ULTRAPOWERS IN THE LIPSCHITZ AND UNIFORM CLASSIFICATION OF BANACH SPACES

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Ultraproducts were introduced into Model Theory by Skolem in the 1930s. For some time they have been used in algebra. Their use in Analysis commenced in the 1970s. A major contribution was made by Stern in 1978 when he derived the Banach space versions of the Lowenheim-Skolem Theorem and the Keisler-Shelah Theorems which appear in this paper. More recently Heinrich and Mankiewicz have made a significant contribution to the use of ultrapowers in the Lipschitz and Uniform Classification of Banach spaces. They considered various Banach spaces (with certain natural properties) which were related by some uniform or Lipschitz mapping. Using Ultrapower techniques many useful results were obtained, including greatly simplified proofs of some difficult results of Ribe [1976], [1978].

In many cases the existence of some (non-linear) uniform or Lipschitz mapping between two Banach spaces (with certain additional properties) guarantees the existence of a linear mapping between them.

In this paper the intention is to provide a look at some of the most critical results in this area, to provide a feeling for the use of them, and to finally prove the result that the ultrapowers of certain uniformly equivalent Banach spaces are in fact linearly equivalent.

FILTERS

A *filter* on an index set I is a non-empty family F of subsets of I which is

1) Closed under finite intersections, i.e. if $A, B \in F$ then $A \cap B \in F$.

and 2) Closed under supersets, i.e. if $A \subseteq B \subseteq I$ and $A \in F$ then $B \in F$.

F is said to be a *proper filter* iff $\phi \notin F$.

For our purposes all filters are assumed to be proper.

An *ultrafilter* is a filter which is maximal w.r.t. ordering by containment. This occurs iff $\forall A \in I$ only one of A and $I \backslash A \in U$. As a corollary to this we have that if $I = I_1 \cup \ldots \cup I_n$ (a finite union) then at least one $I_j \in U$ $(j \in \{1, \ldots, n\})$.

In a Banach space if $(x_i)_{i \in I}$ is a family of elements of X indexed by I then (x_i) converges over U to x, written $\lim_U x_i = x$, iff for every ball centred x radius ε (denoted $B(x,\varepsilon)$)

$$\{i \in I : x_i \in B(x;\varepsilon)\} \in U .$$

THE BANACH SPACE ULTRAPRODUCT

Let $(X_i)_{i \in I}$ be a family of Banach spaces, I being an index set. Consider the Banach space $\ell_\infty(I, X_i)$ which consists of all families (x_i) in $\prod_{i \in I} X_i$ (the cartesian product of the X_i over I) with

$$\|(x_i)\| = \sup_{i \in I} \|x_i\| < \infty .$$

Let

$$N_u = \{(x_i) \in \ell_\infty(I, X_i) : \lim_U \|x_i\| = 0\} .$$

The ultraproduct of the family $(X_i)_{i \in I}$ w.r.t. U is the quotient space $(X_i)_u = \ell_\infty(I, X_i)/N_u$ with the usual quotient norm. If all the X_i are equal to a certain space X then we refer to their ultraproduct w.r.t. U as the *ultrapower* of X w.r.t. U, and denote it by $(X)_u$.

A key result concerning the norm of elements of $(X_i)_u$ is

$$\|(x_i)_u\| = \lim_U \|x_i\| .$$

If X and Y are Banach spaces we will say that X and Y satisfy the *Decomposition Scheme (DS)* if one of the following conditions is met:

1) $X \cong X \oplus X$ and $Y \cong Y \oplus Y$;

or 2) Either X or Y contains a complemented subspace isomorphic to $\ell_p(X)$ ($\ell_p(Y)$ respectively) for some p , $1 \le p < \infty$.

(Here $\ell_p(X)$ denotes the space of all sequences $(x_i) \in X$ s.t.

$$\|(x_i)\| = \left(\sum_{i=1}^{\infty} \|x_i\|^p\right)^{1/p} < \infty .)$$

The importance of this is in the following theorem of Pelczynski:

DECOMPOSITION THEOREM: *Assume that the Banach spaces* X *and* Y *satisfy the* D.S. *If each of the spaces* X,Y *is isomorphic to a complemented subspace of the other, then* X *and* Y *are isomorphic.*

We now present the Banach space version of two fundamental theorems of Model Theory. Their Banach space form is due to Stern [1978].

LOWENHEIM-SKOLEM THEOREM: *Let* X *be a Banach space and let* $Y \subset X$ *be a subspace,* Y *separable. Then there exists a separable subspace* $Z, Y \subset Z \subset X$, *and an ultrafilter* U s.t. $(Z)_u$ *and* $(X)_u$ *are isometrically-isomorphic.*

KEISLER-SHELAH ISOMORPHISM THEOREM: *Let* X *be a Banach space and let* U *and* V *be ultrafilters on* I *and* J *respectively. Then there exists an ultrafilter* W *on an index set* K s.t. *the spaces* $(X)_{uxw}$ *and* $(X)_{vxw}$ *are isometric. (Note that* $(X)_{uxw}$ *and* $(X)_{vxw}$ *can be shown to be equivalent to* $((X)_u)_w$ *and* $((X)_v)_w$ *respectively.)*

The following result can be shown to be true.

LEMMA 1: *If* X *and* Y *are Banach spaces which satisfy the* D.S. *then there exists an ultrafilter* U *on an index set* I s.t. $(X)_u$ *and* $(Y)_u$ *satisfy the* D.S.

Heinrich and Mankiewicz [1980] have derived several results concerning spaces with some uniform mapping between them. For our purpose we present three of their lemmas which are necessary for our final result (Theorem 5).

LEMMA 2: *If* Y *is a super-reflexive Banach space and* $F : X \to Y$ *is a uniform embedding* s.t. F(X) *is the range of a uniform projection in* Y *, then* X *is super-reflexive as well.*

Note: A space is super-reflexive iff it is uniformly convexifiable, and hence is a stronger condition than reflexivity.

Proof: (Sketch only)

It can be shown that the assumptions imply that $(X)_u$ is Lipschitz embeddable into $(Y)_u$ and that $(Y)_u$ must also be super-reflexive. By an infinite dimensional version of the Rademacher Theorem (see Mankiewicz [1973]), each separable subspace of X (regarded as a subspace of $(X)_u$) embeds isomorphically into $(Y)_u$ and so is super-reflexive. This implies that X is also super-reflexive.

LEMMA 3: *If* X *is a super-reflexive Banach space and* $F : X \to Y$ *is a uniform embedding* s.t. F(X) *is the range of a uniform projection in* Y *, then there exists an ultrafilter* U s.t. X *is isomorphic to a complemented subspace of* $(Y)_u$ *(denoted by* $X \hookrightarrow_c (Y)_u$ *).*

Proof: (Sketch only)

By the Lowenheim-Skolem Theorem there exists a separable subspace $\mathcal{Z}$ of X and an ultrafilter U s.t. $(\mathcal{Z})_u$ and $(X)_u$ are isometric. By

Lemma 2, X is super-reflexive and so $Ƶ$ is also. It can then be shown that $Ƶ \hookrightarrow_c (Y)_u$ and so $(Ƶ)_u \hookrightarrow_c (Y)_{u \times u}$. So as $(X)_u$ is isometric to $(Ƶ)_u$, $(X)_u \hookrightarrow_c (Y)_{u \times u}$.

LEMMA 4: *If* Y *is a super-reflexive Banach space uniformly homeomorphic to another Banach space* X*, then there exists an ultrafilter* U *s.t.* $(X)_u \hookrightarrow_c (Y)_u$ *and* $(Y)_u \hookrightarrow_c (X)_u$.

Proof: By Lemma 2 X is also super-reflexive. By Lemma 3 there exist ultrafilters U_1 and U_2 s.t.

(1) $$X \hookrightarrow_c (Y)_{u_1}$$

(2) $$Y \hookrightarrow_c (X)_{u_2} .$$

By the Keisler-Shelah Theorem, there exists an ultrafilter U_3 s.t.

$$(X)_{U_3} \text{ and } (X)_{U_2 \times U_3} \text{ are isometric.}$$

Taking ultrapowers w.r.t. U_3 in (1), we get

(3) $$(X)_{U_3} \hookrightarrow_c (Y)_{U_1 \times U_3} \quad \text{and}$$

(4) $$(Y)_{U_3} \hookrightarrow_c (X)_{U_3} .$$

Applying the theorem again, there is an ultrafilter U_4 s.t.

$$(Y)_{U_1 \times U_3 \times U_4} \text{ is isometric to } (Y)_{U_3 \times U_4} .$$

From (3) and (4) we derive

$$(X)_{U_3 \times U_4} \hookrightarrow_c (Y)_{U_3 \times U_4}$$

and

$$(Y)_{U_3 \times U_4} \hookrightarrow_c (X)_{U_3 \times U_4} .$$

Letting $U = U_3 \times U_4$ completes the proof.

As our final result we see that under certain natural conditions the uniform equivalence of two Banach spaces is sufficient to guarantee that the closely related structures — the ultrapowers of the spaces — are linearly equivalent.

THEOREM 5: *Assume that* X *and* Y *are uniformly homeomorphic Banach spaces which satisfy the* D.S. *Assume that* Y *is super-reflexive. Then there exists an ultrafilter* U s.t. $(X)_u$ *and* $(Y)_u$ *are isomorphic.*

Proof: By Lemma 1 and Lemma 4 there exists an ultrafilter U s.t. $(X)_u$ and $(Y)_u$ satisfy the D.S. and s.t. $(X)_u \hookrightarrow_c (Y)_u$ and $(Y)_u \hookrightarrow_c (X)_u$. So by the Decomposition Theorem $(X)_u$ and $(Y)_u$ are isomorphic.

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