SCALAR TYPE SPECTRAL OPERATORS

IN

LOCALLY CONVEX SPACES

Peter G. Dodds

INTRODUCTION

In this paper, we will present a mini-survey of joint work with Ben de Pagter and Werner Ricker concerning the structure of scalar-type spectral operators in locally convex spaces. The details will appear elsewhere [4], [5], [6], [7]. The central theme is that of reflexivity, which goes back to the following special case of the well known double commutant theorem of von Neumann: if M is a mutually commuting family of self-adjoint projections in a Hilbert space then the strongly closed algebra generated by M consists precisely of those continuous linear operators which leave invariant each M-invariant subspace. A non-trivial extension of this theorem to the setting of Banach spaces was obtained by Bade [1] and this note is concerned with various extensions of the reflexivity theorem of Bade to the more general setting of locally convex spaces.

1. ALGEBRAS GENERATED BY BOOLEAN ALGEBRAS OF PROJECTIONS

In the sequel, X it will denote a locally convex space which is assumed to be quasicomplete. $L(X)$ will denote the space of continuous linear operators on X, equipped with the topology of pointwise convergence on X. It will be assumed throughout that $L(X)$ is sequentially complete. A Boolean algebra of projections in X is a family M of

commuting idempotents in $L(X)$ which is partially ordered with respect to range inclusion and which is a Boolean algebra with respect to the lattice operations defined by setting $Q \vee R = Q + R - QR$ and $Q \wedge R = QR$ for all $Q, R \in M$. It is always assumed that the identity operator I in X is the unit element of M.

An equicontinuous Boolean algebra M of projections in X will be called *Bade complete* if and only if M is complete as an abstract Boolean algebra and whenever $\{Q_\tau\}$ is a downwards filtering system of elements of M for which $\inf_\tau Q_\tau = 0$ in the Boolean algebra M, it follows that $Q_\tau x \to 0$ for each $x \in X$. If M is a Boolean algebra of projections in X, a closed linear subspace $Y \subseteq X$ is called M-invariant if and only if $QY \subseteq Y$ for all $Q \in M$. It is clear that if Y is M-invariant then $T(Y) \subseteq Y$ whenever T belongs to the strongly closed subalgebra of $L(X)$ generated by M. If the Boolean algebra M is Bade complete then the converse to this observation holds in the following strong sense.

THEOREM 1.1 *Let M be a Bade-complete Boolean algebra of projections in X. If T is an everywhere defined linear mapping in X, then the following statements are equivalent.*

(i) *T leaves invariant each closed M-invariant subspace of X.*

(ii) *T belongs to the strongly closed subalgebra generated by M in $L(X)$.*

If the everywhere defined linear mapping T is, in addition, assumed to be continuous, then the result of Theorem 1.1 above is contained in [4] Proposition 3.5 (see also [7], Theorem 1.1). Hence, the novelty of Theorem 1.1 lies in that part of the conclusion which asserts that an everywhere defined linear mapping which leaves invariant each M-invariant

subspace of X is necessarily continuous and this appears to be new, even in the very special case that X is a Hilbert space and the Boolean algebra M consists of self-adjoint projections.

It is instructive to comment briefly on the ideas in [4] which are used to prove Theorem 1.1. The very definition of Boolean algebra of projections suggests that methods from the theory of vector lattices should prove effective. This is indeed the case. If M is an equicontinuous Boolean algebra of projections in X, then the partial order on M can be extended to the strongly closed algebra $\langle M \rangle$ generated by M in $L(X)$ so that $\langle M \rangle$ can be endowed in a very natural way with the structure of a complex locally-solid Riesz space (vector lattice). If in addition M is Bade-complete then $\langle M \rangle$ has the structure of an order complete Riesz space and the given topology on M is locally solid, order continuous and complete. If now $x \in X$, denote by $M[x]$ the smallest closed M-invariant subspace of X which contains x. $M[x]$ will be called the cyclic subspace generated by x. If the Boolean algebra M is equicontinuous, then the order structure on $\langle M \rangle$ induces a natural order structure on each cyclic subspace. If M is Bade complete, then the induced order structure on each cyclic subspace is that of a complex, order complete Riesz space for which the given topology is locally solid, order continuous and complete. The key to the approach of [4] is that if the Boolean algebra M is Bade-complete and if T is an everywhere defined linear mapping which leaves invariant each M-invariant (closed) subspace of X, then the restriction of T to each cyclic subspace is order-bounded and has strong local properties with respect to the induced order. Not only does this observation allow ready application of the recently developed theory of orthomorphisms [13] but the ideas prove fruitful in a more general context, as will be indicated in

If T is an everywhere defined scalar-type spectral operator in X, represented by the spectral integral $\int_\Omega fdP$, then the closure M of the range of the equicontinuous spectral measure P is a Bade complete Boolean algebra of projections in X. It is not difficult to check that T leaves invariant each M-invariant supbspace of X and consequently T belongs to the strongly closed algebra generated by M. In particular, then, T is continuous.

A simple consequence of the preceding remarks is that each continuous scalar-type spectral operator in X belongs to the strongly closed algebra generated by a Bade-complete Boolean algebra of projections in X. The converse to this assertion holds: the strongly closed algebra generated by a Bade-complete Boolean algebra of projections in X consists entirely of scalar-type spectral operators. This was first pointed out in [10] and may be veiwed, at least abstractly, as a consequence of the well-known spectral theorem of Freudenthal. See [6] for details.

2. REFLEXIVITY OF SCALAR-TYPE SPECTRAL OPERATORS

If N is a closed subalgebra of $L(X)$ containing the identity, then N is called reflexive if and only if N consists precisely of those operators in $L(X)$ which leave invariant each closed N-invariant subspace of X. An operator $T \in L(X)$ is called reflexive if the closed subalgebra generated by T and I is a reflexive subalgebra of $L(X)$. Let now M be an equicontinuous Bade-complete Boolean algebra of projections in X and let $\langle M \rangle$ be the strongly closed subalgebra of $L(X)$ generated by M. It is clearly a consequence of Theorem 1.1 that $\langle M \rangle$ is reflective. Much more than this is true, however, as indicated by the following result.

THEOREM 2.1 *Let* M *be an equicontinuous Bade-complete Boolean algebra of projections in* X *and let* $\langle M \rangle$ *be the closed subalgebra of* $L(X)$ *generated by* M. *If* N *is a closed unital subalgebra of* $\langle M \rangle$ *then* N *is reflexive.*

We have the following immediate application to (continuous) scalar-type spectral operators.

COROLLARY 2.2 *Each continuous scalar-type spectral operator in* X *is reflexive.*

Let us remark that the reflexivity of normal operators in a Hilbert space is due to Sarason [11]. That each scalar-type spectral operator in a Banach space is reflexive was first shown by Gillespie [8] by proving Theorem 2.1 in the setting of Banach spaces. Concerning the proof of Theorem 2.1, the general strategy embraced by Gillespie in [8] carries over to the locally convex setting, although the implementation of this strategy is quite different in the more general case. There are two basic ingredients. The first ingredient is, quite naturally, the reflexivity result of Theorem 1.1. The second ingredient is a representation of the topological dual of $\langle M \rangle$, which is due to B. de Pagter and which is of independent interest. We denote by X' the topological dual of X.

PROPOSITION 2.3 *Let* M *be an equicontinuous Bade-complete Boolean algebra of projections in* X *and let* $\langle M \rangle$ *be the strongly closed subalgebra of* $L(X)$ *generated by* M. *If* φ *is a continuous linear functional on* $\langle M \rangle$ *then there exists* $x \in X$ *and* $x' \in X'$ *such that*
$$\varphi(T) = \langle Tx, x' \rangle \text{ for all } T \in \langle M \rangle.$$

If M consists of self-adjoint projections in a Hilbert space, then the preceding representation result is due to R. Pallu de la Barrière [9]. If X is a Banach space, then the existence of such a representation for elements of the dual of $\langle M \rangle$ was shown by Gillespie [8] via an interesting factorization theorem which does not appear to carry over to the locally convex setting. On the other hand, the proof of Proposition 2.3 given in [6] is quite elementary and accordingly yields considerable simplification of method, even for the case that X is Banach.

3. ALGEBRAS OF CLOSED SPECTRAL OPERATORS

Let M be a Bade-complete Boolean algebra of projections in X. If M is displayed as the range of an equicontinuous spectral measure P defined on the Borel subsets of the Stone space Ω of M, then the linear mapping $T : \mathcal{D}(X) \longrightarrow X$ will be called scalar type spectral with respect to M if T has a representation of the form $T = \int_{\Omega} f dP$ with f a complex Borel function on Ω. We denote by $\langle M \rangle_{\infty}$ the collection of all scalar-type spectral operators with respect to M. Of course, the collection of continuous scalar-type spectral operators with respect to M is precisely the closed subalgebra $\langle M \rangle$ of $L(X)$ generated by M. In general, $\langle M \rangle_{\infty}$ is neither a linear space nor an algebra with respect to the usual operator sum and product. Nonetheless, if $S, T \in \langle M \rangle_{\infty}$ have domains $\mathcal{D}(S)$, $\mathcal{D}(T)$ respectively, then it can be shown [5] that there exist unique elements $S \dotplus T$, $S \cdot T \in \langle M \rangle_{\infty}$ such that $S \dotplus T | \mathcal{D}(S) \cap \mathcal{D}(T) = S + T$ and $S \cdot T | \mathcal{D}(T) \cap T^{-1}(\mathcal{D}(S)) = ST$ and with respect to these operations $\langle M \rangle_{\infty}$ is a commutative complex algebra unit. However, more than this is true. The partial ordering on the Dedekind complete Riesz space $\langle M \rangle$ extends to $\langle M \rangle_{\infty}$ so that $\langle M \rangle_{\infty}$ has the

structure of a complex f-algebra (see [13]) which is not only order com-
plete but also laterally complete i.e. arbitrary disjoint systems in the
positive cone of $\langle M \rangle_\infty$ have a least upper bound. For the case that M
consists of self-adjoint projections in a Hilbert space these facts have
already been observed in [3].

Those everywhere defined operators which are scalar type spectral
with respect to M are characterized by Theorem 1.1 above. A similar
characterization holds in general for closed, densely defined operators.
If T is a linear operator in X with domain $\mathcal{D}(T)$, we will say that
T leaves the closed subspace $Y \subseteq X$ invariant if T maps $\mathcal{D}(T) \cap Y$
into Y. We can now formulate the principal result of this section.

THEOREM 3.1 *Let M be an equicontinuous Bade complete Boolean*
algebra of projections in X. Let T be a densely defined closed linear
mapping in X with M-invariant domain. The following statements are
equivalent

    (i) *T leaves invariant each M-invariant subspace of X.*

    (ii) *T is scalar-type spectral with respect to M.*

This characterization of scalar-type spectral operators is proved
in [5] and appears to be new, even in the case that X is Banach. If
M consists of self-adjoint projections in a Hilbert space, then Theorem
3.1 is due to Segal [12]. The key point in the present approach consists
in showing that if T leaves invariant each M-invariant subspace of X
then the restriction of T to each cyclic subspace of X coincides with
the restriction to that subspace of a densely defined multiplication
operator which can be shown to be induced by an element of $\langle M \rangle_\infty$. The

arguments are purely order theoretic and use the theory of extended ortho-morphisms in Riesz spaces to extend the method of [4] employed in proving Theorem 1.1. Needless to say, the main source of additional difficulty is related to various domain problems.

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School of Mathematical Sciences
Flinders University of South Australia
Bedford Park   S.A.   5042
AUSTRALIA.