THE SOLUTION OF SYSTEMS OF OPERATOR EQUATIONS USING CLIFFORD ALGEBRAS

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1. INTRODUCTION

Our aim is twofold. We develop a functional calculus for commuting m-tuples of Banach space operators, and then use this functional calculus to solve a system of operator equations and obtain estimates for the solution. The new ingredient is the use of Clifford algebras.

As a corollary we obtain results on the perturbation of the spectral subspaces of commuting self-adjoint operators. In particular we answer an open question, stated for example on p. 221 of [5], on the spectral perturbation of self-adjoint matrices.

Our idea of using Clifford algebras is derived from the work of R. Coifman and M. Murray [10]. The functional calculus for several operators is a generalization of that developed in S. Kantorovitz [7] and I. Colojoara and C. Foiaş [4] for a single operator. Our results on systems of operator equations extend results of R. Bhatia, Ch. Davis and A. McIntosh [2] concerning single equations. Thanks are due to J. Picton-Warlow with whom we have had several stimulating discussions.

Banach spaces X and Hilbert spaces H and K are defined over the field $\mathbb{F}$, where $\mathbb{F}$ denotes either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$.

2. OPERATOR EQUATIONS

To motivate our discussion of the functional calculus, we state here our results on systems of operator equations.

Throughout this section, $\underset{\sim}{A} = (A_1, \ldots, A_m)$ and $\underset{\sim}{B} = (B_1, \ldots, B_m)$ denote commuting m-tuples of bounded self-adjoint operators defined on Hilbert spaces H and K respectively. The joint spectrum of $\underset{\sim}{A}$ is denoted $\sigma(\underset{\sim}{A})$.

THEOREM 1. *Suppose* $\delta = \text{dist}(\sigma(\underset{\sim}{A}), \sigma(\underset{\sim}{B})) > 0$ *and let* $W_j \in L(K,H)$ *for* $j = 1,2..,m$. *Then the system of operator equations*

$$A_jQ - QB_j = W_j \quad \text{for} \quad j = 1,2,..,m$$

has a solution $Q \in L(K,H)$ *if and only if*

$$A_jW_k - W_kB_j = A_kW_j - W_jB_k \quad \text{for} \quad j,k = 1,2..,m .$$

In this case the solution Q *is unique and satisfies*

$$\|Q\| \leqq c_m \, \delta^{-1} \, \|\underset{\sim}{W}\|_{K\to H^m} .$$

The constant c_m is defined by

$$c_m = \inf(2\pi)^{-m} \int |\hat{g}(\xi)| d\xi$$

where the infimum is taken over all functions $g : \mathbb{R}^m \to \mathbb{R}^m$, each component g_k of which is the Fourier transform of an L_1-function and satisfies $g_k(x) = x_k|x|^{-2}$ if $|x| > 1-\varepsilon$ for some $\varepsilon > 0$.

We remark that $1 < c_m < \infty$.

The Fourier transform being used is the following:

$$\hat{f}(\xi) = \int e^{-i\langle x,\xi\rangle} f(x)dx \quad ;$$

with the Fourier inversion formula

$$f(x) = (2\pi)^{-m} \int e^{i\langle x,\xi\rangle} \hat{f}(\xi) d\xi \quad .$$

In the special case when $\sigma(\underset{\sim}{A}) \subset \bar{B}(0,\kappa)$ and $\sigma(\underset{\sim}{B}) \cap B(0,\kappa+\delta) = \phi$ for some $\kappa \geqslant 0$, the above result has been proved by Bhatia and Davis with c_m replaced by 1.

Theorem 1 will be proved in section 7 after the functional calculus has been developed.

Corollary 1. *Let* $H = K$ *, and for closed subsets* X *and* Y *of* $\mathbb{R}^m$*, let* E_X *and* F_Y *denote the corresponding spectral projections of* $\underset{\sim}{A}$ *and* $\underset{\sim}{B}$ *respectively. Suppose* $\delta = \mathrm{dist}(X,Y) > 0$. *Then*

$$\|E_X F_Y\| \leq c_m \delta^{-1} \|\underset{\sim}{A}-\underset{\sim}{B}\|_{H \to H^m}$$

Proof. Let $H_X = E_X(H)$, $H_Y = F_Y(H)$, $Q = E_X|_{H_Y} \in L(H_Y, H_X)$, $W_j = E_X(A_j - B_j)F_Y \in L(H_Y, H_X)$, and apply theorem 1. ■

The following corollary can be deduced from the first as in the proof of theorem 5.1 of [2] for the case m = 1 .

Corollary 2. *Suppose that* $H = K = \mathbb{C}^N$ *for some* $N < \infty$, *that* $\underset{\sim}{A}$ *has joint eigenvalues* $\underset{\sim}{\alpha}_1, \ldots, \underset{\sim}{\alpha}_N \in \mathbb{R}^m$, *and that* $\underset{\sim}{B}$ *has joint eigenvalues* $\underset{\sim}{\beta}_1, \ldots, \underset{\sim}{\beta}_N \in \mathbb{R}^m$. *If* $\|\underset{\sim}{A} - \underset{\sim}{B}\| \leq \frac{\varepsilon}{c_m}$ *then there exists a permutation* σ *of the index set* $\{1,2,\ldots,m\}$ *such that* $|\underset{\sim}{\alpha}_k - \underset{\sim}{\beta}_{\sigma(k)}| \leq \varepsilon$ *for* $k = 1,2..,N$.

3. CLIFFORD ALGEBRAS, $\mathbb{F}_{(n)}$

The vector space $\mathbb{R}^{n+1}$ is embedded in a 2^n-dimensional algebra $\mathbb{F}_{(n)}$ over $\mathbb{F}$ as follows. Let $e_0, e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^{n+1}$ and denote the basis vectors of $\mathbb{F}_{(n)}$ by e_S , where S is a subset of $\{1,2,\ldots,n\}$. Make the identifications $e_0 = e_\phi$ and $e_j = e_{\{j\}}$ for $1 \leqq j \leqq n$, and define the multiplication on $\mathbb{F}_{(n)}$ by taking e_0 as the unit 1,

$$e_j^{\,2} = -e_0 = -1 \quad \text{for} \quad 1 \leqq j \leqq n \ ;$$

$$e_j e_k = -e_k e_j = e_{\{j,k\}} \quad \text{for} \quad 1 \leqq j < k \leqq n \ ;$$

$$e_{j_1} e_{j_2} \cdots e_{j_s} = e_S \quad \text{if} \quad 1 \leqq j_1 < j_2 < \ldots < j_s \leqq n \quad \text{and}$$

$S = \{j_1, j_2, \ldots, j_s\}$.

The product of two elements $\lambda = \sum_S \lambda_S e_S$, $\lambda_S \in \mathbb{F}$, and $\mu = \sum \mu_T e_T$, $\mu_T \in \mathbb{F}$, is $\lambda\mu$ where

$$\lambda\mu = \sum_{S,T} \lambda_S \mu_T e_S e_T .$$

Note that $e_S e_T$ is again a basis vector of $\mathbb{F}_{(n)}$.

The Clifford algebras $\mathbb{R}_{(1)}$ and $\mathbb{R}_{(2)}$ are the complex numbers and the quaternions respectively. Basic properties of Clifford algebras can be found in Brackx, Delanghe and Sommen [3].

An involution $\lambda \to \bar{\lambda}$ is defined by $\bar{\lambda} = \sum_S \bar{\lambda}_S \bar{e}_S$ where $\bar{\lambda}_S$ is the complex conjugate of λ_S and $\bar{e}_S = \pm e_S$, the sign being chosen so that $e_S \bar{e}_S = \bar{e}_S e_S = 1$.

Not all elements of $\mathbb{F}_{(n)}$ are invertible. One important reason for using Clifford algebras, however, is that non-zero elements $x \in \mathbb{R}^{n+1}$ do have inverses, namely $x^{-1} = |x|^{-2}\,\bar{x} = (\sum_0^n x_k^2)^{-1}\,(x_0 - x_1 e_1 - x_2 e_2 \ldots - x_n e_n)$.

4. CLIFFORD ANALYSIS

Let Ω be an open subset of $\mathbb{R}^{n+1}$. A function $f: \Omega \to \mathbb{F}_{(n)}$ is called *left monogenic* if $Df = 0$. Here $D = \sum_0^n \frac{\partial}{\partial x_j} e_j$ and $Df = \sum_{j=0}^n \sum_S \frac{\partial f_S}{\partial x_j} e_j e_S$ when $f = \sum_S f_S e_S$ for functions $f_S : \Omega \to \mathbb{F}$. Much of the theory of analytic functions in complex analysis generalizes to results concerning left monogenic functions. See [3]. In particular there is an analogue of Liouville's theorem:

THEOREM 2. *If* $f: \mathbb{R}^{n+1} \to \mathbb{F}_{(n)}$ *is a bounded left monogenic function on all of* $\mathbb{R}^{n+1}$ *, then* f *is constant.*

It is not hard to verify that the functions g_z defined for $z, x \in \mathbb{R}^{n+1}$ by

$$g_z(x) = |x-z|^{-n-1}\,\overline{(x-z)}$$

are left monogenic for $x \neq z$.

5. FUNCTIONAL CALCULUS

Let $\underset{\sim}{T} = (T_1,\ldots,T_m)$ be a commuting m-tuple of bounded operators, each acting on a Banach space X over $\mathbb{F}$. We define a kind of joint spectrum $\sigma(\underset{\sim}{T})$ by $\sigma(\underset{\sim}{T}) = \{\lambda \in \mathbb{R}^m : \sum_1^m (T_j-\lambda_j)^2$ is not invertible in $L(X)\}$. This defines a compact subset of $\mathbb{R}^m$ which may reasonably be called a joint spectrum for a large class of m-tuples $\underset{\sim}{T}$. In particular, if the T_j are self-adjoint operators on a Hilbert space, then $\sigma(\underset{\sim}{T})$ is the usual joint spectrum. For a single operator, $\sigma(\underset{\sim}{T})$ is the intersection of the spectrum with the real line.

For $n \geq m$, we identify $\mathbb{R}^m$ with the span of $e_1,e_2,\ldots,e_m$ in $\mathbb{R}^{n+1}$. Then $\mathbb{R}^m \subset \mathbb{R}^{n+1} \subset \mathbb{F}_{(n)}$. We also form the Banach space $X_{(n)} = X \otimes \mathbb{R}_{(n)} = \{u = \sum_S u_S e_S : u_S \in X\}$ and define $T = \sum_1^m T_j e_j \in L(X_{(n)})$ by $T(u) = \sum_{j,S} T_j(u_S)e_j e_S$. It is then possible to prove the following result.

THEOREM 3. $\sigma(\underset{\sim}{T}) = \{\lambda \in \mathbb{R}^m : (T - \lambda I)$ *is not invertible in* $L(X_{(n)})\}$.

In the following, for an algebra A of functions on $\mathbb{R}^m$, let A_0 denote the subspace of functions f with compact support, sptf. For a compact subset K of $\mathbb{R}^m$, let $H(K)$ denote the space of $\mathbb{F}$-valued functions which are real analytic in a neighbourhood of K, taken with its usual topology. For $f \in A_0$ and $g \in H(\text{sptf})$, let $M_f(g) = fg$.

We say that $\underset{\sim}{T}$ has a *functional calculus* $(\underline{T},A)$ *based on* $\mathbb{R}^m$ if the following conditions hold:

A is a topological algebra of functions from $\mathbb{R}^m$ to $\mathbb{F}$, with addition and multiplication defined pointwise, and $\underline{T} : A \to L(X)$ is a continuous algebra homomorphism such that

(a) $C_0^\infty(\mathbb{R}^m) \subset A$;

(b) if $f \in A_0$, then $M_f : H(\mathrm{spt}f) \to A$ is continuous;

(c) $\underset{-}{T}$ has compact support;

(d) $\underset{-}{T}(\Theta p) = p(\underset{\sim}{T})$ for all polynomials $p : \mathbb{R}^m \to \mathbb{F}$,
where $\Theta \in C_0^\infty(\mathbb{R}^m)$ is 1 on a neighbourhood of $\mathrm{spt}\underset{-}{T}$.

The support of $\underset{-}{T}$, $\mathrm{spt}\underset{-}{T}$, is the smallest closed set K such that $\underset{-}{T}(f) = 0$ for all $f \in A_0$ with $K \cap \mathrm{spt}f = \phi$. The support is well-defined in view of condition (a).

Let $A_{(n)}$ be the algebra of functions $f : \mathbb{R}^m \to \mathbb{F}_{(n)}$ of the form $f = \sum_S f_S e_S$ where $f \in A$. The homomorphism $\underset{-}{T}$ extends in a natural way to a homomorphism $\underset{=}{T} : A_{(n)} \to L(X_{(n)})$. Indeed $\underset{=}{T}(f) = \sum \underset{-}{T}(f_S)e_S$, and $\underset{=}{T}$ has the same support as $\underset{-}{T}$.

In the case of a single operator, the following result is due to Foiaş [6].

THEOREM 4. *If* $\underset{\sim}{T}$ *has a functional calculus* $(\underset{-}{T},A)$ *based on* $\mathbb{R}^m$ *then*

$$\sigma(\underset{\sim}{T}) = \mathrm{spt}\underset{-}{T}$$

Proof. It is easy to show that $\sigma(\underset{\sim}{T}) \subset \mathrm{spt}\underset{-}{T}$. We shall prove that $\mathrm{spt}\underset{-}{T} \subset \sigma(\underset{\sim}{T})$, which is what we require for the proof of theorem 1. Let $f \in A_0$ with $\mathrm{spt}f \cap \sigma(\underset{\sim}{T}) = \phi$. We have to show that $\underset{=}{T}(f) = 0$.

Let n be an odd integer, $n \geqq m$, and recall the embeddings $\mathbb{R}^m \subset \mathbb{R}^{n+1} \subset \mathbb{F}_{(n)}$. For $z \in \mathbb{R}^{n+1}$, define functions $h_z : \mathbb{R}^m \to \mathbb{F}_{(n)}$ and $g_z : \mathbb{R}^m \sim \{z\} \to \mathbb{F}_{(n)}$ by $h_z(x) = |x-z|^{n-1}(x-z)$ and $g_z(x) = |x-z|^{-n-1}(\overline{x-z})$. If $\psi \in A_0$, then by condition (a) above, $\psi h_z \in A$, and if $z \notin \mathrm{spt}\psi$, $\psi g_z \in A$. As h_z is a polynomial, $h_z(\underset{\sim}{T})$ is defined and, by theorem 3, is invertible when $z \notin \sigma(\underset{\sim}{T})$. Also $g_z h_z = 1$ except at z .

Construct a function $F : \mathbb{R}^{n+1} \to L(X_{(n)})$ as follows. If $z \notin \operatorname{sptf}$, define $F(z) = \underline{\underline{T}}(fg_z)$, while if $z \notin \sigma(\underset{\sim}{T})$, define $F(z) = h_z(\underset{\sim}{T})^{-1}\underline{T}(f)$. It is straightforward to check that F is well-defined. It takes somewhat more work to check that, for all $u \in X_{(n)}$ and $v \in X^*$, $\langle F(z)u,v\rangle$ is a left monogenic function of z which converges to zero at infinity. By theorem 2, $F(z)$, and hence also $\underline{T}(f)$, is zero. ■

6. OPERATORS WHICH GENERATE BOUNDED GROUPS

In this section $\mathbb{F} = \mathbb{C}$ and $\underset{\sim}{T}$ is an m-tuple of commuting operators $T_j \in L(X)$ which satisfy $\|\exp(isT_j)\| \leq M$ for all $s \in \mathbb{R}$ and some $M > 0$.

Such $\underset{\sim}{T}$ have a functional calculus $(\underline{T}, \check{L}_1(\mathbb{R}^m))$ based on $\mathbb{R}^m$ and defined as follows. The algebra $\check{L}_1(\mathbb{R}^m)$ is the space of inverse Fourier transforms $f = \check{g}$ of functions $g \in L_1(\mathbb{R}^m)$. So $g = \hat{f}$ and we take $\|f\| = (2\pi)^{-m}\|g\|_{L_1}$ which makes $\check{L}_1(\mathbb{R}^m)$ a Banach algebra under pointwise multiplication and addition. The homomorphism $\underline{T} : \check{L}_1(\mathbb{R}^m) \to L(X)$ is defined by

$$\underline{T}(f) = (2\pi)^{-m} \int_{\mathbb{R}^m} e^{i\langle \underset{\sim}{T},\xi\rangle} \hat{f}(\xi)d\xi .$$

Verification of condition (c) of the previous section can be accomplished by an adaptation of the proof of theorem 4 using an approximation argument. Alternatively, it follows from a Paley-Wiener argument as used by Taylor [11] and Anderson [1].

Note that $\underline{\underline{T}}(f)$ is given by the same formula, where $f = \sum_S f_S e_S$, $f_S \in \check{L}_1(\mathbb{R}^m)$ and $\hat{f} = \sum_S \hat{f}_S e_S$.

If $0 \notin \sigma(\underset{\sim}{T})$, then, by theorem 2, $T = \sum_1^m T_j e_j$ is invertible in $L(X_{(n)})$. We then have the following formulae for T^{-1}:

$$T^{-1} = -(\sum_1^m T_j^2)^{-1}T$$
$$= \underline{\underline{T}}(g)$$
$$= (2\pi)^{-m} \int_{\mathbb{R}^m} e^{i<\underset{\sim}{T},\xi>}\hat{g}(\xi)d\xi$$

where $g(x) = |x|^{-2}\bar{x} = -|x|^{-2}x$ on a neighbourhood of $\sigma(\underset{\sim}{T})$ in $\mathbb{R}^m$ and $\hat{g}_k \in L_1(\mathbb{R}^m)$.

7. PROOF OF THEOREM 1

We are now in a position to indicate a proof of theorem 1 concerning the system of operator equations

$$A_jQ-QB_j = W_j \quad \text{for} \quad j = 1,2,\ldots,m$$

where $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are commuting m-tuples of bounded self-adjoint operators defined on Hilbert spaces H and K respectively. We write this system as

$$T(Q) = W$$

where $W = \sum_1^m W_je_j \in X_{(n)}$ with $X = L(K,H)$ and $T = \sum_1^m T_je_j \in L(X_{(n)})$ with $T_j(Q) = A_jQ-QB_j$.

We solve this equation for $Q \in X_{(n)}$ and then determine when the solution is in X (that is, when $Q = Q_0e_0$) .

Consider first the case $\mathbb{F} = \mathbb{C}$. Note that the operators T_j commute and that

$$e^{iT_js}(Q) = e^{iA_js}\,Q\,e^{-iB_js} \quad \text{for} \quad s \in \mathbb{R} .$$

Hence the operators e^{iT_js} are unitary. So the preceding section can be applied to construct a functional calculus $(\underline{T}, \check{L}_1(\mathbb{R}^m))$ for $\underset{\sim}{T}$ based on $\mathbb{R}^m$.

The next task is to find $\sigma(\underset{\sim}{T})$. It will be shown in a fuller version of this paper [8] that $\sigma(\underset{\sim}{T}) = \sigma(\underset{\sim}{A}) - \sigma(\underset{\sim}{B})$. Since we have assumed $\delta = \mathrm{dist}(\sigma(\underset{\sim}{A}) , \sigma(\underset{\sim}{B})) > 0$, it follows that $\sigma(\underset{\sim}{T}) \cap B(0,\delta) = \phi$. So T is invertible, and

$$T^{-1} = -(\sum_1^m T_j^2)^{-1} T = \underline{\underline{T}}(g_{(\delta)})$$

where $g_{(\delta)}(x)=\delta^{-1}g(\delta^{-1}x)$ and $g(x) = -|x|^{-2} x$ for $|x|>1-\varepsilon$ (for some $\varepsilon > 0$) and $\hat{g}_k \in L_1(\mathbb{R}^m)$ for $1 \leqq k \leqq m$.

The solution of the operator equation $T(Q) = W$ is thus

$$\begin{aligned} Q &= T^{-1}(W) \\ &= -(\sum_1^m T_i^2)^{-1} \sum_{j,k} T_j(W_k)e_j e_k \\ &= (\sum_1^m T_i^2)^{-1} \{\sum_{k=1}^m T_k(W_k) - \sum_{1\leqq j<k\leqq m} (T_j(W_k) - T_k(W_j))e_j e_k\} . \end{aligned}$$

So Q belongs to X precisely when $T_j(W_k) = T_k(W_j)$ for all j,k . This is the compatibility condition $A_j W_k - W_k B_j = A_k W_j - W_j B_k$ stated in the theorem.

It remains for $\|Q\|$ to be estimated. If $\delta = 1$ and the compatibility condition is satisfied, then

$$Q = Q_0 = [(2\pi)^{-m} \int e^{i<\underset{\sim}{T},\xi>}\hat{g}(\xi)(W)d\xi]_0$$

the subscript 0 denoting the scalar part (coefficient of e_0) . So

$$Q = -(2\pi)^{-m} \int e^{i<\underset{\sim}{T},\xi>} \sum_{k=1}^m \hat{g}_k(\xi)(W_k)d\xi \quad .$$

Hence,

$$\|Q\| \leqq (2\pi)^{-m} \int |\hat{g}(\xi)|d\xi \, \|\underset{\sim}{W}\|_{K\to H^m}$$

For other values of δ the required estimate follows from a scaling of this one.

This completes the proof for spaces defined over $\mathbb{F} = \mathbb{C}$. The result for $\mathbb{F} = \mathbb{R}$ is obtained by complexifying X to obtain $X_{\mathbb{C}}$ and observing that the operators T and T^{-1} in $L((X_{\mathbb{C}})_{(n)})$ preserve the subspace $X_{(n)}$. ■

8. EPILOGUE

The technique developed in this paper can be applied in the study of more complicated equations (as long as the dependence of W on Q is linear). We can also consider m-tuples of commuting operators $\underset{\sim}{A}$ and $\underset{\sim}{B}$ which themselves have a functional calculus based on $\mathbb{R}^m$, without A_j and B_j necessarily being self-adjoint. Further we can take symmetric norms on Q and W different from the operator norm. Such results will be presented in more detailed papers [8,9]. Also included will be a spectral mapping theorem and a comparison of $\sigma(\underset{\sim}{T})$ with other definitions of joint spectrum.

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