QUENCHING OF SOLUTIONS OF EVOLUTION EQUATIONS

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In this talk we discuss the phenomenon of quenching. Proofs of the results stated can be found in the references. A more detailed exposition, without proof, is [7]; these notes are based in part on that work.

The term "quenching" originated in the study of electric current transients in polarized ionic conductors [5] and the theory of superconductivity [4]. To explain this phenomena, consider the following problem, studied by Kawarada in [5]:

(1) $$u_t = u_{xx} + \frac{1}{1 - u} \quad 0 < x < L,\ t > 0,$$

(2) $$u(x,0) - u(0,t) = u(L,t) \quad 0 \leqslant x \leqslant L,\ t > 0.$$

Kawarada showed that if $L \geqslant 2\sqrt{2}$, then

$$\sup_{0<x<L} u(x,t)$$

increases to one in a finite time T with u_t unbounded for $0 < t < T$. We say, then, that a solution of an evolution equation quenches (in finite or infinite time) if the solution remains bounded while some derivative becomes infinite. For our purpose, quenching will occur when the solution reaches a critical value which we take to be one. Mathematically this corresponds to introducing a nonlinearity $\varphi(u)$ with the properties

φ is defined and increasing on $[-\infty,1]$,

$\varphi(-\infty) \geqslant 0$, $\varphi(1) = \infty$.

Problem (1), (2) can then be generalized by replac ing (1) with

(1)' $$u_t = u_{xx} + \varphi(u) \quad 0 < x < L, \; t > 0.$$

Using the steady-state problem

(3) $$f'' + \varphi(f) = 0 \quad 0 < x < L, \quad f(0) = f(L) = 0$$

as a starting point, Acker and Walter [1,2] and, independently Levine and Montgomery [9], proved the following theorem:

Theorem 1. There exists a solution u _of_ (1)', (2) _at least for some finite time. Moreover the number_

$$L_o = \sup\{L:(3) \text{ has a solution}\}$$

is finite.

(a) _If_ $L < L_o$, _then_ u _exists for all time_, $u(x,t)$ _increases to_ $f(x)$ _as_ $t \to \infty$, u_t _and_ u_{xx} _are bounded, and_ _sup_ $f < 1$; _in short, no quenching._

(b) _If_ $L > L_o$, _then there is a finite_ $T = T(L)$ _such that_ $u(L/2,t)$ _increases to_ 1 _as_ $t \to T$, $u(x,t) < u(L/2,t)$ _for_ $x \neq L/2$ _and_ $0 < t < T$, _and_ $u_t(L/2,t)$ _or_ $u_{xx}(L/2,t)$ _becomes infinite as_ $t \to T$; _in short_ u _quenches at time_ T.

Levine and Montgomery also analysed completely the borderline case $L = L_o$, and they gave an expression for L_o in terms of φ.

Acker and Walter were more concerned with nonlinearities of the form $\varphi(u,Du)$. They also examined a higher dimensional problem:

(4) $$u_t = Au + \varphi(u) \quad x \in D_L,\ t > 0,$$

(5) $$u(x,t) = 0 \quad x \in \partial D_L,\ t > 0,$$

(6) $$u(x,0) = 0 \quad x \in D_L,$$

for a suitable time independent second other elliptic operator A and expanding domains D_L parametrized by L. Although they did not study the case $L = L_o$, they obtained a criterion for quenching in terms of the steady-state problem,

(7) $$Au + \varphi(u) = 0 \text{ in } D_L, \quad u = 0 \text{ on } \partial D_L,$$

parallel to Theorem 1.

Theorem 2. The number

$$L_o = \sup\{L: (7) \text{ has a solution}\}$$

is finite. If $L < L_o$, *then there is no quenching while if* $L > L_o$, *the solution* u *of* (4), (5), (6) *quenches in finite time.*

Proofs of Theorems 1 and 2 rely on the maximum principle. Although there is no corresponding maximum principle for hyperbolic equations, similar results have been obtained in [3] (Theorem 1) and [10] (Theorem 2). The main difference between the hyperbolic and parabolic cases is that for hyperbolic equations, there are numbers $L_1 < L_2$ (with $L_2 < L_o$) such that u does not quench for $L < L_1$ and u quenches in finite time for $L > L_2$. Nothing is known for $L_1 \leqslant L \leqslant L_2$.

The nonlinearity need not occur in the differential equation; instead it may appear in the boundary condition as in the model problem of Levine [6]:

(8) $$u_t = u_{xx} \quad 0 < x < L, \quad t > 0,$$

(9) $$u(x,0) = 0 \quad 0 \leq x \leq L,$$

(10) $$u(0,t) = 0, \quad u_x(L;t) = \varphi(u(L,t)) \quad t > 0.$$

The steady state problem is now

(11) $$f'' = 0 \quad 0 < x < L, \quad f(0) = 0, \quad f'(L) = \varphi(f(L)),$$

which is easily seen to have the solution

$$f(x) = ax/L$$

where a is a root of $a = L\varphi(a)$. Setting

$$L_o = \sup\{L : (11) \text{ has a solution}\}$$

and observing that

(12) $$L_o = \sup_{0<\delta<1} \delta/\varphi(\delta),$$

we now state the criterion for quenching in [6].

Theorem 3. For the problem (8,9,10), if $L \leq L_o$, there is no quenching and u increases to the smallest solution of (11). If $L > L_o$, quenches in finite time T and $u_x(L,T-) = u_t(L,T-) = \infty$.

A higher dimensional analog of (8), (9), (10) was studied in [8]. This work considered the problem

(13) $$u_t = Au \quad \text{in} \quad D \times (0,T),$$

(14) $$u = 0 \quad \text{on} \quad \sigma \times (0,T), \quad \bar{D} \times \{0\},$$

(15) $$\frac{\partial u}{\partial n} = \varphi(u) \quad \text{on} \quad \sum \times (0,T),$$

where A is a suitable time-independent second order elliptic operator on a bounded domain $D \subseteq \mathbb{R}^n$ and ∂D has been decomposed into disjoint sets $\sum$ and σ. The quenching criterion of [8] was given in terms of the steady state solution, and also in terms of the solution of

(16) $$Aw = 0 \quad \text{in} \quad D, \quad w = 0 \quad \text{on} \quad \sigma, \quad \frac{\partial w}{\partial n} = 1 \quad \text{on} \quad \sum.$$

Theorem 4. *Let* u *be a solution of* (13), (14), (15) *and let* w *be a solution of* (16). Define L_o *by* (12), *and set*

$$L_1 = \int_0^1 \frac{ds}{\varphi(s)}, \quad w_o = \sup_D w.$$

(a) *If* $w_o \leq L_o$, *then there is no quenching and* u *increases to a steady state solution bounded from above by a number* $s_o < 1$.

(b) *If* $w_o = L_1$, *then* u *quenches in finite or infinite time. For infinite time quenching,* u *increases to a solution of*

(17) $$Ag = 0 \ \text{in} \ D, \quad g = 0 \ \text{on} \ \sigma, \quad \frac{\partial g}{\partial n} = \varphi(g) \ \text{or} \ g = 1 \ \text{on} \ \sum,$$

with $g = 1$ *wherever* $w = L_1$. *If* (17) *has no solution,* u *quenches in finite time.*

(c) *If* $w_o > L_1$, u *quenches in finite time.*

Note that infinite time quenching cannot occur in Theorem 3 but it is not prohibited in Theorem 4. To illustrate infinite time quenching, we use the example of [8], which also involves other aspects of Theorem 4. Set

$$D = \{x \in \mathbb{R}^2 : 0 < x^1 < 1/4 - (x^2)^2,\ |x^2| < 1/2\},$$

$$\sigma = \{x \in \partial\Omega : x^1 > 0 \quad \text{or} \quad |x^2| = 1/2\},\quad \sum = \partial\Omega \ - \sigma,$$

$$A = \Delta,\ \varphi(u) = 1/(1 - u).$$

It is readily checked that

$$g(x) = 1 - (2(x^1 + |x|))^{1/2}$$

is a solution of (17) with $g = 1$ and $\frac{\partial g}{\partial n} = \infty$ at $x = 0$. A simple calculation gives $L_o = 1/4$ and it can be shown that $w_o = w(0)$. This value can be obtained as the sum of an alternating series with first time

$$16/[\pi^3 \cosh(\pi/2)] < .24;$$

therefore there is no quenching in this case. To demonstrate infinite time quenching, we now use a result of Protter [8] to infer that (17) has a unique solution $\bar{g}$ with maximum less than one. Hence if we take initial values u_o in (13), (14), (15) satisfying $\bar{g} < u_o < g$, the solution quenches in infinite time.

A hypberbolic version of Theorem 3 appears in [6]. No hyperbolic version of Theorem 4 is known to the author.

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