$W^{2,p}$ REGULARITY FOR VARIFOLDS WITH MEAN CURVATURE

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Suppose $V = \underline{\underline{v}}(M,\tilde{\theta})$ is a rectifiable n-varifold in $\mathbb{R}^{n+k}$ with generalised mean curvature $\underline{\underline{H}} \in L^p(\mu, \mathbb{R}^{n+k})$ in U, that is

$$(1) \qquad \int \mathrm{div}_M X \, d\mu = -\int X.\underline{\underline{H}} \, d\mu$$

for all $X \in C_0^1(U, \mathbb{R}^{n+k})$, where $\mu = \mathcal{H}^n \llcorner \tilde{\theta}$. Then, if $\tilde{\theta} \geq 1 \; \mu$-ae in U, $p > n$, $0 \in \mathrm{spt}\,\mu$ and $B_\rho(0) \subset U$, the regularity theorem ([A]) states that there are $\gamma = \gamma(n,k,p)$, $\delta = \delta(n,k,p) \in (0,1)$ such that

$$\frac{\mu(B_\rho(0))}{\omega_n \rho^n} \leq 1 + \delta \;, \qquad \left(\int_U |\underline{\underline{H}}|^p \, d\mu\right)^{1/p} \rho^{1-n/p} \leq \delta$$

imply that $\mathrm{spt}\,\mu \cap B_{\gamma\rho}(0) = q(\mathrm{graph}\; u) \cap B_{\gamma\rho}(0)$ for some linear isometry q of $\mathbb{R}^{n+k}$ and some $u \in C^{1,\,1-n/p}(B_{\gamma\rho}(0), \mathbb{R}^k)$. (Here $B^n_{\gamma\rho}(0) = B_{\gamma\rho}(0) \cap \mathbb{R}^n$.) We show here that a higher regularity prevails, and that u is actually $W^{2,p}$ and that the density function $\tilde{\theta}$ is $W^{1,p}$.

We write (1) in non-parametric form:

$$(2) \qquad \begin{cases} \displaystyle\int \theta \sqrt{g}\, g^{i\ell} D_\ell \eta = -\int \theta \sqrt{g}\, H^i \eta \,, & 1 \leq i \leq n \\ \displaystyle\int \theta \sqrt{g}\, g^{m\ell} D_m u^j D_\ell \eta = -\int \theta \sqrt{g}\, H^{n+j} \eta \,, & 1 \leq j \leq k \,, \end{cases}$$

for $\eta \in C_0^1(\Omega)$, Ω a domain in $\mathbb{R}^n$. Because the results we obtain hold quite generally, as well as in order to simplify the exposition, we consider, instead of (2), the following system:

$$(3) \qquad \int_\Omega \theta\, \Phi_i^s(Du)\, D_i \eta = -\int_\Omega \theta H^s \eta \,, \qquad 1 \leq s \leq n+k \,.$$

Then we have

THEOREM: *Suppose* $\Phi_i^s = \mathbb{R}^{nk} \to \mathbb{R}$ *is* C^1 *for each* $1 \le s \le n+k$, $1 \le i \le n$, *that* Φ_i^s *has a* C^1 *right inverse, that is there is* $\tilde{\Phi}_s^j : \mathbb{R}^{nk} \to \mathbb{R}$ *which is* C^1 *for each* $1 \le s \le n+k$, $1 \le j \le n$ *and satisfies*

$$(4) \qquad \sum_{s=1}^{n+k} \Phi_i^s(p)\, \tilde{\Phi}_s^j(p) = \delta_{ij} \qquad p \in \mathbb{R}^{nk},$$

that the following skew symmetry condition holds:

$$(5) \qquad \sum_{s=1}^{n+k} ((D_{p_m^t} \Phi_i^s(p))\, \tilde{\Phi}_s^j(p) + (D_{p_m^t} \Phi_m^s(p)\, \tilde{\Phi}_s^j(p)) = 0,$$

for all $p \in \mathbb{R}^{nk}$, $1 \le t \le k$, $1 \le i,j,m \le n$, *and that the last* k *equations of* (3) *satisfy the Legendre-Hadamard ellipticity condition, that is*

$$(6) \qquad \sum_{s,t=1}^{k} \sum_{i,j=1}^{n} D_{n_j^t} \Phi_i^{n+s}(p)\, \zeta_s \zeta_t\, \xi^i \xi^j \ge \lambda_P\, |\zeta|^2 |\xi|^2$$

for all $p \in \mathbb{R}^{nk}$ *with* $|p| \le P$, *and all* $\zeta \in \mathbb{R}^k$, $\xi \in \mathbb{R}^n$, *where* $\lambda_P > 0$.

Suppose also that $u = (u^1,\dots,u^k) : \Omega \to \mathbb{R}^k$ *is in* $C^{1,\alpha}(\bar{\Omega}, \mathbb{R}^k)$ *for some* $\alpha \in (0,1)$, *with* $\sup_{\bar{\Omega}} |Du| \le P$, *and that* u *solves* (3), *with* $\theta \in L^\infty(\Omega)$, $1 \le \theta \le L_1$ L^n-a.e. *in* Ω, *and* $(H^1,\dots,H^{n+k}) \in L^{p_0}(\Omega, \mathbb{R}^{n+k})$ *for* $p_0 > n$; *then*

$$u \in W^{2,p_0}_{loc}(\Omega, \mathbb{R}^k), \qquad \theta \in W^{1,p_0}(\Omega),$$

and

$$(7) \qquad D_j\theta(x) = H^s(x) \,.\, \Phi_s^j(Du(x)) \qquad \mathcal{L}^n\text{-a.e. in } \Omega .$$

Proof: (Outline) Let $\Omega_0 \subset\subset \Omega$ with $d = \operatorname{dist}(\bar{\Omega}_0, \partial\Omega)$. Let

$$r = 2\left(\frac{\log \alpha}{\log(1-\alpha)}\right) + 1,$$

and choose domains Ω_i, $1 \le i \le r$, with

$$\Omega_0 \subset\subset \Omega_r \subset\subset \Omega_{r-1} \subset\subset \dots \subset\subset \Omega_1 \subset\subset \Omega,$$

with $\operatorname{dist}(\bar{\Omega}_j, \partial\Omega_{j-1})$, $\operatorname{dist}(\bar{\Omega}_0, \partial\Omega_r)$, $\operatorname{dist}(\bar{\Omega}_1, \partial\Omega) \geq \frac{d}{r+1}$, $j = 2,\ldots,r$.

Let $0 < h < h_0 = \frac{d}{6(r+1)}$, $1 \leq j \leq n$, and replace η in (3) by $\Phi_s^j(Du_h)\eta$ for each $1 \leq s \leq n+k$, where for any $f \in L^1(\Omega)$, f_h denotes its mollification. Then, using (4) and (5), we have

$$\text{(8)} \qquad \int_{\Omega_1} \theta\, D_j\eta = \int_{\Omega_1} f_h^{j,i}\, D_i\eta + \int (f_h^j + \tilde{f}_h^j)\eta$$

$j = 1,\ldots,n$, $\eta \in C_0^1(\Omega_1)$, where

$$\text{(9)} \qquad \begin{cases} f_h^{j,i} = \theta(\Phi_i^s(Du_h) - \Phi_i^s(Du))\ \Phi_s^j(Du_h) , \\ \tilde{f}_h^s = -\ \theta H^s\ \tilde{\Phi}_s^j(Du_h) \\ f_h^j = \theta(\Phi_i^s(Du_h) - \Phi_i^s(Du))\ D_{x_i}\ \tilde{\Phi}_s^j(Du_h) . \end{cases}$$

We digress here to develop some general results concerning functions which satisfy (8) and (9). For $0 < h \leq h_0$ we let

$$\Omega_h = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > h\} ,$$

and for $f \in L^q(\Omega)$, $1 \leq q \leq \infty$, $\gamma \in (0,1)$ and $z \in \bar{B}_1(0)$ we let

$$\Delta_z^h f(x) = f(x + hz) - f(x)$$

and

$$\mathcal{H}^{\gamma,q}(\Omega) = \{f \in L^q(\Omega) : \sup_{\substack{z \in \partial B_1(0) \\ h>0}} \frac{1}{h^\gamma} \|\Delta_z^h f\|_{q,\Omega_h} < \infty\} .$$

(The spaces $\mathcal{H}^{\gamma,q}$ are known as Nikolskii spaces.) The following inequalities are easy to prove:

$$\text{(10)} \qquad \|\Delta_z^h f\|_{q,\Omega_{2h_0}} \leq h\left(\int_{\Omega_{h_0}} \left|\left(\frac{1}{h^n}\int_{B_h(x)} f(y) D_y\rho\left(\frac{x-y}{h}\right)dy\right).z\right|^q dx\right)^{1/q}$$

$$+ 2\sum_{i=1}^{n} \int_0^h \frac{1}{t^{n+1}} \left\|\int_{B_t(x)} f(y) D_{y_i}\left((x_i - y_i)\rho\left(\frac{x-y}{t}\right)\right)dy\right\|_{q,\Omega_{h_0}} dt$$

$$\text{(11)} \qquad \sup_{\Omega_{2h_0}} |\Delta_z^h f(x)| \le h \sup_{x\in\Omega_{h_0}} \left| \frac{1}{h^n} \left(\int_{B_h(x)} f(y)\, D_y \rho\left(\frac{x-y}{h}\right) dy \right) . z \right|$$

$$+ 2 \sum_{i=1}^{n} \int_0^h \frac{1}{t^{n+1}} \sup_{x\in\Omega_{h_0}} \left| \int_{B_t(x)} f(y)\, D_{y_i}\left((x_i - y_i)\rho\left(\frac{x-y}{t}\right)\right) dy \right| dt .$$

Using (10) and (11) we can prove

LEMMA 1: *If* $f \in L^q(\Omega)$, $1 \le q \le \infty$, $f_h^{j,i}, f_h^j \in L^q(\Omega)$, $\tilde{f}_h^j \in L^{q_0}(\Omega)$, $1 \le i,j \le n$, $0 < h \le h_0$ *with* $q_0 = \frac{n}{1-\gamma}$, $\|\tilde{f}_h^j\|_{q_0,\Omega} \le c$,

$$\text{(12)} \qquad \|f_h^{j,i}\|_{q,\Omega} \le ch^\gamma , \qquad \|f_h^j\|_{q,\Omega} \le ch^{\gamma-1}$$

for each $1 \le i,j \le n$ *and each* $0 < h \le h_0$, *where* $\gamma \in (0,1)$ *and* c *is independent of* h *and with*

$$\int f D_j \eta = \int f_h^{j,i} D_i \eta + \int \left(f_h^j + \tilde{f}_h^j \right) \eta$$

for each $\eta \in C_0^1(\Omega)$, $1 \le j \le n$ *and* $0 < h \le h_0$, *then* $f \in H^{\gamma,q}(\Omega_{h_0})$.

Also, by a simple variation of the difference quotient method, we have

LEMMA 2: *Let* $v^s \in C^1(\bar{\Omega})$, $1 \le s \le k$, *be a solution to the system*

$$\int_\Omega \omega(x)\, \Psi_i^s(Dv)\, D_i \eta = \int_\Omega g^s \eta , \qquad 1 \le s \le k ,$$

for $\eta \in C_0^1(\Omega)$, *where* $\omega \in C(\bar{\Omega})$, $\omega \ge 1$, $g^s \in L^2(\Omega)$, $1 \le s \le k$, Ψ_i^s *is* C^1 *in all its variables,* $1 \le i \le n$, $1 \le s \le k$ *and*

$$\sum_{s,t=1}^{k} \sum_{i,j=1}^{n} D_{p_j^t} \Psi_i^s(p)\, \zeta_s \zeta_t \xi^i \xi^j \ge \lambda |\zeta|^2 |\xi|^2 ,$$

for all $\zeta \in \mathbb{R}^k$, $\xi \in \mathbb{R}^n$ *and* $p \in \mathbb{R}^{nk}$ *with* $|p| \le \sup_\Omega |Dv|$, *where* $\lambda > 0$. *Then, if* $\omega \in H^{\gamma,2}(\Omega)$, *we have*

$$D_i v^s \in H^{\gamma,2}(\Omega_0) , \qquad 1 \le i \le n , 1 \le s \le k ,$$

for any $\Omega_0 \subset\subset \Omega$.

We return to the proof of the theorem: Lemma 1, with $q = \infty$, and (8) and (9) immediately imply the Hölder continuity of θ with exponent $\min(\alpha, 1 - \frac{p}{p_0})$. From (9) we have

$$(13) \qquad \begin{cases} \|f_h^{i,j}\|_{2,\Omega_1} \leq c\|Du_h - Du\|_{2,\Omega_1} \\ \|f_h^j + \tilde{f}_h^j\|_{2,\Omega_1} \leq ch^{\alpha-1} \|Du_h - Du\|_{2,\Omega_1} + c \,, \end{cases}$$

so that replacing h by $h^{\frac{1}{1-\alpha}}$ and using Lemma 1, we have

$$\theta \in \mathcal{H}^{\frac{\alpha}{1-\alpha},2}(\Omega_2) \,,$$

provided $\alpha \leq \frac{1}{2}$. Lemma 2 now implies that

$$D_i u^s \in \mathcal{H}^{\frac{\alpha}{1-\alpha},2}(\Omega_3) \,, \qquad 1 \leq i \leq n \,,\ 1 \leq s \leq k \,.$$

Again replacing h by $h^{\frac{\alpha}{1-\alpha}}$ in (13) we see that

$$\theta \in \mathcal{H}^{\frac{\alpha}{(1-\alpha)^2},2}(\Omega_3) \,,$$

provided $\frac{\alpha}{(1-\alpha)^2} \leq 1$. This procedure can be iterated $t = \left[\frac{\log \alpha}{\log(1-\alpha)}\right]$ times (so that $\frac{\alpha}{(1-\alpha)^t} \leq 1$ and $\frac{\alpha}{(1-\alpha)^t} + \alpha > 1$) to obtain

$$D_i u^s \in \mathcal{H}^{\frac{\alpha}{(1-\alpha)^t},2}(\Omega_r) \,, \qquad 1 \leq i \leq n \,,\ 1 \leq s \leq k \,.$$

Letting $h \downarrow 0$ in (8) and noting that none of the quantities depends on Ω_0, gives (7). The regularity of u follows by standard elliptic theory.

It is straightforward to check that the Euler-Lagrange equations arising from arbitrary C^2 elliptic parametric integrands satisfy the hypotheses of the theorem. Thus we have the regularity as stated. Furthermore we see that the theorem readily implies a constancy theorem for any varifold, stationary with respect to a C^2 elliptic parametric integrand, whose support is contained in a $C^{1,\alpha}$ manifold for some $\alpha > 0$.

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