NON-LINEAR CHARACTERIZATIONS OF SUPERREFLEXIVE SPACES

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1. The classical theorem of Weierstrass on approximation says that a real continuous function on a closed, bounded set in a finite dimensional space is the limit of a uniformly convergent sequence of polynomials. While this theorem has very interesting extensions, such as the Stone-Weierstrass Theorem, it does not generalise in this form to infinite dimensional spaces. A.S. Nemirovski and S.M. Semenov [5] have given an example of a real continuous function on a separable, infinite Hilbert space H, possessing uniformly continuous Fréchet derivatives of all orders but, which, on the unit ball of H cannot be approximated uniformly by polynomials. However, they show that every uniformly continuous function on the unit ball of H is the uniform limit of restrictions of functions which are uniformly continuously differentiable on bounded sets. For a discussion of these results see [7]. Results of this type in global analysis on infinite dimensional manifolds raise the question of existence of uniformly continuously differentiable functions on a Banach space which have bounded support. R. Bonic and J. Frampton [2] studied questions of similar nature. If X and Y are Banach spaces, let $C^{p,q}(X,Y)$, $0 \leq q \leq p \leq \infty$, denote those functions in $C^p(X,Y)$ whose derivatives of order less than or equal to q are bounded. Call a Banach space X, $C^{p,q}$-smooth if there exists a nonzero $C^{p,q}$-function on X with bounded support. In this notation, finite dimensional spaces are $C^{\infty,\infty}$-smooth and if an L_p space is C^p-smooth, then it is also $C^{p,q}$-smooth. Consider the space c_0 of all real bounded

null sequences with the supremum norm. There exists a C^∞-function on c_0 which is nonzero in the open ball and zero off it. To see this, let $g : \mathbb{R} \to \mathbb{R}$ be a C^∞-function satisfying

$$g(t) = \begin{cases} 1 & \text{if} \quad |t| \leqq \frac{1}{2} \\ \frac{1}{2} & \text{if} \quad \frac{1}{2} < |t| < 1 \\ 0 & \text{if} \quad |t| \geqq 1 . \end{cases}$$

If $x = (x_1, x_2, \ldots) \in c_0$, let $f_n(x) = \prod_{i=1}^{\infty} g(x_i)$. Then f is the required function on c_0. However, J. Wells [8] showed that if f is a real valued continuous function on c_0 with a uniformly continuous derivative, then the support of f must be unbounded. Thus c_0 is not $C^{2,2}$-smooth. R. Aron [1] has shown that such a result is true for $C(X)$, the space of all real continuous functions on a compact Hausdorff space X.

The question arises, then, as to what type of spaces X have the property that there exist a nonzero real continuous function f on X such that the derivative Df is uniformly continuous and f has a bounded support. Let us call a space <u>U-b-smooth</u> if it possesses the above property. The examples of spaces which are not U-b-smooth, namely c_0 and $C(X)$, are not reflexive. Is reflexivity essential for a space to be U-b-smooth? The answer is yes and it was proved by Sundaresan [6], and also by K. John, H. Torunczyk and V. Zizler [7] using different methods. Actually, the property of U-b-smoothness characterises an important subclass of reflexive spaces, known as superreflexive spaces.

2. SUPERREFLEXIVE SPACES

Let X be a Banach space. X is uniformly convex if, for any pair $\{x_n\}$, $\{y_n\}$ of sequences in the unit ball of X such that $\|\frac{x_n+y_n}{2}\| \to 0$ then $\|x_n - y_n\| \to 0$.

X is uniformly smooth iff the norm of X is uniformly Frechet differentiable. Uniform convexity and uniform smoothness are dual properties in the sense that X is uniformly convex iff X^* is uniformly smooth; and, in either case, the space is reflexive. It was a long-standing open problem whether X having a uniformly convex norm implied that X also had a uniformly smooth norm. The concept of superreflexivity arose from a solution to this problem by R.C. James and Per Enflo. See van Dulst [4] for details. A Banach space Y is finitely-representable in X, iff, for a finite dimensional subspace Y_0 of Y and $\lambda > 1$, there exists an isomorphism T of Y_0 into X such that

$$\lambda^{-1}\|y\| \leqq \|Ty\| \leqq \lambda\|y\|$$

for all $y \in Y_0$.

A Banach space X is called superreflexive if every Banach space Y that is finitely representable in X is reflexive.

For $\varepsilon > 0$, an ε-tree T in a Banach space X is a set of points x_{ij} in X, $i,j = 0,1,2,\ldots,j < 2^i$, such that for each such i, j,

$$x_{ij} = \tfrac{1}{2}(x_{i+1,2j} + x_{i+1,2j+1}) \quad \text{and}$$

$$\|x_{i,2j} - x_{i,2j+1}\| \geqq \varepsilon .$$

If i is restricted to be only $\leqq n$, then we have an $(n-\varepsilon)$ tree, denoted by $T_{n,\varepsilon}$.

It is a beautiful theorem of R.C. James-Per Enflo that the following are equivalent for a Banach space X.

(a) X is superreflexive,

(b) X has an equivalent uniformly convex norm,

(c) X has an equivalent uniformly smooth norm,

(d) X has an equivalent norm which is both uniformly convex and uniformly smooth,

(e) For each $\varepsilon > 0$, there exists $\underline{n}$ such that no $(n-\varepsilon)$ tree, $T_{n,\varepsilon}$, lies in the unit ball of X .

3. ULTRAPOWERS AND U-b-SMOOTHNESS

Let S be an infinite set and $\mathcal{U}$, a non-trivial ultrafilter on S . The limit of a real bounded function f on S with respect to $\mathcal{U}$ is defined by:

$$\lim_{\mathcal{U}} \{f(s)\} = \sup\{\lambda : \{s \in S : f(s) > \lambda\} \subset \mathcal{U}\} .$$

If X is a Banach space and f is a bounded X-valued function on S , let

$$|f| = \lim_{\mathcal{U}} \{\|f(x)\|\} .$$

Then $|\cdot|$ is a semi-norm on the vector space V of all bounded X-valued functions on S . The quotient space of V modulo the kernel of $|\cdot|$, equipped with the quotient-norm is called the ultrapower of X with respect to the pair $(S,\mathcal{U})$ and is denoted by $X(S,\mathcal{U})$. The space X is isometrically embedded in $X(S,\mathcal{U})$. The usefulness of this notion in Banach space theory stems from the following results: (a) The ultrapower of a Banach space is also a Banach space. (b) If a Banach space Y is finitely represented in a Banach space X , then Y is isometric with a subspace of some ultrapower $X(S,\mathcal{U})$. For proofs of these results and other allied results, see [7].

Sundaresan [6] proved that:
If X is U-b-smooth, then an ultrapower $X(S,\mathcal{U})$ of X is also U-b-smooth.

For a proof, see [6] or [7]. It follows from this theorem that, if Y is finitely representable in a U-b-smooth space, then Y is also U-b-smooth.

Since U-b-smooth spaces are reflexive, it follows further that if X is U-b-smooth, then X is superreflexive. The converse result that when X is superreflexive then X is U-b-smooth is also true. To see this, we first note that when X is superreflexive, then X is isomorphic to a uniformly smooth space, and that U-b-smoothness is invariant under isomorphisms. The norm of a uniformly smooth space is uniformly continuously differentiable on regions $\{x : \lambda \leq \|x\| \leq \mu\}$ and by composing the norm with a suitable real C^1-function on the reals, it can be shown that the composition is also uniformly continuously differentiable. One can then construct, given $r, \varepsilon > 0$, a function $f : X \to \mathbb{R}$ such that $0 \leq f \leq 1$, f is uniformly continuously differentiable, and $f \equiv 1$ on an open ball of radius r, centered at 0, while f vanishes outside the closed ball of radius $r + \varepsilon$, center 0. The details can be had from [6], [7].

4. UNIFORMLY SMOOTH PARTITIONS OF UNITY

K. John, H. Torunczyk and V. Zizler approach the problem in a different manner. They introduce the notion of a dual tree as follows and show that a space is superreflexive if it admits partitions of unity formed by functions with uniformly continuous differentials. Let X be a Banach space and $K, \varepsilon, \eta > 0$, $\delta > 1$ be given. Let $|\cdot| \leq K\|\cdot\|$ be a pseudonorm on X. Then a dual tree $D(K,\varepsilon,|\cdot|,\delta,\eta)$ is a set of points $x_{ij} \in X$, $i,j = 0,1,\ldots,j < 2^i$ such that, for each such i,j

$$x_{ij} = \tfrac{1}{2}(x_{i+1,2j} + x_{i+1,2j+1}) , \quad \|x_{i,2j} - x_{i,2j+1}\| \leq 2\delta$$

$$|x_{ij} + t(x_{i+1,2j} - x_{ij})| \geq |x_{ij}| + \varepsilon\delta|t| - \eta$$

for any $|t| \leq 1$. If i is restricted to be only $\leq n$ then we have a *dual n-tree* $D_n(K,\varepsilon,|\cdot|,\delta,\eta)$. Dual trees can be constructed in, say, the space ℓ_1.

With such a definition, these three authors devote their paper [4] proving the following theorem:

The following are equivalent on a Banach space X :

(i) X is superreflexive.

(ii) X is U-b-smooth.

(iii) For any open cover $\mathcal{U}$ of X, there is a locally finite partition of unity on X subordinated to $\mathcal{U}$ and consisting of functions which are uniformly continuously differentiable.

(iv) *Negation* of: There exist $\varepsilon > 0$ and $K > 0$ such that, for any n and $\delta \in (0,1)$, $\eta > 0$, there is a pseudo-norm $|\cdot| \leqq K\|\cdot\|$ on X and a dual-n-tree, $D_n(K,\varepsilon,|\cdot|,\delta,\eta) \subset X$.

For proving the implications, they do not use ultrapowers; the most involved part of the proof is in the implication (i) $\Rightarrow$ (iii). We refer to [4] for details.

5. APPLICATIONS

(1) It is a well known fact that all separable Banach spaces are homeomorphic. It can be shown, using the results of section 3 that there cannot exist uniformly continuously differentiable homeomorphisms for certain Banach space, whether they are separable or not. Specifically, the following result is proved [6]: Suppose E and F are Banach spaces and there exists a uniformly continuously differentiable homeomorphism from E to F, then E is superreflexive.

(2) Let ϕ be a real function on a Banach space X such that $\phi(x) \to 0$ as $\|x\| \to \infty$. Let α be a non-trivial continuous function on $\mathbb{R}$ to $\mathbb{R}$ with compact support, and let $Q : X \to$ space of symmetric bilinear

forms on X, such that Q is bounded and $\alpha(\|x\|)\cdot Q(x) \neq 0$ for at least one point $x \in X$. Then the following differential equation is of interest in theory of dynamical flows: $D^2\phi(x) = \alpha(\|x\|)\cdot Q(x)$. It can be shown [2] that for a non-superreflexive space X, there can be no solution ϕ vanishing at infinity.

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