EIGENSTRUCTURE SPECIFICATION IN HILBERT SPACE

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INTRODUCTION

The solution of the problem of spectrum assignment by linear state feedback for linear finite dimensional systems, is by now a classical result of linear systems theory. The proof was first given in [14]. A statement of the problem and its solution is to be found in good texts on linear systems theory [1], [11]. In the main, the proofs rely on a transformation of the original linear system into a canonical form, wherein the effect of the feedback matrix on the closed loop characteristic polynomial is directly apparent. If the system is completely controllable, it is shown that the coefficients of the characteristic polynomial of the canonical form of the closed loop system, may be arbitrarily specified by choice of the feedback matrix.

There has been recent interest in this problem for infinite dimensional state spaces [2], [3], [4], [9], [10]. In [9], [10] for systems described by a class of linear hyperbolic partial differential equations, an approach analagous to the finite dimensional treatment described above has been adopted. That is, a transformation to a canonical form and a choice of feedback to assign the spectrum of the canonical form. We have two main criticisms to make of this approach. Firstly, it does not seem readily adaptable to other classes of infinite dimensional linear systems which are of interest. Secondly, the feedback constructed for the canonical form does not readily lead to the required feedback for the original system.

Our viewpoint is more geometric than most elucidated thus far. We strongly adopt the position that the closed loop spectrum of the linear system should not be the only concern of a theory of spectrum assignment. Whilst the spectrum provides important qualitative information, the eigenvectors provide equally important quantitative information. Indeed, in many cases the spectral representation of the closed loop system operator given by the closed loop eigenvectors, leads to effective construction of the closed loop system semi-group.

Our dictum is that a general theory of spectrum assignment should include naturally, the generation of the eigenvectors corresponding to the assigned closed loop spectrum. We eschew canonical forms and work directly with the given system. We show the possibility of spectrum assignment depends in a crucial way on the dimension of the control space being sufficiently large in relation to the dimension of the eigenspaces of the linear system operator. This problem was previously considered by Sun [13] for the case of a one dimensional control space. Our methods are unrelated to those of [13] and significantly improve on the main result which appears there.

To make matters concrete, we consider the linear system

$$\dot{x} = Ax + Bu \qquad (1.1)$$

where $x:[0,\infty) \to X$, X a complex, seperable Hilbert space, $u:[0,\infty) \to U$, U a finite dimensional complex inner product space, dim $U = m$, $A: X \to X$, a closed, linear operator with dense domain, $B: U \to X$ a non-singular bounded linear operator. Precise conditions on the pair (A,B) will be given presently. For the moment we assume that A has pure point spectrum $\sigma(A) = \{\lambda_i ; i = 1,2...\}$ and the eigenvectors of A form a basis for X . There arises the question as to whether, given a countable set of complex numbers $\{\mu_i ; i = 1,2,...\}$, there exists a bounded linear operator $F : X \to U$ such that $\sigma(A + BF) = \{\mu_i ; i = 1,2,...\}$. This question arises after the introduction into (1.1) of a control of linear state feedback type, $u = Fx$, wherein (1.1) becomes

$$x = (A + BF)x \qquad (1.2)$$

In the following sections of the paper we state conditions on the pair (A,B) , and the set $\{\mu_i\}$ which provide an affirmative answer to this question. Moreover, we provide a constructive procedure for obtaining F .

Our proof proceeds as follows: Corresponding to the set $\{\mu_i ; i = 1,2...\}$ with corresponding multiplicities $\{\tilde{\nu}_i ; i = 1,2,...\}$, we construct a countable set of vectors in X . Subject to the pair (A,B) being controllable and conditions on the sets $\{\mu_i\}$, $\{\tilde{\nu}_i\}$, this set of vectors is shown to form a Riesz basis for X . The linear operator F is then defined on X and shown to be bounded. The Riesz basis constructed for X is shown to consist of eigenvectors of $A + BF$ corresponding to $\sigma(A + BF) = \{\mu_i\}$ with corresponding multiplicities $\{\nu_i\}$.

Previously almost nothing was known concerning this problem for the case of multiple inputs (dim $U = m > 1$) or when A has eigenvalues of multiplicities greater than one. We provide a complete solution for the general problem by a construction which can provide a basis for computation. Moreover our results improve in a significant way over previous results, even for the single input case.

We state our main result

THEOREM 1 *Let* A *be a discrete spectral operator of scalar type on a Hilbert space* X *and let* A *satisfy conditions* 1,2 below. *Let* B *be a nonsingular operator from* $\mathbb{C}^m$ *into* X *and let the pair* (A,B) *be controllable.*

Then for any countable distinct set of complex numbers $\{\mu_i\}$ *and any countable set of positive integers* $\{\nu_i\}$ *satisfying conditions* 4,5 below *there exists a bounded linear operator* $F : X \to \mathbb{C}^m$ *such that* $A + BF$ *is discrete, spectral and scalar,* $\sigma(A + BF) = \{\mu_i\}$ *and* $\dim \operatorname{Ker}(A + BF - \mu_i) = \tilde{\nu}_i$

MAIN RESULTS. Let $A : X \to X$ be discrete, spectral and of sclar type and $\sigma(A) = \{\lambda_i \ ; \ i = 1,2,\dots\}$ with the following properties

1. $$\inf_{i \neq k} |\lambda_i - \lambda_k| = \delta > 0$$

2. $$\sup_k \sum_{i \neq k} \frac{1}{|\lambda_i - \lambda_k|^2} < \infty$$

Let $E_i = \operatorname{Ker}(\lambda_i - A)$, $\dim E_i = \nu_i < \infty$. The adjoint $A^* : X \to X$ has spectrum $\sigma(A^*) = \{\bar{\lambda}_i \ ; \ i = 1,2,\dots\}$ and corresponding eigenspaces $F_i = \operatorname{Ker}(\bar{\lambda}_i - A^*)$, $\dim F_i = \nu_i < \infty$.

Let $B : \mathbb{C}^m \to X$, $\operatorname{Ker} B = \{0\}$. A crucial property of the linear system (A,B) is that it be *controllable*. We recall the following result [6].

THEOREM 2. (A,B) *is controllable if and only if* $B^* : X \to \mathbb{C}^m$ *is an isomorphism on the subspace* F_i *for each* $i = 1,2,\ldots$.

From the above result it is necessary for controllability of (A,B) that $\nu_i \le m$ for $i = 1,2,\ldots$. If (A,B) is not controllable, then $\dim B^*F_i = r_i < \nu_i$ for some i . Let $F_i' \subset F_i$ be a subspace of X , $\dim F_i' = \nu_i - r_i > 0$, such that $B^*f = 0$ for $f \in F_i'$. Consider

$$(A + BF^*)\, f = (A^* + F^*B^*)f = A^*f = \bar{\lambda}_i f$$

That is, $\bar{\lambda}_i(\lambda_i)$ is an eigenvalue of $(A + BF)^*$ $(A + BF)$, with corresponding eigenspace of dimension $\nu_i - r_i > 0$. We assume that

3. (A,B) is controllable.

We choose an orthonormal basis for each B^*F_i , $\{\gamma_j^i\ ;\ j = 1,\ldots,\nu_i\}$. Each γ_j^i has a unique inverse image $\psi_j^i \in F_i$. That is,

$$B^*\psi_j^i = \gamma_j^i\ ,\quad j = 1,\ldots,\nu_i$$

The collection of eigenvectors of A^* , $\{\psi_j^i\ ;\ i = 1,2\ldots,\ j = 1,\ldots,\nu_i\}$ is a Riesz basis for X [7] , and $\{\phi_j^i\}$ is the unique dual biorthogonal basis for X consisting of eigenvectors of A . That is,

$$\langle \phi_j^i,\ \psi_l^k\ \rangle = \delta_{ik}\delta_{jl},\ i,k=1,2,\ldots,j=1,\ldots,\nu_i;\ l=1,\ldots,\nu_k,$$

for $x \in X$, $x = \sum_{i=1}^{\infty} \sum_{j=1}^{\nu_i} \langle x, \psi_j^i \rangle \phi_j^i$,

$$c\|x\|^2 \le \sum_i \sum_j |\langle x, \psi_j^i \rangle|^2 \le C\|x\|^2 .$$

We complete the set $\{\gamma_j^i\}$ to an orthonormal basis for $\mathbb{C}^m$, $\{\gamma_j^i ; j = 1,\ldots,m\}$,

$$\langle \gamma_j^i , \gamma_l^i \rangle = \delta_{jl} , \quad j,l = 1,\ldots,m$$

We choose a sequence of complex numbers $\{\mu_i\}$ and a sequence of positive integers $\{\tilde{\nu}_i\}$ satisfying

4. $\sum_{i=1}^{\infty} |\mu_i - \lambda_i|^2 < \infty$

5. (i) $\tilde{\nu}_i \le m$

   (ii) For some integer $k > 0$, $\tilde{\nu}_i = \nu_i$ for $i > k$

   (iii) $\sum_{i=1}^{k} \tilde{\nu}_i = \sum_{i=1}^{k} \nu_i$

We initially assume that $\mu_k \in \rho(A)$ and define a sequence of vectors $\{\tilde{e}_l^k ; k = 1,2,\ldots; l = 1,\ldots,\tilde{\nu}_k\}$ by

$$\tilde{e}_l^k = \begin{cases} (\mu_k - \lambda_k) R(\mu_k ; A) B\gamma_l^k , & l \le \nu_k \\ R(\mu_k ; A) B\gamma_l^k , & l > \nu_k \end{cases}$$

for $l = 1,\ldots,\tilde{\nu}_k$.

Consider, $\langle R(\mu_k; A)B\gamma_l^k, \psi_j^i \rangle = \langle B\gamma_l^k, R(\bar{\mu}_k; A^*)\psi_j^i \rangle$

$$= \langle B\gamma_l^k, \frac{\psi_j^i}{\mu_k-\lambda_i} \rangle = \frac{\langle \gamma_l^k, B^*\psi_j^i \rangle}{\mu_k-\lambda_i} = \frac{\langle \gamma_l^k, \gamma_j^i \rangle}{\mu_k-\lambda_i}$$

for $l = 1,\ldots,\nu_k$, $j = 1,\ldots,\nu_i$.

Then,

$$\langle \tilde{e}_l^k, \psi_j^i \rangle = \begin{cases} \left[\dfrac{\mu_k-\lambda_k}{\mu_k-\lambda_i} \right] \langle \gamma_l^k, \gamma_j^i \rangle , & l \le \nu_k , \ j \le \nu_i \\ \\ \dfrac{1}{\mu_k-\lambda_i} \langle \gamma_l^k, \gamma_j^i \rangle , \ l > \nu_k , & l > \nu_k , \ j \le \nu_i \end{cases}$$

for $l = 1,\ldots,\tilde{\nu}_k$.

Consider the expansion

$$\tilde{e}_l^k = \sum_{i=1}^{\infty} \sum_{j=1}^{\nu_i} \langle \tilde{e}_l^k, \psi_j^i \rangle \phi_j^i$$

$$= \begin{cases} \phi_l^k + \sum\limits_{i \ne k} \sum\limits_j \left[\dfrac{\mu_k-\lambda_k}{\mu_k-\lambda_i} \right] \langle \gamma_l^k, \gamma_j^i \rangle \phi_j^i , & l \le \nu_k \\ \\ \sum\limits_{i \ne k} \sum\limits_j \dfrac{1}{\mu_k-\lambda_i} \langle \gamma_l^k, \gamma_j^i \rangle \phi_j^i , & l > \nu_k \end{cases}$$

for $l = 1,\ldots,\tilde{\nu}_k$. We now remove our assumption $\mu_k \in \rho(A)$. If $\mu_k = \lambda_k \in \sigma(A)$, we define $\tilde{e}^k_l$ by the above formula, obtaining

$$\tilde{e}^k_l = \begin{cases} \phi^k_l , & l \le \nu_k \\ \sum_{i\neq k} \sum_j \frac{1}{\lambda_k-\lambda_i} \langle \gamma^k_l, \gamma^i_j \rangle \phi^i_j , & l > \nu_k \end{cases}$$

for $l = 1,\ldots,\tilde{\nu}_k$.

It is our intention to show that the sequence $\{\tilde{e}^k_l\}$ defined above is a Riesz basis for X . To this end we recall the following definitions;

Two sequences $\{x_i\}$, $\{y_i\}$ in X are said to be *quadratically close* if

$$\sum_{i=1} \|x_i-y_i\|^2 < \infty$$

The sequence $\{x_i\}$ is ω-linearly independent if $\sum_{i=1}^{\infty} c_i x_i = 0$ implies $c_i = 0$, $i = 1,2,\ldots$.

THEOREM 3. (Bari [7]). Any ω-linearly independent sequence which is quadratically close to a Riesz basis of X , is also a Riesz basis of X .

We show that $\{\tilde{e}^k_l\}$ is quadratically close to the Riesz basis $\{\phi^i_j\}$. Because of 5(ii), $l > \nu_k$ occurs at most finitely often. Therefore for $k > K$,

$$\|\tilde{e}_l^k - \phi_l^k\|^2 = \| \sum_{i\neq k} \sum_j \left[\frac{\mu_k-\lambda_k}{\mu_k-\lambda_i} \right] \langle \gamma_l^k, \gamma_j^i \rangle \phi_j^i \|^2$$

$$= |\mu_k-\lambda_k|^2 \| \sum_{i\neq k} \sum_j \left[1 + \frac{\mu_k-\lambda_k}{\lambda_k-\lambda_i} \right]^{-1} \frac{\langle \gamma_l^k, \gamma_j^i \rangle}{\lambda_k-\lambda_i} \phi_j^i \|^2$$

$$\leq C|\mu_k-\lambda_k|^2 \| \sum_{i\neq k} \sum_j \frac{\langle \gamma_l^k, \gamma_j^i \rangle}{\lambda_k-\lambda_i} \phi_j^i \|^2 \quad \text{(from 1, 4)}$$

$$\leq C'|\mu_k-\lambda_k|^2 \sum_{i\neq k} \frac{1}{|\lambda_k-\lambda_i|^2} \quad (\text{from } \|\gamma_j^i\| = 1)$$

Therefore

$$\sum_{k>K} \|\tilde{e}_l^k-\phi_l^k\|^2 \leq C'\left(\sum_{k>K} |\mu_k-\lambda_k|^2\right) \left[\sup_{k>K} \sum_{i\neq k} \frac{1}{|\lambda_k-\lambda_i|^2} \right]$$

$$\leq \infty$$

Quadratic closeness to a basis already implies that $\{\tilde{e}_l^k;\ k > N\}$ is ω-linearly independent for N sufficiently large and it is easy to prove that $\{\tilde{e}_l^k;\ k \leq N\}$ is linearly independent for any N. $\{\tilde{e}_l^k\}$ can be made into a basis by replacing at most finitely many vectors. Defining $F:X \to \mathbb{C}^m$ by

$$F\tilde{e}_l^k = \begin{cases} (\mu_k-\lambda_k)\gamma_l^k, & l \leq \nu_k \\ \gamma_l^k, & l > \nu_k \end{cases}$$

for $l = 1,\ldots,\nu_k$, it easily follows that

$$(A+BF)\tilde{e}^k_l = \mu_k \tilde{e}^k_l \ , \ l = 1,\ldots,\tilde{\nu}_k \ .$$

That is $\{\tilde{e}^k_l\}$ are eigenvectors of $A+BF$ corresponding to eigenvalues $\{\mu_i\}$. From this, it rapidly follows that $\{\tilde{e}^k_l\}$ is ω-linearly independent and hence is Riesz basis for X , by the theorem of Bari.

We can extend F to X by

$$x = \sum_{k=1}^{\infty} \sum_{l=1}^{\tilde{\nu}_k} \langle x, \tilde{f}^k_l \rangle \tilde{e}^k_l \ , \quad x \in X$$

and

$$Fx = \sum_{k=1}^{\infty} \sum_{l=1}^{\tilde{\nu}_i} \langle x, \tilde{f}^k_l \rangle F\tilde{e}^k_l$$

where $\{\tilde{f}^k_l\}$ is the unique dual basis of X , biorthogonal to $\{\tilde{e}^k_l\}$ It only remains to show that F is bounded, $A+BF$ has no other eigenvalues than $\{\mu_k\}$ and each eigenspace of $A+BF$ has dimension $\tilde{\nu}_k$, which is easily done. This completes the proof of our main result.

A number of concluding remarks are worth making:

(a) $\{\gamma^i_j\}$, $\{\phi^i_j\}$, $\{\psi^i_j\}$ can be made *explicit*, hence also $\{\tilde{e}^k_l\}$.

(b) F is not unique. Is there a *smallest* F and how is it characterized.?

(c) It would be useful to remove the condition that A, A+BF is scalar. The Riesz bases would then consist of *generalized* eigenvectors. This would allow eigenvalues of (generalized) multiplicity greater than m .

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