AN INTRODUCTION TO THE THEORY OF ABSOLUTELY p-SUMMING OPERATORS BETWEEN BANACH SPACES

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These notes constitute the text of a sequence of lectures given at the University of New England, a brief synopsis of which was presented at the Conference on Analysis in Linear Spaces. *There is nothing new here* in terms of mathematical knowledge; our only hope is that this theory be new to the audience. Readers who like what they sample here will love Professor Gilles Pisier's forthcoming monograph on "Factorization of Operators between Banach Spaces", the present notes are subsummed by the first section of Pisier.

§1. The class of absolutely p-summing operators

This section establishes the basic facts about the class Π_p of absolutely p-summing operators between Banach spaces. After defining the notion of an absolutely p-summing operator and giving a brief discussion regarding the viewpoint of absolutely p-summing operators being operators that increase the degree of summability of a sequence, we show that Π_p is an "operator ideal" and establish the basic inclusion relationship between Π_p's of different index. We close with a basic example of an absolutely p-summing operator and the fundamental Grothendieck-Pietsch theorem.

Let $1 \le p < \infty$. A linear operator $T: X \to Y$ is called <u>absolutely p-summing</u> if there exists a $\rho > 0$ such that given any $x_1, x_2, \ldots, x_n \in X$ we have

$$(*) \qquad \left(\sum_{i=1}^{n} \|Tx_i\|^p\right)^{1/p} \le \rho \sup\left\{\left(\sum_{i=1}^{n} |x^*x_i|^p\right)^{1/p} : \|x^*\| \le 1\right\}.$$

We denote the fact that $T : X \to Y$ is absolutely p-summing by $T \in \Pi_p(X;Y)$; notice that any $T \in \Pi_p(X;Y)$ is bounded. For $T \in \Pi_p(X;Y)$ we define the p-summing norm $\Pi_p(T)$ by

$$\Pi_p(T) = \inf\{\rho > 0 : (*) \text{ holds}\};$$

plainly, $\|T\| \le \Pi_p(T)$ holds for any T.

To better understand just what impositions are put upon an operator when it is assumed to be p-summing we consider two classes of vector-valued sequence spaces : ℓ_p^{strong} and ℓ_p^{weak}. If X is a Banach space and $1 \le p < \infty$, then a sequence (x_n) in X is said to be <u>strongly p-summable</u> if $(\|x_n\|) \in \ell_p$; in case (x_n) is strongly p-summable we say $(x_n) \in \ell_p^{\text{strong}}$ and give (x_n) the length $\|(x_n)\|_{\ell_p^{\text{strong}}} = \|(\|x_n\|)\|_p$. Again, if X is a Banach space and $1 \le p < \infty$, then a sequence (x_n) in X is called <u>weakly p-summable</u> if for each $x^* \in X^*$, $(x^*x_n) \in \ell_p$; the set of all such sequences is denoted by ℓ_p^{weak}. It is an easy consequence of the closed graph theorem that whenever $(x_n) \in \ell_p^{weak}$

$$\|(x_n)\|_{\ell_p^{weak}} \equiv \sup\{(\sum_n |x^*x_n|^p)^{1/p} : x^* \in X^*, \|x^*\| \leq 1\} < \infty .$$

Whenever $T : X \to Y$ is a bounded linear operator, T induces natural operators from $\ell_p^{strong}(X)$ to $\ell_p^{strong}(Y)$ and from $\ell_p^{weak}(X)$ to $\ell_p^{weak}(Y)$ via coordinate-by-coordinate application of T. When this naturally induced operator takes $\ell_p^{weak}(X)$ into $\ell_p^{strong}(Y)$ it is, again by a closed graph argument, bounded and this happens precisely when T is absolutely p-summing with the p-summing norm of T precisely equal to the operator norm of the induced operator from ℓ_p^{weak} to ℓ_p^{strong}. A consequence of this is the fact that $T \in \Pi_p(X;Y)$ precisely when for finitely non-zero sequences (x_n) in X we have for some $\rho \geq 0$ that

$$(**) \qquad \|(Tx_n)\|_{\ell_p^{strong}(Y)} \leq \rho \|(x_n)\|_{\ell_p^{weak}(X)}$$

and, moreover, the tightest fitting ρ in (**) is precisely $\Pi_p(T)$. In addition to possible clarification of the nature of absolutely p-summing operators, the above affords some notational conveniences that are not inconsiderable. That this is so will be made clear throughout the rest of this section.

$\Pi_p(X;Y)$ *is a normed linear space with norm* Π_p

The only real obstacle to be overcome in understanding why this is so is the triangle inequality and its attendant consequence that the sum of two p-summing operators is p-summing. Let $S,T \in \Pi_p(X;Y)$ and let $x_1,\ldots x_n \in X$. Then on considering the finitely non-zero sequence $(x_1,x_2,\ldots,x_n,0,0\ldots)$ we have

$$
\begin{aligned}
\|(Sx_k + Tx_k)\|_{\ell_p^{strong}} &= \left\| (\|Sx_k\| + \|Tx_k\|) \right\|_p \\
&\le \left\| (\|Sx_k\|) \right\|_p + \left\| (\|Tx_k\|) \right\|_p \\
&\le \Pi_p(S)\|(x_k)\|_{\ell_p^{weak}} + \Pi_p(T)\|(x_k)\|_{\ell_p^{weak}} \\
&= [\Pi_p(S) + \Pi_p(T)]\|(x_k)\|_{\ell_p^{weak}}
\end{aligned}
$$

It follows that $S + T$ is absolutely p-summing and that $\Pi_p(S + T) \le \Pi_p(S) + \Pi_p(T)$.

$\Pi_p(X;Y)$ *is a Banach space with norm* Π_p

Let (T_n) be a Π_p-Cauchy sequence. Since $\|T\| \le \Pi_p(T)$ always holds, (T_n) is Cauchy in the classical operator norm as well. Therefore there is a bounded linear operator $T_0\colon X \to Y$ such that $\lim_n \|T_0 - T_n\| = 0$. We claim that $T_0 \in \Pi_p(X;Y)$ and that $\lim_n \Pi_p(T_0 - T_n) = 0$. Let $\varepsilon > 0$ be given. Choose N_ε so that whenever $m,n \ge N$ we have $\Pi_p(T_m - T_n) \le \varepsilon$. Then given any finitely non-zero sequence (x_k) of members of X we have

$$\|(T_m x_k - T_n x_k)_k\|_{\ell_p^{strong}} \le \varepsilon \, \|(x_k)\|_{\ell_p^{weak}}$$

whenever $m,n \ge N_\varepsilon$. If $x_k = 0$ for $k > k_0$, then this translates to mean that once $m,n \ge N$

$$\left(\sum_{k=1}^{k_0} \|T_m x_k - T_n x_k\|^p \right)^{1/p} \le \varepsilon \|(x_k)\|_{\ell_p^{weak}} .$$

But now letting m run off towards ∞ we see that

$$\left(\sum_{k=1}^{k_0} \|T_0 x_k - T_n x_k\|^p \right)^{1/p} \le \varepsilon \, \|(x_k)\|_{\ell_p^{weak}}$$

as well. The arbitrariness of (x_k) and of $\varepsilon > 0$ tells us that in fact $T_0 - T_n \in \Pi_p(X,Y)$ for all $n \ge N$ and that $\lim_n \Pi_p(T_0 - T_n) = 0$.

Of course now $T_0 = (T_0 - T_n) + T_n$ belongs to $\Pi_p(X;Y)$ as well and the completeness of $\Pi_p(X;Y)$ with the norm Π_p is established.

Π_p is an operator ideal, that is, if $S : X \to Y$ and $T : Y \to Z$ are bounded linear operators one of which is absolutely p-summing, then TS is absolutely p-summing.

Suppose $T \in \Pi_p$. Then for any finitely non-zero sequence (x_n) in X:

$$\left(\sum \|TSx_i\|^p\right)^{1/p} \le \Pi_p(T)\|Sx_i\|_{\ell_p^{weak}} \le \Pi_p(T)\|S\|\|x_i\|_{\ell_p^{weak}}$$

and so
$$\Pi_p(TS) \le \Pi_p(T)\|S\|$$

Suppose $S \in \Pi_p$. Then for any finitely non-zero sequence (x_n) in X:

$$\left(\sum \|TSx_i\|^p\right)^{1/p} \le \|T\|\left(\sum \|Sx_i\|^p\right)^{1/p} \le \|T\|\Pi_p(S)\|(x_i)\|_{\ell_p^{weak}}$$

and so
$$\Pi_p(TS) \le \|T\|\Pi_p(S).$$

It follows from this that we have the following:

If $R : W \to X$, $S : X \to Y$ and $T : Y \to Z$ are bounded linear operators with $S \in \Pi_p(X;Y)$, then TSR is absolutely p-summing with $\Pi_p(TSR) \le \|T\|\Pi_p(S)\|R\|$.

How do the classes Π_p compare for differing p's?

If $1 \le p < q < \infty$, then $\Pi_p(X;Y) \subseteq \Pi_q(X;Y)$ and the inclusion map is contractive.

Let $T \in \Pi_p(X;Y)$. Then for any finitely non-zero sequence (x_i) of vectors in X and any sequence (λ_i) of scalars,

$$\left(\sum\|\lambda_i Tx_i\|^p\right)^{1/p} \le \Pi_p(T)\|(\lambda_i x_i)\|_{\ell_p^{weak}} .$$

Suppose r is chosen for the purpose:

$$\frac{1}{r} + \frac{1}{q} = \frac{1}{p} .$$

Then an easy application of Holder's inequality shows that

$$\|(\lambda_i x_i)\|_{\ell_p^{\text{weak}}} \leq \|(\lambda_i)\|_r \, \|(x_i)\|_{\ell_q^{\text{weak}}}$$

If we let $\lambda_i = \|Tx_i\|^{q/r}$, then

$$\begin{aligned}
\|(\lambda_i Tx_i)\|_{\ell_p^{\text{strong}}} &= (\textstyle\sum \|\lambda_i \; Tx_i\|^p)^{1/p} \\
&= \left(\textstyle\sum (\|Tx_i\|^{q/r} \, \|Tx_i\|)^p\right)^{1/p} \\
&= \left(\textstyle\sum \|Tx_i\|^q\right)^{1/p} ;
\end{aligned}$$

where

$$\|(\lambda_i)\|_r = (\textstyle\sum |\lambda_i|^r)^{1/r} = (\textstyle\sum \|Tx_i\|^q)^{1/r} .$$

Since we may as well suppose (λ_i) isn't entirely zeroes, we see then that

$$\begin{aligned}
(\textstyle\sum \|Tx_i\|^q)^{1/q} &= (\textstyle\sum \|Tx_i\|^q)^{1/p} / (\textstyle\sum \|Tx_i\|^q)^{1/r} \\
&= \|(\lambda_i Tx_i)\|_{\ell_p^{\text{strong}}} / \|(\lambda_i)\|_r \\
&\leq \Pi_p(T) \|(\lambda_i x_i)\|_{\ell_p^{\text{weak}}} / \|(\lambda_i)\|_r \\
&\leq \Pi_p(T) \; \|(\lambda_i)\|_r \; \|(x_i)\|_{\ell_q^{\text{weak}}} / \|(\lambda_i)\|_r \\
&= \Pi_p(T) \; \|(x_i)\|_{\ell_q^{\text{weak}}}
\end{aligned}$$

<u>Example</u>. Let Ω be a compact Hausdorff space and μ be a regular Borel probability measure defined on Ω. Then the inclusion map $I_p : C(\Omega) \to L_p(\mu)$

is absolutely p-summing for each $1 \le p < \infty$. In fact, if $f_1, \ldots, f_n \in C(\Omega)$, then

$$\begin{aligned}\left(\sum \|I_p f_i\|_p^p\right)^{1/p} &= \left(\sum \int_\Omega |f(w)|^p \, d\mu(w)\right)^{1/p} \\ &= \left(\int_\Omega \sum |f_i(w)|^p \, d\mu(w)\right)^{1/p} \\ &= \left(\int_\Omega \sum |\delta_w(f_i)|^p \, d\mu(w)\right)^{1/p},\end{aligned}$$

where $\delta_w \in C(\Omega)^*$ is the functional $\delta_w f = f(w)$,

$$\begin{aligned}&\le \int_\Omega \|(f_i)\|_{\ell_p^{\text{weak}}} \, d\mu(w) \\ &= \|(f_i)\|_{\ell_p^{\text{weak}}} .\end{aligned}$$

It follows that $I_p \in \Pi_p(C(\Omega); L_p(\mu))$ and $\Pi_p(I_p) \le 1$.

If μ is just a regular Borel measure on Ω , then the inclusion map of $C(\Omega)$ into $L_p(\mu)$ is still p-summing with p-summing norm $|\mu|(\Omega)^{1/p}$.

In a sense the above example is the prototype of all absolutely p-summing operators. That this is so is part and parcel of the next result which is the most fundamental feature about absolutely p-summing operators.

The Grothendieck-Pietsch Domination Theorem

Let $T : X \to Y$ *be absolutely p-summing. Then there exists a regular Borel probability measure* μ *defined on* $(B_{X^*}, \text{weak*})$ *for which*

$$\|Tx\| \le \Pi_p(T) \left(\int_{B_{X^*}} |x^*(x)|^p \, d\mu(x^*)\right)^{1/p}$$

Proof

We make good use of the fact that $\Pi_p(T)$ is the least $\rho \ge 0$ such that for any $x_1, \ldots x_n \in X$,

$$\sum_{i=1}^{n} \|Tx_i\|^p \leq \rho^p \qquad \sup\{\sum_{i=1}^{n} |x^*x_i|^p : \|x^*\| \leq 1\}$$

How? Well if $x_1, \ldots, x_n \in X$ are given, then the function

$\varphi_{x_1,\ldots,x_n}(x^*) = \Pi_p^p(T) \sum_{i=1}^{n} |x^*x_i|^p - \sum_{i=1}^{n} \|Tx_i\|^p$ is continuous on

$(B_{X^*}, \text{weak*})$. Let $\Phi \subseteq C(B_{X^*}, \text{weak*})$ be the family of all such $\phi_{x_1,\ldots,x_n}$'s. Φ is a convex cone. What's more, because $\Pi_p(T)$ is what it is, each $\phi \in \Phi$ achieves a non-negative value somewhere on $(B_{X^*}, \text{weak*})$. Therefore, Φ is disjoint from the convex cone N of all continuous functions ψ on $(B_{X^*}, \text{weak*})$ that're everywhere negative, a set with non-empty interior. It follows from the Hahn-Banach theorem that there is a $\mu \in C(B_{X^*}, \text{weak*})^*$ such that

$$\mu(\psi) < 0 \leq \mu(\phi)$$

for all $\psi \in N$ and all $\phi \in \Phi$. Plainly we can assume $\|\mu\| = 1$. Since $\mu(\psi) < 0$ for all $\psi \in N$, μ is a non-negative Borel measure on $(B_{X^*}, \text{weak*})$; $\|\mu\| = 1$ just says μ is a probability measure. Checking $0 \leq \mu(\phi)$ for $\phi = \phi_x$ finishes the proof.

§2. More examples of absolutely p-summing operators

In this lecture we give some more examples of absolutely p-summing operators. To begin, we show that the absolutely 2-summing operators between Hilbert spaces are precisely those operators of Hilbert-Schmidt type even up to coincidence of norms. Then we employ Khintchine's inequalities to show that the natural inclusion map of ℓ_1 into ℓ_2 is absolutely 1-summing; since every Hilbert-Schmidt operator admits this natural inclusion as a factor, the ideal property of absolutely 1-summing operators alerts us to the fact that Hilbert-Schmidt operators are always absolutely

1-summing. Finally, we note that, thanks to the Grothendieck-Pietsch domination theorem, absolutely p-summing operators are always weakly compact and completely continuous, features of the class that are any-thing but obvious from the definition. One upshot of this observation is a transparent proof of the celebrated Dvoretsky-Rogers theorem: the identity operator on an infinite dimensional Banach space is never absolutely p-summing for any $1 \le p < \infty$.

Example. *If* H *and* K *are Hilbert spaces, then* $\Pi_2(H:K)$ *coincides with the class of Hilbert-Schmidt operators from* H *to* K*; moreover, the* Π_2 *norm and the Hilbert-Schmidt norm are the same.*

To establish the above claim, it's convenient to recall a few erstwhile facts about the Hilbert-Schmidt operators. Let H, K be Hilbert spaces and $T : H \to K$ be a bounded linear operator. We say T is a Hilbert-Schmidt operator if for some complete orthonormal system $(e_i)_{i \in I}$ in H, $\sum_i \|Te_i\|^2 < \infty$; it is not too difficult to establish that if T is a Hilbert-Schmidt operator then $\sum_i \|Te_i\|^2 < \infty$ for every complete orthornormal system $(e_i)_{i \in I}$ for H and that the sum $\sum_i \|Te_i\|^2$ is the same regardless of the choice of $(e_i)_{i \in I}$. The number $\sigma(T) = (\sum \|Te_i\|^2)^{\frac{1}{2}}$ is called the Hilbert-Schmidt norm of T. Every Hilbert-Schmidt operator is compact and admits of a representation in the form $\sum_n \lambda_n < \cdot \,, e_n > f_n$ for some $(\lambda_n) \in \ell_2$, and some orthornormal sequences (e_n) and (f_n) in H and K respectively with $\|(\lambda_n)\|_2$ coinciding with the Hilbert-Schmidt norm of the operator.

Suppose $T : H \to K$ is absolutely 2-summing. Let $e_1, e_2, \ldots, e_n$ be orthonormal vectors. Then for any $x \in H$

$$\sum |<x, e_i>|^2 \le \|x\|^2$$

by Bessel's inequality. Therefore,

$$(\sum\|Te_i\|^2)^{\frac{1}{2}} \le \Pi_2(T)\ \|(e_i)\|_{\ell_2^{weak}}$$
$$= \Pi_2(T)\ \sup\{(\sum_1^n |x^*e_i|^2)^{\frac{1}{2}} : \|x^*\| \le 1\}$$
$$= \Pi_2(T)\ \sup\{(\sum_{i=1}^n |<x, e_i>|^2)^{\frac{1}{2}} : \|x\| \le 1\}$$
$$\le \Pi_2(T)\ \sup\{(\|x\|^2)^{\frac{1}{2}} : \|x\| \le 1\} = \Pi_2(T).$$

It follows that T is a Hilbert-Schmidt operator and $\sigma(T) \le \Pi_2(T)$.

Now suppose T is a Hilbert-Schmidt operator and represent T in the form

$$Tx = \sum_n \lambda_n <x, e_n> f_n ,$$

where $(\lambda_n) \in \ell_2$ with $\sigma(T) = \|(\lambda_n)\|_2$ and with (e_n) and (f_n) orthonormal sequences in H and K respectively. For any $x \in H$ we have

$$\|Tx\|^2 = \sum_n |\lambda_n|^2\ |<x, e_n>|^2,$$

so for $x_1, \ldots, x_k \in H$ we see that

$$\sum_{i=1}^k \|Tx_i\|^2 = \sum_n |\lambda_n|^2 \sum_{i=1}^k |<x_i, e_n>|^2$$
$$\le \sigma^2(T)\ \sup\{\sum_{i=1}^k |<x_i, u>|^2 : \|u\| \le 1\}$$

Therefore, $T \in \Pi_2(H; K)$ and $\Pi_2(T) \le \sigma(T)$.

The prototype for several of the most striking results in the theory of absolutely p-summing operators is the following fact, itself an interesting interpretation of Khinchine's Inequalities.

Example. *The natural inclusion* $\ell_1 \hookrightarrow \ell_2$ *is absolutely 1-summing.*

Let $x_1, \ldots, x_n \in \ell_1$. Then

$$\sum_i \|x_i\| = \sum_i \Big(\sum_k |x_{ik}|^2\Big)^{\frac{1}{2}}$$

$$\leq C \sum_i \Big\| \sum_k x_{ik}\, r_k \Big\|_1 ,$$

where (r_k) denotes the sequence of Rademacher functions,

$$= C \sum_i \int_0^1 \Big| \sum_k x_{ik}\, r_k\,(t) \Big|\, dt$$

$$= C \int_0^1 \sum_i \Big| \sum_k x_{ik}\, r_k\,(t) \Big|\, dt$$

But given $0 \leq t \leq 1$, $(r_k(t))$ belongs to B_{ℓ_∞} so

$$\sum_i \Big| \sum x_{ik}\, r_k(t) \Big| \leq \sup\{\sum_i |x^* x_i| : x^* \in B_{\ell_\infty}\} .$$

Therefore

$$\sum_i \|x_i\|_2 \leq C \int_0^1 \sum_i \Big| \sum_k x_{ik}\, r_k(t) \Big|\, dt$$

$$\leq C \int_0^1 \sup \{\sum_i |x^* x_i| : x^* \in B_{\ell_\infty}\}\; dt$$

$$= C \sup\{ \sum_i |x^* x_i| : x^* \in B_{\ell_\infty}\} = C \,\|(x_i)\|_{\ell_1}^{\text{weak}}$$

It follows that $\ell_1 \hookrightarrow \ell_2$ is absolutely 1-summing and has Π_1-norm $\leq C$.

Every Hilbert-Schmidt operator admits the natural inclusion $\ell_1 \hookrightarrow \ell_2$ *as a factor.*

Let $T : H \to K$ be such an operator. Then these are sequences (λ_n), (e_n) and (f_n), in ℓ_2, H and K, respectively, for which

$$Tx = \sum \lambda_n \langle x, e_n \rangle f_n$$

where $\sigma(T) = \|(\lambda_n)\|_2$ and (e_n), (f_n) are orthonormal. Define $T_1 : H \to \ell_1$ by

$$T_1 x = (\lambda_n < x, e_n >) ,$$

and note that

$$\|T_1 x\|_1 = \sum_n | \lambda_n < x, e_n > | \le \|(\lambda_n)\|_2 \|(< x, e_n >)\|_2$$

so that $\|T_1\| \le \sigma(T)$. Define $T_2 : \ell_2 \to K$ by

$$T_2((\mu_n)) = \sum_n \mu_n f_n ,$$

and note that $\|T_2\| \le 1$. Plainly T is naught but T_1 composed with $\ell_1 \hookrightarrow \ell_2$ followed by T_2.

Corollary. *Let* H, K *be Hilbert spaces and* $1 \le p \le 2$. *Then*

$$\Pi_p(H; K) = HS(H; K) = \Pi_2(H; K).$$

Proof. We've already commented on the fact that $HS(H; K) = \Pi_2(H; K)$ and on the inclusion of $\Pi_1(H; K)$ in $\Pi_2(H; K)$. Now if $T : H \to K$ is a Hilbert-Schmidt operator then there exist bounded linear operators $L : H \to \ell_1$ and $R : \ell_2 \to K$ such that the diagram

$$\begin{array}{ccc} H & \xrightarrow{T} & K \\ {\scriptstyle L}\downarrow & & \uparrow{\scriptstyle R} \\ \ell_1 & \hookrightarrow & \ell_2 \end{array}$$

commutes. Since $\ell_1 \hookrightarrow \ell_2$ is absolutely 1-summing so too is $R \hookrightarrow L = T$.

We remark that the above offers an interesting mapping property of Hilbert-Schmidt operators unknown to their earliest proponents, namely, Hilbert-Schmidt operators take unconditionally convergent series into absolutely convergent series.

Now for a really informative insight into the structure of p-summing operators we prove the following remarkable result.

<u>Theorem</u>. *Let* $1 \leq p < \infty$. *Then every absolutely p-summing operator is weakly compact and completely continuous.*

<u>Proof</u>. Let $S \in \Pi_p(X; Y)$. By the Grothendieck-Pietsch domination theorem there is a regular Borel probability measure μ on (B_{X^*}, weak^*) such that for any $x \in X$

$$\|Sx\| \leq \Pi_p(S) \left(\int_{B_{X^*}} |x^*x|^p \, d\mu(x^*) \right)^{1/p} .$$

Suppose we consider elements of X as functions defined on B_{X^*}; because each x is weak* continuous on B_{X^*}, the above inequality can be read as follows

$$\|Sx\| \leq \Pi_p(S) \, \|x(\cdot)\|_{L_p(\mu)} .$$

In other words, S acts as a bounded linear operator from $(X, L_p(\mu)$-topology) to Y; on completing X in $L_p(\mu)$ and extending S to the resultant Banach space X_p we have the following factorization:

$$\begin{tikzcd} X \arrow[rr, "S"] \arrow[dr, hook] & & Y \\ & X_p \arrow[ur, "\hat{S}"'] & \end{tikzcd}$$

where $\hat{S}$ is the unique bounded linear extension of S to X_p. It's the inclusion map $X \hookrightarrow X_p$ that we're interested in because it is weakly compact and completely continuous.

$X \hookrightarrow X_p$ is weakly compact. If $1 < p < \infty$, then this follows from the reflexivity of $L_p(\mu)$ and the attendant reflexivity of its closed linear subspaces. If $p = 1$, then the fact that the inclusion map of X into $L_1(\mu)$ has to pass continuously through $L_2(\mu)$ on its way ensures its weak compactness and hence that of $X \hookrightarrow X_1$.

$X \hookrightarrow X_p$ is completely continuous. Indeed, if (x_n) is a weakly null sequence in X, then for all n and all $x^* \in B_{X^*}$, $|x_n(x^*)| \leq \sup\|x_n\| < \infty$ and for all $x^* \in B_{X^*}$, $\lim\limits_n x_n(x^*) = 0$; it follows from the Lebesgue

Bounded Convergence Theorem that $\lim_n \|x_n\|_p = 0$.

<u>Corollary</u>. *If* $1 \leq p < \infty$ *and* $T \in \Pi_p(X; X)$, *then* T^2 *is compact.*

<u>Proof</u>. If (x_n) is a bounded sequence, then (Tx_n) has a subsequence (Tx_n') which is weakly convergent; therefore, $(T^2x_n') = (T(Tx_n'))$ is norm convergent.

<u>Dvoretsky-Rogers Theorem</u>. *Suppose for some* $1 \leq p < \infty$ *we have* $\sum |x^*x_n|^p < \infty$ *for each* $x^* \in X^*$ *implying that* $\sum \|x_n\|^p < \infty$. *Then* $\dim X < \infty$.

<u>Proof</u>. If $\sum_n \|x_n\|^p < \infty$ holds whenever $\sum_n |x^*x_n|^p < \infty$ for each $x^* \in X^*$, then the identity operator $id : X \to X$ on X is absolutely p-summing. It follows that $id^2 = id$ is compact. But the classical Riesz lemma assures us that the identity operator on a normed linear space is compact only in case of a finite dimensional space.

Actually, a bit stronger statement about composing operators can be said.

<u>Corollary</u>. *If* $T \in \Pi_p(X; Y)$ *and* $S \in \Pi_q(Y; Z)$ *for some* $1 \leq p,q < \infty$, *then* ST *is compact.*

§3 A few more applications of the Grothendieck-Pietsch theorem

This lecture provides several additional applications of the Grothendieck-Pietsch theorem. Hints as to the breadth and depth of usage to which this wonderful theorem can be put might be gleaned from this last truly introductory lecture.

We start by introducing the important notion of cotype 2. After observing that Hilbert spaces have cotype 2, we follow Bernard Maurey to the conclusion that absolutely p-summing operators into such spaces are 2-summing for any $p > 2$. We then close the circle of ideas concerning Hilbert-Schmidt operators by showing that this class coincides with the p-summing operators for any $p \geq 1$. Next, we take up an extension property enjoyed by the absolutely 2-summing operators. After recovering from the surprise discovery that regardless of the geometry of a finite dimensional Banach space E the identity operator on E always has 2-summing norm equal to the square root of E's dimension, we continue to follow Stan Kwapien's lead in proving that every n-dimensional subspace of a Banach space is the range of a linear projection of operator norm not exceeding $\sqrt{n}$. This is the famous theorem of Kadec and Snobar.

A Banach space Y is said to have cotype 2 if there is a $k \geq 0$ such that given $y_1, \ldots, y_n \in Y$, then

$$(*) \left(\sum_{i=1}^{n} \|y_i\|^2 \right)^{\frac{1}{2}} \leq k\left(\int_0^1 \|\sum_{i=1}^{n} r_i(t) y_i\|^2 \, dt\right)^{\frac{1}{2}}$$

where the r_i's are the usual Rademacher functions. The least $k \geq 0$ for which (*) holds for all $y_1, \ldots, y_n \in Y$ is called the cotype 2 constant of Y and is denoted by $c_2(Y)$.

It is a classical result of W. Orlicz that if $1 \leq p \leq 2$, then $L_p(\mu)$ has cotype 2 for any measure μ. Naturally, Khintchine's inequalities play a significant role in Orlicz's theorem and we recommend Orlicz's paper be read. However, at this moment we will have to contain our enthusiasm and be happy with the following.

Theorem. *Hilbert spaces have cotype 2 with cotype 2 constant* = 1.

Proof. Let $h_1, \ldots, h_n$ be members of the Hilbert space H.

Then

$$\int \Big\| \sum_{i=1}^{n} r_i(t)h_i \Big\|^2 dt = \int_0^1 \Big< \sum_{i=1}^{n} r_i(t)h_i, \sum_{i=1}^{n} r_i(t)h_i \Big> dt$$

$$= \int_0^1 \sum_{i,j=1}^{n} r_i(t)r_j(t) < h_i, h_j > dt$$

$$= \sum_{i,j=1}^{n} < h_i, h_j > \int_0^1 r_i(t)r_j(t)\, dt$$

$$= \sum_{i=1}^{n} <h_i, h_i> \quad ,$$

because $\displaystyle\int_0^1 r_i(t)r_j(t)\, dt = \delta_{ij}$.

The importance of spaces having cotype 2 derives, in part, from the following.

Theorem. *Let* Y *be a Banach space having cotype 2. Then, for any* $p > 2$ *and any Banach space* X *,*

$$\Pi_2(X;\ Y) = \Pi_p(X;\ Y).$$

Proof. Suppose $T \in \Pi_p(X;\ Y)$. By the Grothendiech-Pietsch Domination Theorem, there is a regular Borel probability measure μ on $(B_{X^*}, \text{weak*})$ such that for each $x \in X$

$$\|Tx\| \le \Pi_p(T) \left(\int_{B_{X^*}} | x^*x \ ^p| d\ (x^*) \right)^{1/p} .$$

Let (x_n) be a finitely non-zero sequence of members of X. Since $c_2(y)$ is the cotype 2 constant of Y

$$\left(\sum \|Tx_j\|^2\right)^{\frac{1}{2}} \leq c_2(Y) \left\{ \int_0^1 \|\sum r_j(t)Tx_j\|^2 \, dt\right\}^{\frac{1}{2}}$$

$$\leq c_2(Y) \left\{ \int_0^1 \|\sum r_j(t)Tx_j\|^p \, dt\right\}^{1/p}$$

$$= c_2(Y) \left\{ \int_0^1 \|T(\sum r_j(t)x_j)\|^p \, dt\right\}^{1/p}$$

$$\leq c_2(Y) \, \Pi_p(T) \left\{\int_0^1\int_{B_{X^*}} |\sum r_j(t) \, x^*(x_j)|^p \, d\mu(x^*) \, dt\right\}^{1/p}$$

$$= c_2(Y) \, \Pi_p(T) \left\{ \int_{B_{X^*}} \int_0^1 |\sum r_j(t) \, x^*(x_j)|^p \, dt \, d\mu(x^*)\right\}^{1/p}$$

$$\leq c_2(Y) \, \Pi_p(T) \int_{B_{X^*}} \left\{ \int_0^1 |\sum r_j(t) \, x^* x_j|^p \, dt\right\}^{1/p} d\mu(x^*),$$

which by Khintchine's inequality is

$$\leq c_2(Y) \, \Pi_p(T) \, a_p \int_B \left(\sum |x^* x_j|^2\right)^{\frac{1}{2}} d\mu(x^*)$$

for some $a_p > 0$. This in turn is

$$\leq c_2(Y) \, \Pi_p(T) \, a_p \, \|(x_j)\|_{\ell_2^{weak}}$$

It follows that $T \in \Pi_2(X; Y)$ with

$$\Pi_2(T) \leq c_2(Y) \, \Pi_p(T) \, a_p$$

Since $\Pi_2(X; Y) \subseteq \Pi_p(X; Y)$ wherever $p \geq 2$ we have completed the proof.

The above result is due to Bernard Maurey and is but a small part of his analysis of spaces with nontrivial cotype.

<u>Corollary</u>. *If* H *and* K *are Hilbert spaces, then for any* $p \geq 1$ *the class of absolutely p-summing operators from* H *to* K *coincides with the Hilbert-Schmidt class.*

Indeed we saw in §2 that all the Π_p's coincide with HS for $1 \leqq p \leqq 2$ and in §1 we noted that if $p \geqq 2$ then Π_2 is contained in Π_p; all that remains is to notice that Maurey's theorem in tandem with the cotype 2 character of Hilbert spaces gives us equality of Π_p with Π_2 for $p \geqq 2$.

A small remark ought to be made here. As we remarked in §2 the theory of absolutely p-summing operators uncovered new mapping properties valid for Hilbert-Schmidt operators, namely if $1 \leqq p < 2$ then Hilbert-Schmidt operators take weakly p-summable series into strongly p-summable series. In light of the developments of this chapter, we are now able to give very weak conditions that will ensure a given operator between Hilbert spaces is of Hilbert-Schmidt type: if for some $p > 2$, an operator between Hilbert spaces takes weakly p-summable series into strongly p-summable series then it is a Hilbert-Schmidt operator.

On the way to polishing off the next gem an interesting variation on the principal example of §1 presents itself.

Example. *Suppose* $(\Omega, \sum, \mu)$ *is a probability space and* $1 \leqq p < \infty$. *Then the natural inclusion map of* $L_\infty(\mu)$ *into* $L_p(\mu)$ *is absolutely p-summing with p-summing norm* = 1.

This is a consequence of some magic derived from M.H. Stone's representation theorem for Boolean algebras. Recall what this wonderful theorem says: if $\mathbb{A}$ is a Boolean algebra then there is a compact Hausdorff totally disconnected space $\hat{\Omega}$ such that $\mathbb{A}$ is isomorphic as a Boolean algebra to the Boolean algebra $\hat{\mathbb{A}}$ of all simultaneously closed and open ("clopen") subsets of $\hat{\Omega}$.

What's Stone's theorem got to do with us? Well, take the σ-algebra Σ; it's a Boolean algebra possessed of a special ideal $N = \{A \in \Sigma : \mu(A) = 0\}$. If we factor N out of Σ, then we get a new Boolean algebra $\mathbb{A} = \Sigma/N$ wherein two members from Σ find themselves in the same coset if the measure of their symmetric difference is 0. Notice that μ can be viewed as acting on $\mathbb{A}$, too; after all, if $A, B \in \Sigma$ are sent to the same cosets modulo N, then $\mu(A) = \mu(B)$. View μ as acting on $\mathbb{A}$. Apply the Stone representation theorem to $\mathbb{A}$ and find a compact, Hausdorff, totally disconnected space $\hat{\Omega}$ whose algebra $\hat{\mathbb{A}}$ of clopen subsets is isomorphic as a Boolean algebra to $\mathbb{A}$; if $A \in \mathbb{A}$ then denote by $\hat{A}$ its isomorphic image in $\hat{\mathbb{A}}$. So far we've done little but complicate the situation. The magic is about to begin.

Start with a simple function $\sum_{i=1}^{n} a_i \chi_{A_i}$ in $L_\infty(\mu)$; suppose $A_1, \ldots, A_n$ are pairwise disjoint sets of positive measure. Look at $\sum_{i=1}^{n} a_i \chi_{\hat{A}_i}$: it's <u>continuous</u> on $\hat{\Omega}$. Because two simple functions in $L_\infty(\mu)$ are regarded as the same in $L_\infty(\mu)$ only when they agree except on a set of N, the mapping J_∞ that takes $\sum_{i=1}^{n} a_i \chi_{A_i}$ to $\sum_{i=1}^{n} a_i \chi_{\hat{A}_i}$ is well-defined. Observe that the $L_\infty(\mu)$ norm of $\sum_{i=1}^{n} a_i \chi_{A_i}$ is just $\sup_{1\le i\le n}|a_i|$ as is the norm of $\sum_{i=1}^{n} a_i \chi_{\hat{A}}$ in $C(\hat{\Omega})$. In other words, J_∞ is an isometry. Now simple functions are dense in $L_\infty(\mu)$ so J_∞ extends to an isometry of all of $L_\infty(\mu)$ into $C(\hat{\Omega})$ whose range contains all functions on $\hat{\Omega}$ of the form $\sum_{i=1}^{n} a_i \chi_{\hat{A}_i}$; for notational sanity we'll denote this extended isometry by J_∞, too. Okay, let's look at the range of J_∞: as an isometric image of a Banach space, it's closed. But here's the punch-line: the collection of functions of the form $\sum_{i=1}^{n} a_i \chi_{\hat{A}_i}$, i.e., the image under J_∞ of the simple functions in $L_\infty(\mu)$, is an algebra of continuous functions on $\hat{\Omega}$ that contains the constants and, thanks to $\hat{\Omega}$'s total disconnectedness, separates points; moreover, in case the scalars are the complex numbers this algebra is self-adjoint; therefore the range of J_∞ is dense in $C(\hat{\Omega})$ by the Stone-Weierstrass theorem.

J_∞ *is an isometric isomorphism of* $L_\infty(\mu)$ *onto* $C(\hat{\Omega})$.

No, we haven't forgotten about $L_p(\mu)$. Recall that μ can be viewed as living on $\mathbb{A}$; if we define $\hat{\mu}$ on $\hat{\mathbb{A}}$ by $\hat{\mu}(\hat{A}) = \mu(A)$, then it is quickly established that $\hat{\mu}$ is a bounded additive measure on $\hat{\mathbb{A}}$ with non-negative values (the same values that μ has) and total mass 1. What's more, $\hat{\mu}$ is plainly regular and countably additive! Think about it: regularity is trivial because all of $\hat{\mu}$'s arguments are already both compact and open, while countable additivity follows from the fact that there can only be finitely many disjoint clopen subsets of the compact $\hat{\Omega}$ whose union is also clopen (hence compact). We can extend $\hat{\mu}$ to the Baire σ-field of $\hat{\Omega}$ in a unique fashion reserving regularity, countable additivity, non-negativity and total mass as we do so. Again we can extend this extension in precisely one regular, countably additive fashion to the Borel σ-field of $\hat{\Omega}$; the result of this extension work is a regular Borel probability measure $\hat{\mu}$ defined on $\hat{\Omega}$. Regularity ensures, by the way , the density of $C(\hat{\Omega})$ as a linear subspace of $L_p(\hat{\mu})$. Now we're in business! Take a simple function $\sum_{i=1}^n a_i \chi_{\hat{A}_i}$ in $L_p(\mu)$ and consider its counterpart $\sum_{i=1}^n a_i \chi_{\hat{A}_i}$ in $C(\hat{\Omega})$. Compute their L_p-norms.

$$\left\| \sum_{i=1}^n a_i \chi_{A_i} \right\|_p = \left(\sum_{i=1}^n |a_i|^p \mu(A_i) \right)^{1/p}$$

$$\left\| \sum_{i=1}^n a_i \chi_{\hat{A}_i} \right\|_p = \left(\sum_{i=1}^n |a_i|^p \hat{\mu}(\hat{A}_i) \right)^{1/p}$$

(we're assuming the A_i's, and hence the $\hat{A}_i$'s, to be proved disjoint).

Because $\mu(A_i) = \hat{\mu}(\hat{A}_i)$ the L_p-norms of $\sum_{i=1}^{n} a_i \chi_{A_i}$ and $\sum_{i=1}^{n} a_i \chi_{\hat{A}_i}$ coincide. Therefore the map J_p that takes $\sum_{i=1}^{n} a_i \chi_{A_i}$ in $L_p(\mu)$ to $\sum_{i=1}^{n} a_i \chi_{\hat{A}_i}$ in $L_p(\hat{\mu})$ is an isometry. By density of simple functions, J_p can be extended to an isometric embedding of $L_p(\mu)$ into $L_p(\hat{\mu})$ whose range contains a dense linear subspace of the dense subspace $C(\hat{\Omega})$. Therefore, J_p's extension, still denoted by J_p, is an isometry of $L_p(\mu)$ onto $L_p(\hat{\mu})$.

To conclude we need but realize the commutativity of the following diagram;

$$\begin{array}{ccc} L_\infty(\mu) & \hookrightarrow & L_p(\mu) \\ {\scriptstyle J_\infty}\downarrow & & \uparrow{\scriptstyle J_p^{-1}} \\ C(\hat{\Omega}) & \hookrightarrow & L_p(\hat{\mu}) \end{array}$$

The inclusion $C(\hat{\Omega}) \hookrightarrow L_p(\hat{\mu})$ is absolutely p-summing with p-summing norm $= 1$; this we know from §1. Both J_∞ and J_p^{-1} are isometric hence have norm one. The rest follows from the ideal property of Π_p.

We're ready for this next result

<u>Theorem</u>. *Let $T:X \to Y$ be absolutely 2-summing. Suppose X is a closed linear subspace of Z. Then T admits a linear extension $S:Z \to Y$ which is also absolutely 2-summing with $\Pi_2(S) \leq \Pi_2(T)$.*

<u>Proof</u>. By the Grothendieck-Pietsch domination theorem, there is a regular Borel probability measure μ on (B_{X^*}, weak^*) such that

$$\|Tx\| \leq \Pi_2(T) \, \|x(.)\|_2$$

holds for each $x \in X$. As in §2 we see that if X_2 is the closure of X's image in $L_2(\mu)$, then the inequality above allows us to "extend T" to an operator $\hat{T}:X_2 \to Y$ with $\|T\| \leq \Pi_2(T)$.

Diagramatically

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, hook] & & Y \\ & X_2 \arrow[ur, "\hat{T}"'] & \end{tikzcd}$$

Now X_2 is a closed linear subspace of $L_2(\mu)$, a Hilbert space. As such X_2 has an orthogonal complement $X_2^{\perp}$; we can extend $\hat{T}$ to all of $L_2(\mu)$ simply by defining it on $X_2^{\perp}$ to be zero. Now we have the diagram above, on suitable modification looking like this:

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, hook] & Y \\ L_\infty(\mu) \arrow[r, hook] & L_2(\mu) \arrow[u, "\hat{T}"'] \end{tikzcd}$$

where $X \hookrightarrow L_\infty(\mu)$ is the isometric embedding $x \to x(.)$, $L_\infty(\mu) \hookrightarrow L_2(\mu)$ is the absolutely 2-summing natural inclusion which as an ordinary bounded linear operator has norm one and $\hat{T}$ is a bounded linear operator with $\|T\| \leq \Pi_2(T)$. Don't take our insertion of $L_\infty(\mu)$ into the above diagram too lightly.

Why? Well, $L_\infty(\mu)$ is a very special Banach space. Suppose for the moment that all the spaces under consideration are <u>real</u> Banach spaces. Then $L_\infty(\mu)$ has a lattice structure as well as being a Banach space; it is easy to see that as a lattice $L_\infty(\mu)$ is Dedekind complete, i.e., every bounded set in $L_\infty(\mu)$ has a least upper bound in $L_\infty(\mu)$. Moreover, the closed unit ball of $L_\infty(\mu)$ has a biggest element, the constantly 1 function. Now peak at the standard proof of the Hahn-Banach theorem BUT allow yourself the freedom of operators, instead of functionals, with values in $L_\infty(\mu)$. With the aforementioned comments on $L_\infty(\mu)$'s Banach lattice structure clearly in mind, you will soon convince yourself that operators into $L_\infty(\mu)$ can be linearly extended from subspaces to superspaces with nary an enlargement of norm. In case you're in the complex setting, try to apply the old Sobczyk trick used in deriving the complex version of the Hahn-Banach theorem; with proper attention to details, it works! To sum up the special character of $L_\infty(\mu)$ is highlighted by the following:

if X *is a closed linear subspace of the Banach space* Z *and* u *is a bounded linear operator from* X *into* $L_\infty(\mu)$*, then there is a bounded linear operator* $v:Z \to L_\infty(\mu)$ *for which* $v|_X = u$ *and* $\|v\| = \|u\|$.

Apply this message to the diagram

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d, phantom, "\cap|"] \arrow[rrr, bend left, "T"] & L_\infty(\mu) \arrow[r, hook] & L_2(\mu) \arrow[r, "\hat{T}"] & Y \\
Z \arrow[ur, dashed, "E"'] & & &
\end{tikzcd}$$

extending $X \hookrightarrow L_\infty(\mu)$ to the norm one operator $E:Z \to L_\infty(\mu)$ we get the absolutely 2-summing operator $\hat{T} \circ [L_\infty(\mu) \hookrightarrow L_2(\mu)] \circ E$ which extends T and has 2-summing norm $\leq \Pi_2(T)$ thanks to the ideal structure of Π_2.

Next, we present a surprising and beautiful computation, due to Stan Kwapien.

<u>Example</u>: *If* E *is a finite dimensional Banach space, then* $\Pi_2(id_E) = \sqrt{\dim E}$.

First, we'll show that given $x_1, \ldots x_n \in E$, then

$$\left(\sum_{i=1}^{n} \|x_i\|^2\right)^{\frac{1}{2}} \leq \sqrt{\dim E} \sup_{\|x^*\| \leq 1} \left(\sum_{i=1}^{n} |x^* x_i|^2\right)^{\frac{1}{2}};$$

this will show that $\Pi_2(id_E) \leq \sqrt{\dim E}$. Let $x_1, \ldots x_m$ be given. Consider $T:\ell_2^m \to E$ given by $T((a_1, \ldots, a_m)) = \sum_{i=1}^{m} a_i x_i$.

It is plain that $T^*:E^* \to (\ell_2^m)^* = \ell_2^m$ exhibits the following behaviour

$$T^* x^* = ((x^* x_1, \ldots, x^* x_m)).$$

Therefore,

$$\|T\| = \|T^*\| = \sup_{\|x^*\| \leq 1} \|T^* x^*\|_2$$

$$= \sup_{\|x^*\| \leq 1} \|(x^* x_1, \ldots, x^* x_m)\|_2$$

$$= \sup_{\|x^*\| \leq 1} \left(\sum_{i=1}^{m} |x^* x_i|^2 \right)^{\frac{1}{2}},$$

a quantity to be reckoned with. Now consider the diagram (and its consequences):

$$\begin{tikzcd} \ell_2^m \arrow[rr, "T"] \arrow[dr, "Q"'] & & E \\ & \ell_2^m/\ker T \arrow[ur, "\hat{T}"'] \arrow[d, equal] & \\ & \ell_2^d & \end{tikzcd}$$

$$\Pi_2(T) = \Pi_2(\hat{T}\, id_{\ell_2^d}\, Q)$$

$$\leq \|T\| \, \Pi_2(id_{\ell_2^d}) \, \|Q\|$$

$$\leq \|T\| \sqrt{d}$$

$$\leq \|T\| \sqrt{\dim E}$$

where an understanding of the diagram and the accompanying computation may well be aided by the remarks that Q is the natural quotient of ℓ_2^m onto the factor space $\ell_2^m/\ker T$ (and so has norm 1) and $\hat{T}$ is the 1-1 linear operator induced by T on that factor space (thereby ensuring $\hat{T}$ has the same norm as T). It follows that,

$$\left(\sum_{i=1}^{m} \|x_i\|^2 \right)^{\frac{1}{2}} = \left(\sum_{i=1}^{m} \|Te_i\|^2 \right)^{\frac{1}{2}}$$

$$\leq \Pi_2(T) \sup_{\|e^*\|_{\ell_2^m} \leq 1} \left(\sum_{i=1}^{m} |e^* e_i|^2 \right)^{\frac{1}{2}}$$

$$= \Pi_2(T) \leq \|T\| \sqrt{\dim E}$$

$$= \sqrt{\dim E} \cdot \sup_{\|x^*\| \leq 1} \left(\sum_{i=1}^{m} |x^* x_i|^2 \right)^{\frac{1}{2}}$$

VOILA!

To prove the reverse inequality, we apply the Grothendieck-Pietsch domination theorem to id_E; the result is that there is a compact Hausdorff space K, a regular Borel probability measure μ on K and operators $i:E \to C(K)$, $j:C(K) \hookrightarrow L_2(\mu)$ and $G:L_2(\mu) \to E$ with $\|i\|$, $\Pi_2(j) \leq 1$ and $\|G\| \leq \Pi_2(id_E)$. Pictorially, we have

$$
\begin{tikzcd}
E \arrow[r, "i"] \arrow[dr, "i"] \arrow[rrr, bend left, "id_E"] \arrow[drr, bend right=60, "A"'] & C(K) \arrow[r, hook, "j"] & L_2(\mu) \arrow[r, "G"] & E \\
 & iE \arrow[r, "j"] & jiE \arrow[u, hook] \arrow[ur] \arrow[ur, bend right=60, "A^{-1}"'] & \\
 & & \ell_2^{\dim E} \arrow[u, equal] &
\end{tikzcd}
$$

Reflecting on these implicitly defined operators, we see that

$$
\begin{aligned}
\sqrt{\dim E} &= \text{Hilbert-Schmidt norm } (id_{\ell_2^{\dim E}}) \\
&= \Pi_2 \, (id_{\ell_2^{\dim E}}) \\
&= \Pi_2 \, (A\, A^{-1}) \\
&\leq \Pi_2 \, (A) \, \|A^{-1}\| \\
&\leq \Pi_2 \, (ji) \, \|G\| \\
&\leq \|i\| \ \Pi_2(j) \ \|G\| \\
&\leq \Pi_2(id_E).
\end{aligned}
$$

<u>Corollary (Kadec-Snobar)</u>. *If* E *is a finite dimensional subspace of a Banach space* X, *then there exists a linear projection* P *of* X *onto* E *for which* $\|P\| \leq \sqrt{\dim E}$.

<u>Proof</u>. We know that the absolutely 2-summing operator $id_E : E \to E$ can be extended to an absolutely 2-summing operator $P : X \to E$ with $\Pi_2(P) \leq \Pi_2(id_E) = \sqrt{\dim E}$; of course, P is the sought-after projection.

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