Abstract. The Maxwell-Dirac equations give a model of an electron in an electromagnetic (e-m) field, in which neither the Dirac or the e-m fields are quantized. The two equations are coupled via the Dirac current which acts as a source in the Maxwell equation, resulting in a nonlinear system of partial differential equations (PDE’s). In this way the self-field of the electron is included.

We review our results to date and give the four real consistency conditions (one of which is conservation of charge) which apply to the components of the wavefunction and its first derivatives. These must be met by any solutions to the Dirac equation. These conditions prove to be invaluable in the analysis of the nonlinear system, and generalizable to higher dimensional supersymmetric matter.

In earlier papers, we have shown analytically that in an isolated stationary system, the surrounding electron field must be equal and opposite to the central (external) field. The nonlinearity forces electric neutrality, at least in the static case. We illustrate these properties with a numerical family of orbits which occur in the (static) spherical and cylindrical ODE cases. These solutions are highly localized and die off exponentially with increasing distance from the central charge.

1. The Maxwell-Dirac Equations and QED

The coupled Maxwell-Dirac equations can be written as follows:

\[
\gamma^\alpha (\partial_\alpha - ieA_\alpha)\psi + im\psi = 0 \quad \text{where} \quad \alpha = 0, \ldots, 3
\]

\[
F_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta}
\]

\[
\partial^\alpha F_{\alpha\beta} = -4\pi e j_\beta
\]

where \( j_\alpha = \overline{\psi}\gamma_\alpha\psi \).

Note that \( \psi \in C^4 \) is the Dirac wave-function or \( 4\)-spinor and \( \overline{\psi} \) the Dirac conjugate. These are acted upon by \( \gamma^\alpha \) which are the usual gamma matrices (representations of a Clifford algebra) \( \Gamma \) and \( A_\alpha \) is the 4-potential.

These equations model an electron in an electromagnetic field. The two equations are coupled via the Dirac current \( j_\alpha \) i.e. we include
the nonlinearity of the self-field see Equation (1). These equations form the foundation of quantum electrodynamics (QED) the theory of electrons interacting with fields. QED is one of the most successful physical theories explaining the Lamb shift and the anomalous magnetic moment of the electron. The calculations of QED are achieved by quantizing the fields and using perturbation theory. In so doing, well-known mathematical problems occur which have yet to be resolved at the fundamental level.

It is possible that the full nonlinearized equations must be analysed rigorously before we can hope to resolve these deep problems. For example, Lieb and Loss observe that the stability of matter requires that the electron be defined with a Dirac operator with the magnetic vector potential instead of the free Dirac operator (without ). That perturbation theory must start from the dressed electrons (including their own self-field) might be “fundamentally important in a non-perturbative QED”.

This view is shared by many analysts working on this problem including Flato, Simon and Taflin who established global existence for the M-D equations as recently as 1997 following many years of sustained interest in the problem. In showed that the nonlinear representation is integrable to a global nonlinear representation of the Poincaré group on a differential manifold of small initial conditions. This established the existence of global solutions for initial data at . They go on to show that the asymptotic representations are also nonlinear and draw conclusions for the infrared tail of the electron. Their results show that “in the classical case (also) one obtains infrared divergencies if one requires free asymptotic fields as it is needed in QED”. In other words, the fields must remain coupled via the self-field if we are to resolve the infrared problem.

In the case where we assume that the system is static and/or stationary (see Section 2) we can make some simplifications. The system is elliptic in the stationary case (no time dependence). Esteban, Georgiev and Séré showed existence of soliton-like solutions (that is, solutions which are spatially localized) in this case. Furthermore, the wavefunction together with all its derivatives decreases exponentially at infinity.

2. **Some simplified versions of the problem**

The full 4-dimensional nonlinear problem is somewhat intractable – as stated in Section 1, global existence has only been established recently (and references therein). If we want to get some idea of
the types of behaviour we might expect in the 4-dimensional problem we might begin by looking at various subcases.

Some simplified versions of the problem are as follows:

1. The static case in which we assume that there exists a Lorentz frame in which there is no current “flow” i.e. $j_\alpha = \delta^{\alpha}_0 j_0$ [29] [4]$
2. The stationary case in which we assume $\psi(x_0, \mathbf{x}) = e^{i\omega x_0} \phi(\mathbf{x})$
3. The static spherically symmetric case [29]$
4. The static cylindrically symmetric case [4] [8]$
5. The static case with $z$ dependence only [6]$
6. The $1+1$ case which was solved exactly for a massless electron by Schwinger in [31]$
7. The static axi-symmetric case$
8. The circular current case in which we assume (in spherical coordinates) that $j_\alpha = (j_0, 0, j_\phi, 0)$.

It appears that the static and stationary assumptions are rather strong since the resulting system becomes elliptic rather than hyperbolic. In [5] and [30] we proved electric neutrality in the static case (for an isolated system). While this is interesting in that it implies that solutions of this type must consist of an inner charge say surrounded by an equal and oppositely charged electron field i.e. the solutions must be atom-like it raises the problem of finding a solution that represents a single charged particle. We need to find a weaker ansatz perhaps the circular current assumption with which we have sufficient simplification without losing the important properties of the 4-dimensional system. It is possible of course that electric neutrality could be shown to be fully generall which would supporting the conjecture that the total charge of the universe is zero. The larger (quantum cosmological) problem here is the Einstein-Maxwell-Dirac problem and related problems such as Einstein-Dirac [17] and Einstein-Yang-Mills [3].

Following the methods of the analysts working in relativity theory our aim in [29] [4] etc was to enumerate the subcases starting with all possible ODE cases (see Section 4) and then progressing to (static) axi-symmetric and other two-dimensional cases. Meanwhile the three dimensional static case proved to be somewhat tractable.

3. Using the Clifford Algebra to Derive Some Constraints

In this section we make use of the properties of the Clifford algebra (which the $\gamma^\alpha$ represent as $4 \times 4$ matrices) to solve the Dirac equation for the potential and to show that there are some useful consistency
conditions upon the wavefunction and its first derivatives. We want to write the potential in terms of the wavefunction so that we can substitute it into the Maxwell equation.

A complex Clifford algebra $\mathcal{A}(n)$ consists of all possible products of the $n$ basis vectors (of an $n$-dimensional vector space) which obey the following:

\[(e_i)^2 = -1, \text{ if } i \in \{0, \ldots, n-1\},\]
\[e_i e_j + e_j e_i = 0, \text{ if } i \neq j.\]

Using the isomorphism [11]
\[\mathcal{A}(n+2) \cong \mathcal{A}(n) \otimes \mathcal{A}(2)\]
we can construct the representation of $\mathcal{A}(4)$ from the Pauli matrices which represent $\mathcal{A}(2)$. For the construction of higher dimensional Clifford algebras from lower dimensions see for example [14]. One possible representation of $\mathcal{A}(4)$ is:

\[
\gamma^0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad \gamma^1 = i \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},
\]

\[
\gamma^2 = i \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \gamma^3 = i \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},
\]

We note that in this representation $\gamma^0$ is anti-symmetric (a-s) and $\gamma^i, i = 1, 2, 3$ are symmetric.

Whatever representation we use we can always invert the Dirac equation to express the spatial components of the potential $A^i, i = 1, 2, 3$ in terms of $A^0 \Gamma$ the wavefunction $\psi$ and its first derivatives. Multiplying (1) on the left $\psi^t i \gamma^i$ (where $\psi^t$ is the transpose of $\psi$) we have

\[
\psi^t i \gamma^i \gamma^0 A_\alpha \psi = \frac{1}{ie} \psi^t i \gamma^i (\gamma^0 \partial_\alpha + im) \psi.
\]

But the product $\gamma^i \gamma^j, i \neq j$ is (a-s) $\gamma^j \gamma^i = -1$ and $\gamma^0 \gamma^0$ is symmetric so that using the argument following (7) we can write:

\[
\psi^t \psi A_i = \psi^t \gamma^0 \gamma^i A_0 \psi + \frac{1}{ie} \psi^t \gamma^i (\gamma^0 \partial_\alpha + im) \psi.
\]

In a similar way (by multiplying the Dirac equation by $\overline{\psi} \gamma^0 \gamma^1 \gamma^2 \gamma^3$ on the left and subtracting the conjugate equation premultiplied by
\( \psi \gamma^0 \gamma^1 \gamma^2 \gamma^3 \) we can solve for \( A_0 \). There is a condition to be met here see [29] i.e. that \( j^a \) is not a null vector which has the physical interpretation that the electron is not travelling faster than the speed of light.

In 1957 one of Dirac's students, Eliezer, solved for the potential in this way [15]. He went on to show that when we solve for the potential \( \Gamma \) we must also adopt a consistency condition which applies to \( \psi \). In that paper there was a contribution by Dirac who streamlined some of Eliezer's calculations. Although they used a different representation of the algebra \( \Gamma \) the argument was essentially the following.

If \( \Gamma \) is an antisymmetric (a-s) \((4 \times 4)\) matrix then

\[
\psi^\dagger \Gamma \psi = 0, \text{since this quantity is an a-s scalar.}
\]

If we can find a \( \Gamma \) such that \( \Gamma \gamma^a \) are all a-s \( \Gamma \) then

\[
\psi^\dagger \Gamma (\gamma^a A_\alpha) \psi = 0,
\]

since \( A_\alpha \) is a scalar \( \Gamma \) so that if \( \psi \) satisfies (1) then

\[
\psi^\dagger \Gamma (\gamma^a \partial_\alpha + im) \psi = 0,
\]

which gives us a consistency condition on \( \psi \) and its first derivatives.

The same is true of the complex Dirac equation and we have

\[
\overline{\psi}^\dagger \Gamma (\gamma^a \partial_\alpha - im) \overline{\psi} = 0.
\]

We can extend Dirac's argument one step further by premultiplying the Dirac equation by \( \overline{\psi} \) and the complex equation by \( \psi \) and noting that since \( \Gamma \gamma^a \) is a-s \( \Gamma \)

\[
\left( \overline{\psi}^\dagger \Gamma \gamma^a A_\alpha \psi \right)^t = -\psi^\dagger \Gamma \gamma^a A_\alpha \overline{\psi}.
\]

This gives us another condition on \( \psi \) namely

\[
\left( \overline{\psi}^\dagger \Gamma (\gamma^a \partial_\alpha + im) \psi \right)^t + \psi^\dagger \Gamma (\gamma^a \partial_\alpha - im) \overline{\psi} = 0.
\]

There is only one possible element of the Clifford algebra which when premultiplying all of the \( \gamma^a \) yields an a-s matrix. In the representation used here \( \Gamma \) this is the product

\[
\Gamma = \gamma^1 \gamma^2 \gamma^3.
\]

Using this in (8) (9) and (10) \( \Gamma \) we have two complex conditions (or four real conditions) on the components of \( \psi \).

As is well-known (see [11] for example) even dimensional complex Clifford algebras are simple \( \Gamma \) that is \( \Gamma \) they can not be decomposed into the direct sum of two nontrivial subspaces which obey closure under algebraic multiplication. If we want to decompose the 4-dimensional into a 2-dimensional algebra and use 2-spinors \( \Gamma \) then when must accept
the additional structure in which the two 2-spinor spaces are conjugate dual spaces. See for example [28] in which the 2-spinor formalism is given in terms of Infield-van der Waerden symbols. An argument equivalent to Dirac/Eliezer’s but giving all of the consistency conditions was given by Radford using the 2-spinor formalism in 1996 [29] and subsequently in [4] although Radford referred to them as “reality conditions” in keeping with the conventions in [28].

We can think of the Dirac equation and its conjugate equations as eight equations in four unknowns (the four real scalars $A_\alpha$). If we solve for these $A_\alpha$ then we must have four additional (real) constraints which correspond to the two complex consistency conditions confirming that there are no further constraints upon the system. In [7] we outline these conditions and then go on to generalize to higher dimensional cases. Allowing these higher Clifford algebras enables us to pursue the same arguments when applied to supersymmetric matter [7]. See [22] and references therein.

4. ODE solutions

Within the static system there are three interesting ODE cases: spherically symmetric, cylindrically symmetric and dependence on $z$ only. The spherical and cylindrical cases were examined extensively in [29] [4]. The case where dependence is on $z$ only is similar in some respects [5]. We first apply our consistency conditions to the electromagnetic potential $A^\alpha$ which has been expressed in terms of Dirac spinors and their first derivatives (by solving the Dirac equations for the potential as outlined in Section 3). These reality conditions allow us some simpler expressions which are then inserted into the Maxwell equation resulting in fourth order ODE’s. We will also note here that in the $1+1$ case [12] [13] [6] the system was also reduced to fourth order ODE’s which in some cases are solved explicitly [12] [13] whilst in others we are currently developing more numerical results [6].

When we assume that the Dirac current is static we lose three (real) degrees of freedom in $\psi\Gamma$ since three components of $j^\alpha$ are set to zero. See [29] and [4] in which we also fix the gauge by choosing

$$\psi^0 = -Ye^{i(x-\eta)} \quad \psi^2 = -Ye^{-i(x+\eta)}$$

$$\psi^1 = Xe^{i(x+\eta)} \quad \psi^3 = Xe^{-i(x-\eta)},$$

where $\eta X$ and $Y$ are real functions. These expressions were substituted into the potential (which was solved for in terms of Dirac spinors
and their first derivatives yielding:

\begin{align}
(12) \quad A^0 &= \cos \chi + \left( \frac{X^2 - Y^2}{X^2 + Y^2} \right) \frac{\partial \eta}{\partial t} + \frac{(\nabla \chi) \cdot I}{(X^2 + Y^2)} \\
(13) \quad A &= \frac{1}{(X^2 + Y^2)} \left[ \frac{\partial \chi}{\partial t} I + (X^2 - Y^2) \nabla \eta - \nabla \times I \right].
\end{align}

We now use the consistency conditions one of which is conservation of charge which is obeyed automatically as stated in [29]. The other three conditions in the variables required for the static case are given below.

\begin{align}
(14) \quad \frac{\partial}{\partial t} (X^2 + Y^2) &= 0 \\
(15) \quad \nabla \cdot I &= -(X^2 + Y^2) \sin \chi \\
(16) \quad \frac{\partial I}{\partial t} + (\nabla \chi) \times I &= 0.
\end{align}

where \( I = (2XY \cos \eta, 2XY \sin \eta, X^2 - Y^2) \).

The Maxwell equations act upon \( A \) as defined above and the current vector becomes \( j^a = (2(X^2 + Y^2), 0, 0, 0) \).

We showed in [5] that the static equations in the gauge given by Eq. 11 are stationary if and only if \( \frac{\partial \eta}{\partial t} = 0 \) and \( \frac{\partial X}{\partial t} = 0 \) (or \( \frac{\partial Y}{\partial t} = 0 \)). In the stationary case \( \frac{\partial \chi}{\partial t} = 0 \) and \( \frac{\partial I}{\partial t} = 0 \). Now in the stationary case the third reality condition Eq. 16 tells us that \( \nabla \chi \) is proportional to \( I \) and we choose the function \( \alpha \) such that \( I = \frac{\alpha}{r \sin \theta} \nabla \chi \). Substituting this into the expression for the potential Eq. 12 and noting that \( X^2 + Y^2 = |I| \) then

\begin{align}
(17) \quad A^0 &= \cos \chi \pm \frac{\nabla \chi \cdot I}{(X^2 + Y^2)} = \cos \chi \pm \sqrt{\frac{\chi^2}{r^2} + \frac{\chi^2}{r^2} \sin^2 \theta}.
\end{align}

From here it is a straightforward calculation to apply symmetry arguments and calculate the resulting ODE’s [29] [4] [5].

Then in dimensionless variables [29] [4] the equations reduce to:

\begin{align}
\frac{d\chi}{dx} &= A - \cos \chi \\
\frac{dF}{dx} &= Z \\
\frac{dA}{dx} &= F \\
\frac{dZ}{dx} &= -Z \sin \chi,
\end{align}

(18)
where

\[
f(x) = \begin{cases} 
  x^2 & \text{in the spherical case} \\
  x & \text{in the cylindrical case} \\
  1 & \text{in the } z \text{ dependent case}
\end{cases}
\]

and

\[
x = \begin{cases} 
  r & \text{in the spherical case} \\
  \rho & \text{in the cylindrical case} \\
  z & \text{in the } z \text{ dependent case}
\end{cases}
\]

\(r\) being the distance from the origin and \(\rho\) the distance from the central axis.

It is easily shown that these four first order equations can be written as the fourth order

\[
\frac{d^2}{dx^2} \left( f(x) \left( \frac{d^2 \chi}{dx^2} - \sin \chi \frac{d\chi}{dx} \right) \right) \\
+ \frac{d}{dx} \left( f(x) \left( \frac{d^2 \chi}{dx^2} - \sin \chi \frac{d\chi}{dx} \right) \right) \sin \chi = 0.
\]

(19)

In the cylindrical case \(Z(\rho)\) is the charge per unit ring radius \(\rho \Gamma F(\rho)\) is the charge within a radius \(\rho \Gamma\) and \(A\) is the scalar potential \(\Gamma A^0\). See [4]. Similar physical quantities are represented in the spherical and \(z\) cases. Most importantly we are looking for solutions whose charge density \(Z(\rho)\) decreases rapidly towards infinity so that we can find solutions which are localized or particle-like. Our zero total charge result [5] tells us to expect that \(F \to 0\) as \(\rho \to \infty\) and likewise \(\Gamma A \to \text{const as } \rho \to \infty\).

As pointed out by Chris Cosgrove (private communication) Equation (19) has non-integer resonance numbers [1] [2] and we do not expect to find an integrable system (in the soliton sense) here. Instead we show that there are a family of orbits (in the sense of [10]) all of which approach the trivial (constant) solution at infinity. As an example we look at the cylindrical case noting that similar results hold in the spherical and \(z\) case. The numerical orbits in the Figure 1 complete the results in [4] in which a single member (analytic in \(1/\rho\)) of these families was shown to exist and calculated numerically.

5. Vacuum Maxwell singularities near the origin

In [4] it was shown that in the cylindrical case

**Lemma 1.** Suppose \((\chi, F, A, Z)\) is a solution to Equation (18) on \(I = (0, \rho_1)\), for some \(\rho_1, 0 < \rho_1 < 1\). Suppose also that \(Z \geq 0\) is continuous and bounded on \(I\). Then,
(i) $F$ is $C^1$ on $I$ and has a well defined, finite limit as $\rho \to 0$. $Z$ has a well-defined limit as $\rho \to 0$.

(ii) if $F(0) \neq 0$ then $A$ is unbounded as $\rho \to 0$. In particular, $A = \Omega(\rho) \ln(\rho)$, where $\Omega$ is $C^2$ and bounded on $I$, $\Omega \to F(0)$ as $\rho \to 0$.

Also, $\chi$ is bounded as $\rho \to 0$.

Lemma 2. Suppose $(\chi, F, A, Z)$ is a solution to Equation (18) on $\rho \in (0, \infty)$. Suppose also that $Z \geq 0$ with $F$ continuous and bounded on the interval. Then

(i) If $F(\rho_1) \geq 0$ for some $\rho_1 \in [0, \infty)$, then $\chi \to \infty$, $A \to \infty$, and $F \to \infty$ as $\rho \to \infty$.

(ii) If $F < 0$ on $(0, \infty)$ then $F \to 0$ as $\rho \to \infty$. In addition, if $A$ and $Z$ have well-defined limits as $\rho \to \infty$ then $Z \to 0$ and $A \to A_\infty$ as $\rho \to \infty$, with $-1 \leq A_\infty \leq 1$.

Similar results were established for the spherical case in [29]. In [29][4], Radford and the current author have shown that the behaviour of solutions near the origin resembles the vacuum solutions of the Maxwell equations. Given that the Maxwell equation can be written as the square of a Dirac operator (see for example [26]) it should be possible to formulate these results together with exponential decrease in terms of the properties of $k$-monogenic functions — those functions which are solutions to

$$D_k = D + k \epsilon_0 = 0,$$

where $\epsilon_0$ is a basis vector (corresponding to the time coordinate) of a complex Clifford algebra. The fundamental solution of the k-Dirac operator has the same singularity at the origin and decreases exponentially as $x \to \infty$. In this case the operator has coefficients in $\Lambda^1 \Gamma$ the subspace of vectors $\gamma^0$. We are currently working on a clarification of this point.

6. Numerical Solutions

In [29][4][8] we showed examples of numerical solutions exhibiting the characteristics referred to in Section 5. Earlier attempts at finding numerical solutions [32][25] were marred by a “simplifying assumption” which proved to be invalid — solutions did not exist in that case. See [29].

In [8] we noted that there are families of orbits parameterized by the constant $c|x|$ where we assume that solutions are of the form $e^{-c|x|} g(|x|) \Gamma$ with $g(|x|)$ an analytic function. By varying the value of $c\Gamma$ the boundary conditions are perturbed to neighboring solutions.
(orbits) all decreasing exponentially at infinity. See Figure 1. All values of $c$ yield solutions which satisfy the two Lemmas and the zero total charge result. That is, all solutions surround a central wire along the axis of symmetry.

Similar numerical solutions can be found in the spherical case [6]. The spherical solutions surround a central Coulomb field but the static condition forces a monopole at the origin [29]-[30] that is an unbounded $A^\theta$ component. Further ODE solutions occur in the $z$ case and $1 + 1$ case, neither of which have been fully examined in the previous literature [12]-[13]. These results will be forthcoming also in [6].

Similar spherically-symmetric solutions were found in [27]. This time the Schrödinger-Newton equations provided an identical coupling as in the static Maxwell-Dirac case which is essentially an elliptic system. More work must be done to investigate the stability of both of these systems. In [27] implications have been developed in the context of quantum gravity which work in this area is far from complete. We are currently considering the problem in this context. The solutions found in [27] blow-up at larger distances from the origin. However in the Maxwell-Dirac case this behaviour appeared only as a numerical anomaly. When the total charge became slightly positive (due to the step-size of the numerical solution) we entered a regime described in Lemma 2 of [4] in which all solutions become unbounded as $\rho \to \infty$. These solutions were illustrated numerically in [4] but discarded as being of less interest than the bounded solutions.

The solution shown in Figure 1 was calculated using the MATLAB ODE solver ODE113. The relative error tolerance was set at $1e-4$ and the absolute error tolerance at $1e-8$. The same behaviour was observed when the tolerances were decreased to $1e-6$ and $1e-12$ or increased to $1e-3$ and $1e-6$. All solutions remained stable whether calculated as a function of increasing or decreasing radius (i.e. shooting away from or towards the central axis).

The two dimensional cases (static axi-symmetric, circular current axi-symmetric, the massive $1 + 1$ case) still require reliable numerical results. Those available to date [32]-[25] have been flawed by the imposition of “approximations” which were shown in [29] to be possible only in trivial cases (in which the equations are no longer coupled).

7. Vanishing Total Charge

In the solutions in Figure 1 the variable $F(\rho)$ representing the total charge within a ring radius $\rho$ tends towards zero at $\infty$. Lemma 1 also states that the potential must tend towards a solution which is
Figure 1. A family of orbits exhibiting the properties of Lemma 1 and Lemma 2.

logarithmic in $\rho \Gamma$ near the central axis. This corresponds to a central charged wire (along the $z$ axis) which is a solution to the vacuum Maxwell equations. As such we can think of this part of the solution as representing an external field. (The scalar potential could be separated at this point into $A_{\text{external}} + A_{\text{fermion+interaction}} \Gamma$ since the $A_{\text{external}} \Gamma$ as the homogenous solution contributes nothing to the coupling between the Maxwell and Dirac equations.) Physically this means that the (inner) external field must be surrounded by an equal and oppositely charged field. The total electric charge of the system

$$\lim_{\rho \to \infty} \frac{1}{4\pi} \int_{S_{\rho}} (\nabla A^0) \cdot dS, \quad \text{with} \quad S_{\rho} \quad \text{the unit ring of radius} \quad \rho,$$

must vanish.
In [5] we were able to show that this is the case for all stationary solutions given that they are in a “reasonable” function space. If we define an isolated system as one for which all sources are contained in some ball $B_k$ ($k < \infty$) and for which the fields die off as $|x| = r \to \infty$ then we showed in [5] and [30] that an isolated static Maxwell-Dirac system is electrically neutral. A similar result holds for the total electric charge per unit length in the $z$ direction (or unit ring) in the cylindrical case. This shows mathematically that a (stationary) solution must be atom-like (in the sense that any central charge must be surrounded by an equal and opposite charge. In addition it shows that there cannot be a stationary solution representing an isolated electron (or even one that rotates around the axis with constant velocity).

Note also that the signs of the charges can be reversed giving us negatively charged singularities and positively charged “electron” fields. All results remain unchanged under such a reversal of sign. The total charge must vanish overall. A more difficult problem presents itself if we are to allow for two fermion fields of opposite charge for example the solution corresponding to positronium. We are currently considering ways in which we can have have solutions of opposite charge interacting together.

8. Discussion and Conclusions

Until the mathematical problems of QED have been resolved all methods of addressing the equations governing the interaction between electrons and fields are significant. A non-perturbative QED must consist of classical and/or semi-classical arguments which aim to justify the Feynman integral method and the quantized field approach.

For the coupled M-D system we have existence (for small initial data) and existence for the stationary case and descriptions of various subcases. But analysis of the system is far from complete. We have indicated within this review paper the directions we are currently pursuing.

References


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