REGULARITY FOR SOLUTIONS TO OBSTACLE PROBLEMS

J.H. Michael

This is a report on some research done jointly with William P. Ziemer at the Centre for Mathematical Analysis. The research establishes interior regularity for a solution to a classical obstacle problem of general type.

1. INTRODUCTION

Let Ω be a bounded non-empty open set of R^n. Let K be the convex subset of the Sobolev space $W^{1,\alpha}(\Omega)$ consisting of all v, such that v agrees with a boundary function θ on $\partial\Omega$ in a suitable way and

$$v(x) \geqq \psi(x)$$

for almost all $x \in \Omega$, where ψ is a function defined on Ω (the "obstacle"). Put

$$I(v) = \int_\Omega F(x,v(x),Dv(x))dx \tag{1}$$

for $v \in K$, where F is a function with suitable properties. Let

$$\sigma = \inf_{v\in K} I(v) \tag{2}$$

and suppose there is a function $u \in K$, such that

$$I(u) = \sigma . \tag{3}$$

The above is a general description of a classical obstacle problem and u is a solution. A great deal of research has been done on the regularity of such solutions [1,2,4]. Our research assumes much less about the function ψ than has been assumed in earlier work.

Actually our results are obtained in a slightly more general setting. It is well known that if $u \in K$ is such that (3) holds and the function F satisfies appropriate conditions, then

$$\sum_{i=1}^{n} \int_{\Omega} \frac{\partial F}{\partial p_i} (x,u(x),Du(x)) \frac{\partial \phi}{\partial x_i} (x)\,dx$$

$$+ \int_{\Omega} \frac{\partial F}{\partial z} (u,u(x),Du(x))\phi(x)\,dx \geqq 0 \ , \tag{4}$$

for all $\phi \in W_0^{1,\alpha}(\Omega)$ with

$$\phi(x) \geqq \psi(x) - u(x) \tag{5}$$

for almost all $x \in \Omega$.

This is a special case of the weak inequality:

$$\sum_{i=1}^{n} \int_{\Omega} A_i(x,u(x),Du(x)) \frac{\partial \phi}{\partial x_i} (x)\,dx$$

$$+ \int_{\Omega} B(x,u(x),Du(x))\phi(x)\,dx \geqq 0 \tag{6}$$

for all $\phi \in W_0^{1,\alpha}(\Omega)$ with

$$\phi(x) \geqq \psi(x) - u(x) \tag{7}$$

for almost all $x \in \Omega$. Our research is concerned with this more general inequality. It will be assumed that $u \in W^{1,\alpha}(\Omega)$ (where $1 < \alpha < \infty$)

$$u(x) \geqq \psi(x) \tag{8}$$

for almost all $x \in \Omega$ and u satisfies the inequality (6) for all ϕ satisfying (7). It will also be assumed that ψ is an upper semi-continuous function on Ω satisfying the approximate continuity condition:

$$\psi(x) = \lim_{\rho\to 0+} \fint_{|\xi-x|<\rho} \psi(\xi)\,d\xi \;. \tag{9}$$

[The symbol $\fint$ denotes the integral average.] The coefficients A_i and B are Borel measurable functions on $\Omega\times R\times R^n$ and they satisfy the following standard conditions.

$$|A(x,z,p)| \leqq \mu|p|^{\alpha-1} + \mu|z|^{\alpha-1} + \nu \;, \tag{10}$$

$$p\cdot A(x,z,p) \geqq |p|^{\alpha} - \mu|z|^{\alpha} - \nu \;, \tag{11}$$

$$|B(x,z,p)| \leqq \mu|p|^{\alpha-1} + \mu|z|^{\alpha-1} + \nu \tag{12}$$

for $x\in\Omega$, $z\in R$, $p\in R^n$, where μ, ν are non-negative constants.

2. DISCUSSION OF THE RESULTS.

We observe to begin with that as a consequence of the upper semicontinuity, ψ is locally bounded above.

A standard iteration procedure followed by an interpolation argument (see [3] and [5]) yields the following.

2.1 LEMMA *Let* $M_0 > 0$ *and* $\gamma > 0$. *There exists a constant* $c > 0$ *and such that, for every* $x_0\in\Omega$, *every* $\rho\in(0,1]$ *for which* $\overline{B_\rho(x_0)}\subset\Omega$ *and every constant* M *for which* $|M|\leqq M_0$, *it is true that*

(i) *the inequality*

$$\operatorname*{ess\,sup}_{|x-x_0|<\frac{1}{2}\rho} (u(x)-M)^- \leqq C\left[\fint_{|x-x_0|<\rho} \{(u(x)-M)^-\}^{\gamma}dx\right]^{\frac{1}{\gamma}} + C\rho$$

always holds and

(ii) *the inequality*

$$\operatorname*{ess\,sup}_{|x-x_0|<\frac{1}{2}\rho} (u(x)-M)^+ \le C\left[⨍_{|x-x_0|<\rho} \{(u(x)-M)^+\}^\gamma dx\right]^{\frac{1}{\gamma}} + C\rho$$

holds, provided that $\psi(x) \le M$ *for all* $x \in \overline{B_\rho(x_0)}$.

It follows immediately from 2.1 that u is locally bounded on Ω .

By using a standard iteration, combined with the John-Nirenberg lemma, we are able to prove

2.2 LEMMA *Let* $M_0 > 0$. *There exist* $B > 0$, $c > 0$, $\gamma \in (0,1]$, *such that for every* $x_0 \in \Omega$, *every* $\rho \in (0,1]$ *for which* $\overline{B_\rho(x_0)} \subset \Omega$ *and every* M *for which* $|M| \le M_0$ *and* $u(x) \ge M$ *for almost all* $x \in B_\rho(x_0)$, *the inequality*

$$\operatorname*{ess\,inf}_{|x-x_0|<\frac{1}{2}\rho} (u(x)-M) \ge C\left[⨍_{|x-x_0|<\rho} (u(x)-M)^\gamma dx\right]^{\frac{1}{\gamma}} - B\rho$$

holds.

Consider an arbitrary $x_0 \in \Omega$ and a $\rho \in (0,1]$ such that $\overline{B_\rho(x_0)} \subset \Omega$. Put

$$m_\lambda = \operatorname*{ess\,inf}_{|x-x_0|<\lambda} u(x)$$

for $0 < \lambda \le \rho$. By 2.2

$$m_{\frac{1}{2}\rho} - m_\rho \ge C\left[⨍_{|x-x_0|<\rho} (u(x)-m_\rho)^\gamma dx\right]^{\frac{1}{\gamma}}$$

and hence

$$m_{\frac{1}{2}\rho} - m_\rho \geqq C(M-m_\rho)^{-(\frac{1-\gamma}{\gamma})} \left[⨍_{|x-x_0|<\rho} (u(x)-m_\rho)\,dx \right]^{\frac{1}{\gamma}} , \tag{13}$$

where M is an upper bound for u . But, since u is locally bounded above, m_ρ approaches a limit as $\rho \to 0+$. Hence

$$⨍_{|x-x_0|<\rho} (u(x)-m_\rho)\,dx \to 0$$

as $\rho \to 0+$. Then

$$\lim_{\rho\to 0+} ⨍_{|x-x_0|<\rho} u(x)\,dx \quad \text{exists and}$$

$$= \operatorname{ess} \liminf_{\rho\to 0+} u(x) . \tag{14}$$

We now define

$$u(x_0) = \lim_{\rho\to 0+} ⨍_{|x-x_0|<\rho} u(x)\,dx \tag{15}$$

for all $x_0 \in \Omega$. Then

$$u(x) \geqq \psi(x)$$

for all $x \in \Omega$. It follows from (14) and (15) that u is lower semicontinuous on Ω .

Put

$$H = \{x; x \in \Omega \text{ and } u(x) = \psi(x)\} \tag{16}$$

and

$$\Omega_0 = \Omega \sim H . \tag{17}$$

Then H is closed relative to Ω and Ω_0 is open. Standard regularity theory for solutions to quasi-linear partial differential equations gives

2.3 LEMMA *There exists a* $\delta \in (0,1)$ *and such that, for every compact subset* K *of* Ω_0 *,* u *is Hölder continuous with exponent* δ *on* K .

Consider a point x_0 of the contact set H. Let

$$\Gamma > u(x_0) = \psi(x_0)$$

and let $\rho \in (0,1]$ be such that $\overline{B_\rho(x_0)} \subset \Omega$ and

$$\sup_{|x-x_0|\leqq\rho} \psi(x) < \Gamma .$$

By 2.1 (ii), with $\gamma = 1$,

$$\sup_{|x-x_0|<\frac{1}{2}\rho} (u(x)-\Gamma)^+ \leqq C \fint_{|x-x_0|< \rho} (u(x)-\Gamma)^+ dx + C\rho . \tag{18}$$

Now let

$$w(x) = \inf\{u(x),\Gamma\} .$$

Then

$$u = (u-\Gamma)^+ + w$$

so that by (15)

$$\fint_{|x-x_0|<\rho} (u(x)-\Gamma)^+ dx + \fint_{|x-x_0|<\rho} w(x)\,dx \to u(x_0) \tag{19}$$

as $\rho \to 0+$. But

$$w(x) \geqq \inf_{|x-x_0|<\rho} u(x)$$

when $|x-x_0|<\rho$, so that by (14) and (15)

$$\liminf_{\rho\to 0+} \fint_{|x-x_0|<\rho} w(x)\,dx \geqq u(x_0) .$$

Therefore, by (19)

$$\limsup_{\rho\to 0+} \fint_{|x-x_0|<\rho} (u(x)-\Gamma)^+ dx \leqq 0$$

and hence by (18)

$$\limsup_{x\to x_0} u(x) \leqq \Gamma . \tag{20}$$

Since Γ was arbitrary and we already know that u is lower semicontinuous at x_0. Thus u is continuous on Ω.

Now we consider a point x_0 of the contact set H at which ψ is Hölder continuous; i.e., we suppose there exists a $\delta \in (0,1)$ and an E, such that

$$|\psi(x) - \psi(x_0)| \leqq E|x-x_0|^{\delta} \tag{21}$$

for all $x \in \Omega$. By (13),

$$m_{\frac{1}{2}\rho} - m_\rho \geqq C'\left[⨍_{|x-x_0|<\rho} (u(x)-m_\rho)\,dx \right]^{\frac{1}{\gamma}}, \tag{22}$$

so that (putting $\Lambda = (C')^{-\gamma}$),

$$⨍_{|x-x_0|<\rho} u(x)\,dx \leqq m_\rho + \Lambda(m_{\frac{1}{2}\rho} - m_\rho)^{\gamma}$$
$$\leqq u(x_0) + \Lambda(\psi(x_0) - m_\rho)^{\gamma}.$$

But

$$m_\rho \geqq \inf_{|x-x_0|<\rho} \psi(x) \geqq \psi(x_0) - E\rho^{\delta}$$

and hence

$$⨍_{|x-x_0|<\rho} u(x)\,dx \leqq u(x_0) + \Lambda E^{\gamma}\rho^{\delta\gamma}. \tag{23}$$

Put

$$\Gamma_\rho = \sup_{|x-x_0|<\rho} \psi(x)$$

and

$$w_\rho(x) = \inf\{u(x), \Gamma_\rho\}.$$

Then

$$u = w_\rho + (u-\Gamma_\rho)^{+}$$

so that by (23)

$$⨍_{|x-x_0|<\rho} w_\rho(x)\,dx + ⨍_{|x-x_0|<\rho} (u(x)-\Gamma_\rho)^+ dx \leqq u(x_0) + \Lambda E^\gamma \rho^{\delta\gamma} \quad . \tag{24}$$

But

$$⨍_{|x-x_0|<\rho} w_\rho(x)\,dx \geqq ⨍_{|x-x_0|<\rho} \psi(x)\,dx \geqq \psi(x_0) - E\rho^\delta$$

and therefore by (24)

$$⨍_{|x-x_0|<\rho} (u(x)-\Gamma_\rho)^+ dx \leqq \Lambda E^\gamma \rho^{\delta\gamma} + E\rho^\delta .$$

Hence, by Lemma 2.1 (ii),

$$\sup_{|x-x_0|<\frac{1}{2}\rho} (u(x)-\Gamma_\rho) \leqq C\Lambda E^\gamma \rho^{\delta\gamma} + CE\rho^\delta + C\rho \quad .$$

Therefore

$$\sup_{|x-x_0|<\frac{1}{2}\rho} (u(x)-u(x_0)) \leqq C\Lambda E^\gamma \rho^{\delta\gamma} + CE\rho^\delta + C\rho + E\rho^\delta \quad . \tag{25}$$

Since

$$\inf_{|x-x_0|<\frac{1}{2}\rho} (u(x)-u(x_0)) \geqq \inf_{|x-x_0|<\frac{1}{2}\rho} (\psi(x)-\psi(x_0))$$

$$\geqq - E\rho^\delta$$

it follows from (25) that u is Hölder continuous at x_0 .

It is now possible to prove the following theorem:

2.3 THEOREM *Suppose* $\delta \in (0,1)$ *is such that* ψ *is Hölder continuous with exponent* δ *on every compact subset of* Ω *. Then there exists a* $\delta' \in (0,1)$ *such that* u *is Hölder continuous with exponent* δ' *on every compact subset of* Ω *.*

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