TYPE I ABELIAN GROUPS WITH MULTIPLIERS

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1. INTRODUCTION

Let G be a second countable locally compact group and N a closed normal subgroup of G. The group G then acts naturally by conjugation on N and hence on the dual $\hat{N}$ of N - the unitary equivalence classes of irreducible representations of N with the hull-kernel topology (cf.[3]Ch.3). If N is type I and the action of G on $\hat{N}$ is smooth then it is possible to analyse the irreducible and factor representations of G in terms of those of N and the so-called "little group" H/N (see, for example, [4] Ch.3). Here, for a fixed element π of $\hat{N}$, H is the stabilizer of π under the conjugation action. It turns out, however that one needs to extend the concept of representation of H even to deal with ordinary representations of G. The appropriate concept is that of a multiplier representation. A <u>multiplier</u> on G is a Borel map $\omega: G \times G \to T$ satisfying

(i) $\omega(x,y)\ \omega(xy,z) = \omega(x,yz)\,\omega(y,z) \qquad (x,y,z \in G);$

(ii) $\omega(x,e) = \omega(e,x) = 1 \qquad (x \in G);$

(iii) $\omega(x^{-1},y^{-1}) = \omega(x,y)^{-1} \qquad (x,y \in G),$

and an ω-<u>representation</u> of G is a Borel map π from G into the unitary group $U(G)$ of a Hilbert space G with the strong operator topology which satisfies

$$\pi(xy) = \omega(x,y)\ \pi(x)\ \pi(y) \qquad (x,y \in G)$$

In the content of Mackey's "little group" analysis, we are forced to consider ω-representations of H/N in order to describe the ordinary representations of G. The ideas of equivalence, irreducible representation and factor representation are extendible in the obvious way to ω-representations of a group G. The pair (G,ω) will be type I if every factor ω-representation of G generates a type I von Neumann algebra.

One hopes that the little group analysis will reduce the problem of studying the representations of a group successively to smaller and simpler groups. In particular, it will often be possible to arrive at an abelian little group. It is important then to know whether the little group with the multiplier in question is type I. Indeed, type I-ness of the pair $(H/N,\omega)$ for all choices of ω that arise in the Mackey analysis, forces type I-ness of the full group G.

This paper is an announcement of results concerning the structure of type I pairs (G,ω) for certain classes of abelian groups G. From now on G will be assumed to be abelian. For such groups Baggett and Kleppner ([1]) have given necessary and sufficient conditions on a multiplier ω for (G,ω) to be type I, but these do not explicitly describe the group G or the multiplier in relation to G. A particular case of a type I abelian group occurs when $G = H\times\hat{H}$ where H is a second countable locally abelian group and $\hat{H}$ is its Pontryagin dual. If we define for $(h,\hat{h}),(h_1,\hat{h}_1) \in H\times\hat{H}$ a function ω by

$$\omega((h,\hat{h}),\ (h_1,\hat{h}_1)) = \hat{h}(h_1),$$

then ω is a multiplier on G and (G,ω) is type I. We call these multipliers cross-multipliers. It has been conjectured that all type I multipliers on abelian groups arise in essentially this way. It is this particular problem that we address here.

2. STATEMENT AND DISCUSSION OF RESULTS

To begin with, we need to be more explicit about the way in which a multiplier is essentially a cross-multiplier. Two pairs (G,ω), (G',ω') are isomorphic if there is a topological isomorphism $\phi:G\to G'$ such that $\omega'(\phi(x),\phi(y)) = \omega(x,y)$ for $x,y \in G$, and two multipliers ω and ω' on the same group G are similar if there is a Borel map $\phi:G\to\Pi$ such that $\omega'(x,y) = \omega(x,y)\ \phi(x)\ \phi(y)\ \phi(x+y)^{-1}$ for all $x,y\in G$. A multiplier is trivial if it is similar to the constant multiplier.

It is convenient to express much of the information about ω in terms of its skew-symmetrical form $\tilde{\omega}$, where

$$\tilde{\omega}(x,y) = \omega(x,y)\omega(y,x)^{-1} \qquad (x,y \in G).$$

The maps $x\to\tilde{\omega}(x,y_o)$ and $y\to\tilde{\omega}(x_o,y)$ are then both continuous homomorphisms from G to T for any choice of x_o and y_o. If S is a subset of G, we let

$$S\omega = \{x : \tilde{\omega}(x,y) = 1 \qquad (y\in S)\}.$$

This is always a closed subgroup of G. In the case when $G\omega = \{1\}$ we say that ω is a totally skew multiplier and that the pair (G,ω) is totally skew.

It is always possible to reduce to this case by factoring out $G\omega$ and observing that ω is similar to a multiplier lifted from $G/G\omega$.

Observe that the multiplier ω is trivial if and only if $\tilde{\omega} \equiv 1$, and that we may regard $\tilde{\omega}$ as a homomorphism $\tilde{\omega}: G \to \hat{G}$ by writing

$$\tilde{\omega}(x)(y) = \tilde{\omega}(x,y) \qquad (x,y \in G).$$

The key result of Baggett and Kleppner ([1]) can now be stated.

THEOREM 1 (Baggett and Kleppner)

The pair (G,ω) is type I if and only if $\tilde{\omega}(G)$ is a closed subgroup of $\hat{G}$.

In the totally skew situation, $\tilde{\omega}(G)$ is easily seen to be dense in $\hat{G}$ and so type I-ness is equivalent to $\tilde{\omega}(G) = \hat{G}$. In particular, for any type I totally skew pair (G,ω), G must be self-dual.

A subgroup H of G is <u>isotropic</u> if $\tilde{\omega}(x,y) = 1$ for all $x,y \in H$, and <u>maximal isotropic</u> if it is maximal among such groups. Evidently a maximal isotropic subgroup is closed.

Our methods are structure theoretic and rely heavily on being able to eliminate the connected component of the group. The lemma that enables us to do this is the following.

LEMMA 1

Let ω be a totally skew and type I multiplier on G, and suppose that H is a closed isotropic subgroup which is a topological direct summand of G. Then G is topologically isomorphic to $H \times \hat{H} \times (H\omega/H)$ and ω is similar to a multiplier ω' of the form

$$\omega'((h,\hat{h},x),(h_1,\hat{h}_1,x_1)) = \hat{h}(h_1)\,\sigma(x,x_1)$$

where σ is a totally skew type I multiplier on $H\omega/H$.

The effect of this lemma, then, is to reduce the problem to one on $H\omega/H$. However, it is limited by the requirement that H be a topological direct summand. In the case when G is finite, this restriction has no effect and we can easily deduce the following corollary.

COROLLARY

Let ω be a totally skew multiplier on a finite abelian group G. Then G is isomorphic to $H \times H^{\wedge}$ and under this isomorphism ω is carried to one similar to a cross-multiplier.

In dealing with general locally compact abelian groups, one can use the lemma and the structure theorem to eliminate copies of $\mathbb{R}$ and reduce the problem to the case where G has a compact open subgroup. If the connected component of G is a topological direct summand, which is always the case if G is divisible, then the lemma can be used to reduce further to the situation where G is totally disconnected.

When G is a totally disconnected group it is possible to invoke a structure theorem of Braconnier ([2]) and hence deduce the next theorem.

THEOREM 2

Let G be totally disconnected and suppose that (G,ω) is a type I pair with ω totally skew. Then there is a compact open maximal isotropic subgroup H of G and a local direct product decomposition

$$(G,H) = \underset{p \in P}{LP}\ (G_p, H_p)$$

where each group G_p *consists of the elements* x *of* G *such that* $x^{p^n} \to 1$ *as* $n \to \infty$ *and* P *is the set of all primes. Each* G_p *admits a totally skew type I multiplier* ω_p *and*

$$\tilde{\omega}((g_p),(g'_p)) = \prod_{p \in P} \tilde{\omega}_p(g_p, g'_p)$$

for $(g_p),(g'_p) \in G$. *This product is always finite.*

Combining this result with applications of the lemma we are able to obtain the following partial resolution of the conjecture.

THEOREM 3

Let G *be a locally compact second countable abelian group and let* ω *be a totally skew multiplier on* G. *Suppose that* (G,ω) *is type I and let* C *denote the connected component of* G. *If* $C\omega/C$ *is divisible then* G *is of the form* $H \times \hat{H}$ *and* ω *is similar to a cross-multiplier.*

COROLLARY

Let G *be a locally compact second countable abelian group. If* G *is divisible and* ω *is a totally skew multiplier such that* (G,ω) *is type I then* G *is isomorphic* $H \times \hat{H}$ *for some closed subgroup* H *of* G *and under this isomorphism* ω *is carried to a multiplier similar to a cross-multiplier.*

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