FLOW BY MEAN CURVATURE OF CONVEX SURFACES INTO SPHERES

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In this talk we consider a uniformly convex n-dimensional $(n \geq 2)$ surface M, which is smoothly imbedded in $\mathbb{R}^{n+1}$. Let us assume that M is locally given by a diffeomorphism

$$F_0 : U \subset \mathbb{R}^n \longrightarrow F_0(U) \subset M \subset \mathbb{R}^{n+1} .$$

Then we want to find a whole family of diffeomorphisms $F(\cdot,t)$ satisfying the evolution equation

$$\begin{aligned} \frac{\partial}{\partial t} F(\vec{x},t) &= \Delta_t F(\vec{x},t) , \quad \vec{x} \in U \\ F(\cdot,0) &= F_0 \end{aligned} \qquad (1)$$

where Δ_t is the Laplace-Beltrami operator on the manifold M_t, which is given by $F(\cdot,t)$. We have

$$\Delta_t F(\vec{x},t) = -H(\vec{x},t) \cdot \nu(\vec{x},t)$$

where $H(\cdot,t)$ is the (positive) mean curvature and $\nu(\vec{x},t)$ the (outer) unit normal on M_t : The surfaces M_t are moving along their mean curvature vector. Since problem (1) is parabolic, we know that it has a smooth solution at least on some short time interval.

In our case the mean curvature is positive and the surfaces M_t are contracted in direction of their inner normal. We want to show that the shape of M_t approaches more and more the shape of a sphere. In particular, no singularities will occur, before the surfaces M_t shrink down to a single point in finite time:

<u>THEOREM 1</u> Let $n \geqq 2$ and assume that M_0 is uniformly convex, i.e. the eigenvalues of its second fundamental form are strictly positive everywhere. Then the evolution equation (1) has a smooth solution on a finite time interval $0 \leqq t < T$ and the surfaces M_t converge to a single point $\mathbb{O}$ as $t \to T$.
If we homothetically expand the surfaces M_t around $\mathbb{O}$ such that the new surfaces $\tilde{M}_t$ have constant area $|M_0|$, then $\tilde{M}_t$ converges to a sphere of area $|M_0|$ in the C^∞-topology.

<u>REMARK</u>

(i) The corresponding one dimensional problem for convex curves in $\mathbb{R}^2$ has been solved recently by Gage and Hamilton, see [2].

(ii)Motion by mean curvature is used to model grain boundaries in annealing pure metal. The interested reader should look at Brakke's book, [1], where grain boundaries and motion by mean curvature are studied from the view point of geometric measure theory.

The approach to Theorem 1 is inspired by Hamilton's paper on 'Three-manifolds with positive Ricci curvature', [3]. There the metric of a compact three dimensional Riemannian manifold with positive Ricci curvature was evolved in direction of the Ricci tensor and a metric of constant positive curvature was obtained in the limit. The evolution equations for the curvature quantities in our problem turn out to be similar to the equations in [3] and we can use some of the methods developed there.

In the following we sketch the proof of Theorem 1

a) Evolution equations

Let us denote by $g = \{g_{ij}\}$ and $A = \{h_{ij}\}$ the metric and the second fundamental form on M_t :

$$g_{ij}(\vec{x}) = \left(\frac{\partial F(\vec{x})}{\partial x_i}, \frac{\partial F(\vec{x})}{\partial x_j}\right)$$

$$\vec{x} \in U \subset \mathbb{R}^n$$

$$h_{ij}(\vec{x}) = \left(\nu(\vec{x}), \frac{\partial^2 F(\vec{x})}{\partial x_i \partial x_j}\right)$$

Then (1) determines the evolution of g and A. Using the Gauß-Weingarten relations one can compute

LEMMA 1 We have the evolution equations

(i) $$\frac{\partial}{\partial t} g_{ij} = -2H \cdot h_{ij}$$

(ii) $$\frac{\partial}{\partial t} h_{ij} = \Delta h_{ij} - 2H\, h_{i\ell} g^{\ell m} h_{mj} + |A|^2 \cdot h_{ij}$$

(iii) $$\frac{\partial}{\partial t} H = \Delta H + |A|^2 \cdot H$$

(iv) $$\frac{\partial}{\partial t}|A|^2 = \Delta|A|^2 - 2|\nabla A|^2 + 2|A|^4$$

b) Preserving convexity

From Lemma 1(iii) and the maximum principle one concludes immediately that the mean curvature stays strictly positive for all time. Now let us use the notation $T_{ij} \geqq 0$ if all eigenvalues of a symmetric tensor $T = \{T_{ij}\}$ are non-negative. Then the assumption of Theorem 1 implies that there is some $\varepsilon > 0$ such that

$$h_{ij} \geqq \varepsilon \cdot H \cdot g_{ij} \tag{2}$$

holds at time $t = 0$. Using a maximum principle for parabolic systems which is proved in [3] we conclude from Lemma 1(ii) that inequality (2) (i.e. uniform convexity) remains to be true with the same $\varepsilon > 0$ as long as a solution of (1) exists.

c) The eigenvalues of A.

In this step we consider the quantity

$$|A|^2 - \frac{1}{n} H^2 = |h_{ij} - \frac{1}{n} Hg_{ij}|^2$$

which measures how far the eigenvalues of the second fundamental form diverge from each other. We show that the eigenvalues approach each other at least at those points where the mean curvature becomes large:

THEOREM 2 There are constants $\delta > 0$ and $C < \infty$ depending only on M_0 such that

$$|A|^2 - \frac{1}{n} H^2 \leqq C \cdot H^{2-\delta}$$

holds for all times where the solution of (1) exists.

To prove Theorem 2 it is enough to show that with some small δ the quantity

$$f_\delta = \frac{|A|^2 - \frac{1}{n}H^2}{H^{2-\delta}}$$

is bounded for all times. The proof of this bound is very technical and depends heavily on the fact that estimate (2) holds for all time. Using (2) and Lemma 1 one derives the following inequality for the time derivative of f_δ

LEMMA 2 We have

$$\frac{\partial}{\partial t} f_\delta \leqq \Delta f_\delta + \frac{2(1-\delta)}{H} \langle \nabla_i H, \nabla_i f_\delta \rangle$$

$$-\varepsilon^2 \cdot \frac{1}{H^{2-\delta}} \cdot |\nabla H|^2 + \delta \cdot |A|^2 \cdot f_\delta$$

Now it is not possible to simply apply the maximum principle since the absolute term in this inequality is positive. But the negative gradient term on the right hand side can be exploited by the divergence theorem and from the Sobolev inequality and an iteration method developed by De Georgi and Stampacchia one is led eventually to Theorem 2.

d) Comparing H_{max} to H_{min}

In order to compare the maximum value of the mean curvature H_{max} to the minimum value H_{min} on M_t, one derives the following gradient bound for the mean curvature from Theorem 2 in much the same way as the gradient estimate for the scalar curvature was derived in [3].

THEOREM 3 For any $\eta > 0$ there is a constant $C(M_0, \eta)$ such that

$$|\nabla H|^2 \leqq \eta H^4 + C(\eta, M_0) .$$

Now let $[0,T)$ be the maximal time interval where the smooth solution of (1) exists. Then we must have that

$$\max_{M_t} |A|^2 \text{ becomes unbounded as } t \to T . \qquad (3)$$

Otherwise it would be possible to show the existence of a smooth limit surface M_T and to further extend the solution of (1) in contradiction to the maximality of T. As $|A|^2 \leq H^2$, we have in particular that H_{max} becomes unbounded as $t \to T$.

Then, since estimate (2) implies a lower bound on the Ricci curvature of the surfaces M_t, one may use Theorem 3 together with Meyer's theorem as in [3] to compare the mean curvature at different points of the surfaces and derives

THEOREM 4 $\quad H_{max}/H_{min} \longrightarrow 1 \quad$ as $t \to T$.

Once this is established it is obvious from (3) that the diameter of the surfaces M_t goes to zero for $t \to T$, thus proving the first part of Theorem 1. By Theorem 4 we have $H_{min} \to \infty$ as $t \to T$ and thus we conclude from Theorem 2 that the eigenvalues of the second fundamental form approach each other for $t \to T$. Having made this observation one derives the second part of Theorem 1 with some straight forward interpolations.

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