Gamma \frac{F''(x)}{\phi'(x)} S(k,h)\circ\phi(x)\,dx \right. \tag{3.17}$$

$$- \sum_{j\in\mathbb{Z}} \left[\frac{1}{h}\delta_{kj}^{(2)} - \frac{\phi^{(2)}(z_j)}{\phi^{(1)}(z_j)^2}\delta_{kj}^{(1)} \right.$$

$$\left. \left. + \frac{h}{\phi'(z_j)} (1/\phi')''(z_j)\delta_{kj}^{(0)} \right] F(z_j) \right|$$

$$\leq e^{-\pi d/h} \left\{ \left| \frac{1}{d^2 \tanh(\pi d/h)} + \frac{\pi}{2hd} + \frac{h}{\pi d^3} \right| N(\phi' F, \mathcal{D}) + \right.$$

$$+ \left[\frac{1}{2d\tanh(\pi d/h)} + \frac{h}{2\pi d^2} \right] N(\phi^{(2)} F/\phi', \mathcal{D})$$

$$\left. + \frac{h}{2\pi d} N((1/\phi')'' F, \mathcal{D}) \right\} .$$

Moreover, if

$$|F(x)| \leq C' e^{-\alpha|\phi(x)|} \quad , \ x \in \Gamma \ , \tag{3.9'}$$

taking $h = [\pi d/(\alpha N)]^{\frac{1}{2}}$, then replacing the infinite sums $\Sigma_{j\in\mathbb{Z}}$ in (3.18) and (3.19) by finite sums $\Sigma_{j=-N}^{N}$ then the right hand sides of the resulting (3.18) and (3.19) become $C_1' N^{\frac{1}{2}} e^{-(\pi d\alpha N)^{\frac{1}{2}}}$ and $C_2' N e^{-(\pi d\alpha N)^{\frac{1}{2}}}$ respectively, where C_1' and C_2' are contants which are independent of N.

Corollary 3.7: (a) If F' and $\phi' F$ are in $B(\mathcal{D})$, then

$$\left| \frac{hF'(z_k)}{\phi'(z_k)} + \sum_{j\in\mathbb{Z}} \delta_{kj}^{(1)} F(z_j) \right| \leq C_1 e^{-\pi d/h} ; \qquad (3.18)$$

(b) If F''/ϕ' and $\phi' F$ are in $B(\mathcal{D})$, and if $F'(x)/[\phi(x)\phi'(x)] \to 0$ as $x \to a$ and as $x \to b$ along Γ, then

$$\left| \frac{hF''(z_k)}{\phi'(z_k)^2} - \sum_{j\in\mathbb{Z}} \left[\frac{1}{h} \delta_{kj}^{(2)} - \frac{\phi^{(2)}(z_j)}{\phi^{(1)}(z_j)^2} \delta_{kj}^{(1)} + \frac{h}{\phi'(z_j)} (1/\phi')''(z_j) \right] F(z_j) \right| \qquad (3.19)$$

$$\leq \frac{C_2}{h} e^{-\pi d/h} .$$

In (3.18) and (3.19) C_1 and C_2 are constants independent of h.

Theorem 3.8: Let $F \in B(\mathcal{D})$. Then

$$\left| \int_a^{z_k} F(t)dt - h \sum_{j\in\mathbb{Z}} [\tfrac{1}{2} + \sigma_{k-j}] \frac{F(z_j)}{\phi'(z_j)} \right| \qquad (3.20)$$

$$\leq \frac{2hN(F,\mathcal{D})}{\pi d \sinh(\pi d/h)} ,$$

where the integral on the left hand side is taken along Γ and where σ_k is defined as in Eq.(2.16) of [6].

Proof: Using the formula (2.16) of [6], we get

$$\int_a^{z_k} S(j,h)\circ\phi(t)\phi'(t)dt = h[\tfrac{1}{2}+\sigma_{k-j}] \ . \qquad (3.21)$$

Also, we have for $\xi \in \mathbb{R}$,

$$\int_{-\infty}^{kh} \frac{\sin(\pi t/h)dt}{t-\xi\pm id} = \sum_{j=-\infty}^{k} \int_{(j-1)h}^{jh} \sin(\pi t/h)\left\{\frac{t-\xi}{(t-\xi)^2+d^2} \mp \frac{id}{(t-\xi)^2+d^2}\right\} dt \ . \qquad (3.22)$$

Hence, by taking the largest term of each alternating series on the right hand side, we find that the left hand side of (3.22) is bounded by $4h/d$. By combining these results with (3.5), we get (3.20).

Finally, we remark that if F/ϕ' satisfies (3.9), then by choosing $h = [\pi d/(\alpha N)]^{\frac{1}{2}}$ and replacing the infinite sum $\Sigma_{j\in\mathbb{Z}}$ (3.20) by $\Sigma_{j=-N}^{N}$, we may replace the right hand side of the resulting (3.20) by $C''e^{-(\pi d\alpha N)^{\frac{1}{2}}}$, where C'' is a constant.

3.4 Formulas of Type II

Whereas the formulas of the previous section are based on the interpolating function $S(k,h)\circ\phi$, the new formulas of the present section are based on the interpolating functions s_k, where s_k is defined by

$$s_k(x) = \frac{h}{\pi}\,\frac{\sin\{\pi[\phi(x)-kh]/h\}}{x-z_k} \ , \ k \in \mathbb{Z} \ . \qquad (3.23)$$

If $\phi' F \in B(\mathcal{D})$, then we may write (3.5) in the form

$$|F(x) - \sum_{k \in \mathbb{Z}} F(z_k) S(k,h) \circ \phi(x)| \leq \frac{N(\phi' F, \mathcal{D})}{2\pi d \sinh(\pi d/h)} , \qquad (3.24)$$

i.e., the sum in (3.24) converges uniformly for all $x \in \Gamma$. This will also be the case for the interpolation formulas derived in this section. However, the formulas of this section will also converge, although not necessarily uniformly, even if only $F \in B(\mathcal{D})$. We are thus led to interpolation in the sense of a relative error.

Let $F \in B(\mathcal{D})$, and let us set

$$N(F/(\cdot - x), \mathcal{D}) = \int_{\partial} \left| \frac{F(x)}{z-x} dz \right| , \quad x \in \Gamma . \qquad (3.25)$$

If $\sup_{(x \in \Gamma)} N(F/(\cdot - x), \mathcal{D}) < \infty$, we set

$$M(F, \mathcal{D}) = \sup_{x \in \Gamma} N(F/(\cdot - x), \mathcal{D}) . \qquad (3.26)$$

Theorem 3.9: Let $F \in B(\mathcal{D})$. Then for $x \in \Gamma$,

$$\eta_1(x) \equiv F(x) - \sum_{k \in \mathbb{Z}} \frac{F(z_k)}{\phi'(z_k)} S_k(x) \qquad (3.27)$$

$$= \frac{1}{2\pi i} \int_{\partial \mathcal{D}} \frac{F(z) \sin[\pi \phi(x)/h] dz}{(z-x) \sin[\pi \phi(z)/h]} .$$

Moreover,

$$|\eta_1(x)| \leq \frac{N(F/(\cdot - x), \mathcal{D})}{2\pi \sinh(\pi d/h)} . \qquad (3.28)$$

Proof: We omit the proof, which is similar to previously obtained results above.

Theorem 3.10: Let $F \in B(\mathcal{D})$. Then

$$\eta_2 \equiv \int_\Gamma F(x)s_k(x)dx - hF(z_k) \tag{3.29}$$
$$= \frac{(-1)^k ih}{2\pi} \int_{\partial\mathcal{D}} \frac{F(z)\exp\{[i\pi\phi(z)/h]\,\mathrm{sgnIm}\phi(z)\}dz}{z-z_k} .$$

In particular,

$$|\eta_2| \le \frac{h}{2\pi} N(F/(\cdot - z_k),\mathcal{D})e^{-\pi d/h} . \tag{3.30}$$

<u>Theorem 3.11</u>: (a) If F and F'/ϕ' are in $B(\mathcal{D})$, then

$$\eta_3 \equiv \frac{hF'(z_k)}{\phi'(z_k)} + h\left\{-\frac{\phi^{(2)}(z_k)}{\phi'(z_k)^2}F(z_k) + \sum_{j\neq k} \frac{(-1)^{j-k}F(z_j)}{(z_j-z_k)\phi'(z_j)}\right\}$$

$$= \frac{i}{2}\int_\partial \exp\{[i\pi\phi(z)/h]\,\mathrm{sgnIm}\phi(z)\}\left\{\frac{F'(z)(-1)^k\sin[\pi\phi(z)/h]}{\phi'(z)(z-z_k)}\right.$$
$$+ F(z)\left[\frac{(-1)^k\cos[\pi\phi(z)/h]}{z-z_k} + \right. \tag{3.31}$$
$$\left.\left. + \frac{(-1)^k h}{\pi}\sin[\pi\phi(z)/h]\,\frac{d}{dz}\,\frac{1}{\phi'(z)(z-z_k)}\right]\right\}dz .$$

In particular,

$$|\eta_3| \le \frac{1}{2}\,e^{-\pi d/h}\left\{N([F'/\phi']/(\cdot-z_k),\mathcal{D}) + \frac{N(F/(\cdot-z_k),\mathcal{D})}{\tanh(\pi d/h)}\right.$$
$$\left. + \frac{h}{\pi}N(F/[\phi'(\cdot-z_k)]',\mathcal{D})\right\} ; \tag{3.32}$$

(b) If F''/ϕ'^2 and F are in $B(\mathcal{D})$, then

$$\eta_4 = \frac{h^2 F''(z_k)}{\phi'(z_k)^2} - h^2\left\{\left[-\frac{\pi^2}{3h^2} - \frac{5}{3}\,\frac{\phi^{(3)}(z_k)}{\phi'(z_k)^3} + \frac{6\phi^{(2)}(z_k)^2}{\phi'(z_k)^4}\right]F(z_k)\right.$$

$$\left. + \sum_{j\neq k}\left[-\frac{3\phi^{(2)}(z_j)}{\phi'(z_j)^3(z_j-z_k)} - \frac{2}{\phi'(z_j)^2(z_j-z_k)^2}\right](-1)^{j-k}F(z_j)\right\} \tag{3.33}$$

$$= \frac{(-1)^k ih}{2}\int_{\partial\mathcal{D}} \frac{\exp\{[i\pi\phi(z)/h]\,\mathrm{sgn}\,\mathrm{Im}\,\phi(z)\}}{\sin[\pi\phi(z)/h]}\left\{\frac{F''(z)\sin[\pi\phi(z)/h]}{\phi'(z)^2(z-z_k)}\right.$$

$$+ F(z)\left[\left\{\frac{\phi^{(2)}(z)}{(z-z_k)\phi'(z)^2} + 2\phi'(z)\frac{d}{dz}\left(\frac{1}{(z-z_k)\phi'(z)^2}\right)\right\}\cos[\pi\phi(z)/h]\right.$$

$$\left.\left. + \left\{-\frac{(\pi/h)}{z-z_k} + (h/\pi)\,\frac{d^2}{dz^2}\,\frac{1}{\phi'(z)^2(z-z_k)}\right\}\sin[\pi\phi(z)/h]\right]\right\}dz\ .$$

In particular,

$$|\eta_4| \le \frac{h}{2}\,e^{-\pi d/h}\left\{\left[N(F''/\{\phi'^2(\cdot-z_k)\},\mathcal{D}) + \right.\right.$$

$$+ N(F\phi^{(2)}/[\phi'^2(\cdot-z_k)],\mathcal{D})$$

$$\left. + 2N(\phi' F[1/\{(\cdot-z_k)\phi'\}',\mathcal{D}\right]\frac{1}{\tanh(\pi d/h)} +$$

$$+ (\pi/h)N(F/(\cdot-z_k),\mathcal{D})$$

$$\left. + h/\pi\ N(F[1/\{\phi'^2(\cdot-z_k)\}''],\mathcal{D})\right\}\ .$$

4. APPLICATIONS FOR SPECIAL TRANSFORMATIONS

Various transformations ϕ and their relationships with formulas of type I have been dealt with in [6]. We now illustrate some of these transformations, and then discuss the role of these transformations with respect to the formulas of type II. In addition to having F analytic in $\mathcal{D}$, we shall assume that for some constants $C > 0$ and $\alpha > 0$

$$|F(x)| \leq C\, e^{-\alpha|\phi(x)|} \quad , \; x \in \Gamma . \qquad (4.1)$$

As for formulas of type I, if the function to be approximated does not go to zero at end-points of Γ, we first subtract off a linear part (see [6,p.188]).

Ex. 4.1: $\Gamma = [0,1]$. In this case we take

$$\mathcal{D} = \{z \in \mathbb{C} : |\arg[z/(1-z)]| < d\} , \quad 0 < d < \pi$$

$$\phi(z) = \log[z/(1-z)] \qquad (4.2)$$

$$z_k = e^{kh}/(1+e^{kh}) \; , \quad k \in \mathbb{Z} .$$

The inequality (4.1) is satisfied if for all $z \in \mathcal{D}$,

$$|F(z)| \leq C|z(1-z)|^{\alpha} . \qquad (4.3)$$

Ex. 4.2: $\Gamma = [-1,1]$. In this case we take

$$\mathcal{D} = \{z \in \mathbb{C} : |\arg[(1+z)/(1-z)]| < d\} , \; 0 < d < \pi$$

$$\phi(z) = \log[(1+z)/(1-z)] \qquad (4.4)$$

$$z_k = (e^{kh}-1)/(e^{kh}+1) \; , \; k \in \mathbb{Z} .$$

The inequality (4.1) is satisfied if for all $z \in \mathcal{D}$,

$$|F(z)| \leq C|(1-z^2)|^{\alpha} . \qquad (4.5)$$

Ex. 4.3: $\Gamma = [0,\infty]$. In this case we take

$$\mathcal{D} = \{z \in \mathbb{C} : |\arg z| < d\}$$

$$\phi(z) = \log z \tag{4.6}$$

$$z_k = e^{kh}, \quad k \in \mathbb{Z}.$$

The inequality (4.1) is satisfied if for all $z \in \mathcal{D}$,

$$|F(z)| \leq C|z/(1+z)^2|^{\alpha}. \tag{4.7}$$

Ex. 4.4: $\Gamma = [0,\infty]$. In this case we take

$$\mathcal{D} = \{z \in \mathbb{C} : |\arg \sinh(z)| < d\}, \quad 0 < d < \pi$$

$$\phi(z) = \log \sinh(z) \tag{4.8}$$

$$z_k = \log[e^{kh} + (1+e^{2kh})^{\frac{1}{2}}], \quad k \in \mathbb{Z}.$$

The inequality (4.1) is satisfied if for all $z \in \mathcal{D}$,

$$|F(z)| \leq C|z\, e^{-z}|^{\alpha}. \tag{4.9}$$

Lemma 4.5: For each transformation of Examples 4.1 to 4.4, we have

$$|s_k(x)| \leq e^{h}, \quad x \in \Gamma. \tag{4.10}$$

Proof: Since $|\sin\{\frac{\pi}{h}[\phi(x)-kh]\}| \leq \frac{\pi}{h}[\phi(x)-kh]$ we have,

$$|s_k(x)| \leq \frac{|\phi(x)-kh|}{\phi'(z_k)\,|x-z_k|} = \frac{\phi'(\xi)}{\phi'(z_k)} \tag{4.11}$$

where ξ lies between z_k and x. Now for each of the above examples we have for $x_{n-1} \leq x \leq x_{k+1}$

$$|\phi'(\xi)| \leq \max \phi'(z_{k\pm1}). \tag{4.12}$$

By considering each of the four cases separately, (4.10) follows from (4.11) and (4.12) if $z_{k-1} \leq x \leq z_{k+1}$.

Next, if $x \in \Gamma$ but $x \notin [z_{k-1}, z_{k+1}]$, we use the inequality

$$|\sin[(\pi/h)[\phi(x)-kh]\}| \leq 1 ,$$

to get

$$|s_k(x)| \leq \frac{(h/\pi)}{\phi'(z_k)|x-z_k|} . \qquad (4.13)$$

Once again, by considering each of the four cases separately, we find that (4.10) is satisfied for $x \in \Gamma$, $x \notin [z_{k-1}, z_{k+1}]$. For example, for the case of $\phi(x) = \log x$, we have

$$\frac{(h/\pi)}{\phi'(z_k)|x-z_k|} \leq \frac{(h/\pi)}{\phi'(z_k)|z_{k\pm 1}-z_k|} = \frac{h/\pi}{|e^{\pm h}-1|} \leq e^h/\pi .$$

We omit the other cases, which are dealt with similarly.

Lemma 4.5 enables us to prove

<u>Theorem 4.6</u>: Let ϕ be defined by one of Ex.4.1 to 4.4 above. Let F be analytic in $\mathcal{D}$ and let F satisfy (4.9) on Γ . Let N be a positive integer, and select h by the formula $h = [\pi d/(\alpha N)]^{\frac{1}{2}}$. Let η_i be defined by any one of Eqs. (3.27), (3.31) or (3.33), and let $\eta_{i,N}$ denote the left hand sides of these equations obtained by replacing $\Sigma_{j\in\mathbb{Z}}$ by $\Sigma_{j=-N}^{N}$. Then

$$|\eta_{i,N}| \leq C_i \; N^{\frac{1}{2}} \; e^{-(\pi d\alpha N)^{\frac{1}{2}}} ,$$

where C_i are constants independent of N .

5. FUNCTION SPACES FOR REGIONS IN $\mathbb{R}^n$

The types of regions V in $\mathbb{R}^n$ which we shall consider are a union of regions of the form

$$K_n = \{(\xi^1,\ldots,\xi^n) \in \mathbb{R}^n : u^1 \le \xi^1 \le v^1,\ u^2(\xi^1) \le \xi^2 \le v^2(\xi^1)\ , \quad (5.1)$$
$$\ldots, u^n(\xi^1,\ldots,\xi^{n-1}) \le \xi^n \le v^n(\xi^1,\ldots,\xi^{n-1})$$

where it is assumed that the u^i and v^i are either identically infinite or else they are bounded a.e. on their domain of definition. The sinc approximation methods of the previous section are best suited for product-type approximations over rectangular regions of the form

$$V_n = \{(x^1,\ldots,x^n) \in \mathbb{R}^n : u^i \le x^i \le v^i, i=1,2,\ldots,n\}, \quad (5.2)$$

where in (5.2), (u^i,v^i) is any one of $(0,1)$, $(-1,1)$, $(0,\infty)$ or $(-\infty,\infty)$.

Let us first consider a suitable function space on V_n . To this end, let α and d be positive constants. Let ϕ_i be a conformal map of the region $\mathcal{D}_i \subset \mathbb{C}$ onto $\mathcal{D}_d$ (Eq. (3.1)), let $\psi_i = \phi_i^{-1}$ denote the inverse map, such that $\Gamma_i = [u_i, v_i] = \{\psi_i(\omega) : \omega \in \mathbb{R}\}$. Let $(x^1,\ldots,x^{i-1}, x^{i+1},\ldots,x^n)$ be a fixed point in the region $\prod_{j\neq i} \Gamma_j$, and denote by $F_i = F_i(z)$ the function $F(x^1,\ldots,x^{i-1},z,x^{i+1},\ldots,x^n)$. Let $B_\alpha(V_n)$ denote the family of all functions F defined on V_n such that for $i=1,2,\ldots,n$, F_i is analytic in analytic in $\mathcal{D}_i$ and such that G_i defined by

$$G_i = -\frac{e^{-\phi^i/2}}{e^{\phi_i/2}+e^{-\phi_i/2}} F_i(u^i) - \frac{e^{\phi^i/2}}{e^{\phi_i/2}+e^{-\phi^i/2}} F_i(v^i) + F_i \quad (5.3)$$

satisfies

$$|G^i(z)| \leq C\, e^{-\alpha|\phi^i(z)|}$$

for all z in $\mathcal{D}_i$, and such that ϕ^i, α, C and $\mathcal{D}_i$ are independent of $(x^1,\ldots,x^{i-1},x^{i+1},\ldots,x^n) \in \prod_{j\neq i} \Gamma_j$.

In this case the function F and its derivatives may be accurately approximated on V_n via products of the one-dimensional formulas derived in this paper, and by [7] the error of approximation is that of one-dimensional formulas.

Given a problem over the region K_n defined by (5.1), we first transform K_n into the region V_n of the form (5.2). This can always be accomplished via simple linear transformations. For example, if u^i and v^i are finite everywhere, we set

$$\xi^i = u^i + (v^i-u^i)x^i \quad (5.5)$$

so that, if e.g. this is the case for $i=1,2,\ldots,n$ the new region in the variables x^i is now the cube $\prod_{i=1}^{n} [0,1]$. If $u^i=-\infty$, $v^i=\infty$ we do not make a transformation, unless some other advantages may be gained. If u^i is a function and $v^i=\infty$, we set

$$\xi^i = u^i + x^i\,, \quad (5.6)$$

and if $u^i=-\infty$ and v^i is a function, we set

$$\xi^i = v^i - x^i . \tag{5.7}$$

The transformed function under (5.5), (5.6) or (5.7) in the variables x^i will belong to $B_\alpha(V^n)$ if the u^i and v^i appearing in these transformations belong to $B_\alpha(V^{i-1})$ ($i > 1$; the case of $i=1$ is trivial).

We recommend care in the selection of the subregions V_n of V such that these conditions are satisfied, for if so, then the resulting sinc approximations of this paper are accurate, while if not, the accuracy is apt to be very poor. Corresponding to a given problem, we also recommend keeping small the number of V_n whose union is V in order to keep small the order of the system of equations derived via the sinc formulas of this paper. Finally, if the number of regions V_n whose union is V is greater than 1, we must take care in identifying the common unknowns on the common boundary of such regions.

6. EXAMPLES OF APPLICATIONS

In this section we illustrate the application of the methods of type I to the solution of some ordinary and partial differential equation boundary value problems. We have selected relatively simple boundary conditions in order to keep the examples simple; more complicated boundary conditions are discussed in [4].

Ex. 6.1: The solution f to the boundary value problem

$$\begin{gathered} \varepsilon^2 f''(x) - f(x) + \frac{1}{x(1-x)} = 0 , \quad 0 < x < 1 \\ f(0) = f(1) = 0 \end{gathered} \tag{6.1}$$

is approximated by

$$f_N(x) = \sum_{k=-N}^{N} f_k S(k,h) \circ \phi(x) \; ; \; \phi(x) = \log[x/(1-x)] \; . \qquad (6.2)$$

By taking $N = 16$, $h = .75/N^{\frac{1}{2}}$, and using the approximations (3.18) and (3.19) we get a linear system of order 33 for the f_k, whose solution yields 5 places of accuracy if $\varepsilon = 1/5$ and 3 places if $\varepsilon = 1/10$. The best dependence of h on ε and N has not been investigated.

<u>Ex. 6.2</u>: We consider the nonlinear ordinary differential equation boundary value problem

$$f''(r) = f(r) - f^3(r)/r^2 \qquad 0 < r < \infty \qquad (6.3)$$

$$f(0) = f(\infty) = 0 \qquad (6.4)$$

which is a steady-state radially symmetric version of the Klein-Gordon equation

$$\nabla^2 u - \frac{\partial u}{\partial t} = u - u^3 \; . \qquad (6.5)$$

It may be shown that the bounded solutions of (6.3)-(6.4) are analytic and bounded in the region $\mathcal{D}$ of Eq. (4.8), and they satisfy (4.9), with $\alpha = 1$. Hence, making the substitution

$$f_N(r) = \sum_{k=-N}^{N} f_k S(k,h) \circ \log \sinh r \qquad (6.6)$$

and taking $N = 16$, $h = .75/N^{\frac{1}{2}}$, we get a 5 decimal in the approximation to the positive solution of (6.3)-(6.4).

Ex. 6.3 [2]: We next consider the Poisson problem

$$u_{xx} + u_{yy} = -\frac{1}{[xy(1-x)(1-y)]^{\frac{1}{2}}}, \quad (x,y) \in S=[0,1]\times[0,1]. \quad (6.7)$$

$$u = 0 \quad \text{on} \quad \partial S . \quad (6.8)$$

Here we use the approximation

$$u_N(x,y) = \sum_{i=-N}^{N} \sum_{j=-N}^{N} u_{ij} S(i,h)\circ\phi(x) S(j,h)\circ\phi(y) \quad (6.9)$$

where $\phi(t) = \log[t/(1-t)]$. In this case we get a system of equations

$$BU + UB^T = W \quad (6.10)$$

in which $U = [u_{ij}]$ and B and W are square matrices of order $2N+1$. The diagonalization of B yields a solution to (6.10) which, with $h = .75/N^{\frac{1}{2}}$, and $N = 16$ yields 5 dec. accuracy in (6.9).

Ex. 6.4 [2]: The initial value problem

$$u_{xx} = u_t \ , \quad 0 < x < 1 \ , \quad 0 < t < \infty \quad (6.11)$$

$$u(x,0) = \sin(\pi x) \ , \ u(0,t) = u(1,t) = 0 \quad (6.12)$$

has the exact solution

$$u(x,t) = e^{-\pi^2 t}\sin(\pi x) \ . \quad (6.13)$$

We attempt to solve (6.11)-(6.12) via a boundary value procedure. Setting

$$u = v + e^{-4t}\sin(\pi x) \ , \quad (6.14)$$

the resulting function v now satisfies

$$v(x,0) = 0 \ , \ v(0,t) = v(1,t) = 0 \ , \ v(x,t) \to 0 \quad \text{as} \quad t \to \infty \ .$$

We can thus approximate the solution to (6.11)-(6.12) by

$$u_N(x,t) = e^{-4t}\sin(\pi x) + \sum_{i=-N}^{N} \sum_{j=-N}^{N} v_{ij} S(i,h)\circ\phi(x) S(j,h^*)\circ\phi^*(t). \qquad (6.15)$$

where $\phi(x) = \log[x/(1-x)]$, $\phi^*(\bar{t}) = \log t$.

In this case we again take $N = 16$, $h = .75/N^{\frac{1}{2}}$, $h^* = .5/N^{\frac{1}{2}}$ to get the system of equation

$$BV + VC = W \quad , \quad V = [v_{ij}] \qquad (6.16)$$

of order $(2N+1)$, where B is the same as in (6.10). Diagonalizing both B and C , we get 4 dec. accuracy in the approximation (6.15).

Ex. 6.5: We illustrate yet another method of solving Poisson's equation over a planar region with boundary consisting of a finite number of analytic arcs. For example, we consider the problem

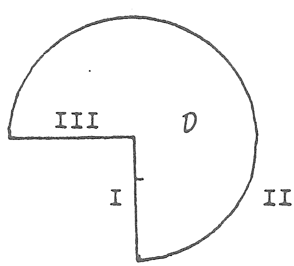

Fig. 6.1
The Region $\mathcal{D}$ of Ex. 6.5.

$$u_{xx} + u_{yy} = -f(x,y) \quad , \quad (x,y) \in \mathcal{D} \qquad (6.17)$$

where $\mathcal{D}$ is the region in Fig. 6.1, which consists of the unit disc with $\frac{1}{4}$ of it cut away. We solve (6.17) subject to boundary conditions

$$\begin{cases} u = 1+y^2 \quad \text{on the arc} \quad I \\ u = 2\, e^{\theta+\pi/2}, \quad -\frac{\pi}{2} \leq \theta \leq \pi \quad \text{on the arc} \quad II \\ u = (1+x) - 2x\, e^{3\pi/2} \quad \text{on the arc} \quad III . \end{cases} \tag{6.18}$$

and for f we take $f(x,y) = e^{xy}$.

We now approximate the solution to (6.17)-(6.18) by

$$u(z) \cong u_N(z) = v(z) + p_2(z)$$
$$+ \sum_{n=1}^{3} \sum_{j=-N}^{N} \mu_{nj} \operatorname{Im}\left\{\frac{e^{i\pi[\phi_n(z)-jh]/h}-1}{\pi[\phi_n(z)-jh]/h}\right\} .$$

In (6.19) $z = (x,y)$ ($\Leftrightarrow z = x+iy$) $v(z)$ is any solution to the non-homogeneous problem. One such solution is given by

$$v(z) = \frac{1}{\pi} \iint \log|z-\zeta| f(\zeta)\, d\xi d\eta , \quad \zeta = (\xi,\eta) . \tag{6.20}$$

Also, $p_2(z)$ is a polynomial of degree 2 in $z = x+iy$, which is harmonic in $\mathcal{D}$. Finally the functions ϕ_n in (6.19) are determined so that ϕ_n maps the n'th boundary arc (defined in a "counterclockwise sense) of Fig. 6.1 conformally onto $\mathbb{R}$, and such that ϕ_n is analytic in $\mathcal{D}$. The resulting double sum in (6.19) is harmonic in $\mathcal{D}$ (see Eq. (2.32) of [6]).

The polynomial p_2 and the unknowns μ_{nj} may be determined by iteration. We first determine p_2 such that $p_2 = p_2{}^{\circ}$ interpolates $u - v$ at the corner points of $\mathcal{D}$. The solution for the μ_{nj} then takes the form $C_1\, \underset{\sim}{\mu}{}^{\circ} = \underset{\sim}{b}{}^{\circ}$ where C_1 is a block diagonal matrix,

$$C_1 = \begin{bmatrix} I & A & V \\ C & I & D \\ E & F & I \end{bmatrix}$$

in which each of the 9 square blocks is of order $2N+1$ and the matrices I are unit matrices, and the entry b_{nj} of $\underset{\sim}{b}^{\circ}$ is the function $u - v - p_2^{\circ}$ evaluated at $z = z_{nj}$, with $\phi_n(z_{nj}) = jh$. Once $\underset{\sim}{\mu} = \underset{\sim}{\mu}^{\circ}$ has been determined, we compute a new $p_2 = p_2^{(1)}$ which interpolates the values of $u - v - \omega$ at the corner points of $\mathcal{D}$, where ω denotes the double sum in (6.19) in which $\underset{\sim}{\mu} = \underset{\sim}{\mu}^{\circ}$. We then determine a new set of values μ_{nj} by solving $C_1 \underset{\sim}{\mu}^1 = \underset{\sim}{b}^1$ where the entry b_{nj} of $\underset{\sim}{b}^1$ is the function $u - v - p_2^{(1)}$ evaluated at the corner points of $\mathcal{D}$.

For example, if we take $N = 8$, $h = 2/N^{\frac{1}{2}}$ we get 3 significant figures of accuracy. Notice that we are able to ignore the effect of the singularities at the corners.

Finally, we cite two additional examples where sinc methods of type I have been used successfully on differential equations. In [3] a sinc method was used to compute eigenvalues of a second order ordinary differential equation over $(0,\infty)$, and in [4], a sinc method was used to reconstruct a surface. This latter procedure involved constructing an approximate solution to a system of 3 nonlinear second order partial differential equations related to the given Gaussian curvature of the surface.

Acknowledgement

The author wishes to acknowledge support for this work from U.S. Army Research Contract No. DAAG 29 83 K 0012, and from the Centre for Mathematical Analysis, Australian National University. Excellent typing by Mrs Dorothy Nash is also gratefully acknowledged.

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Department of Mathematics
University of Utah
Salt Lake City, Utah, 84112, U.S.A.