VILENKIN BASES IN NON-COMMUTATIVE L_p-SPACES

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ABSTRACT. We study systems of eigenspaces arising from the representation of a Vilenkin group on a semifinite von Neumann algebra. In particular, such systems form a Schauder decomposition in the reflexive non-commutative L_p-spaces of measurable operators affiliated with the underlying von Neumann algebra. Our results extend classical results of Paley concerning the familiar Walsh-Paley system to the non-commutative setting.

1. INTRODUCTION

It is a classical theorem of Paley [Pa] that the Walsh system, taken in the Walsh-Paley ordering, is a Schauder basis in each of the reflexive L_p-spaces on the unit interval. Although the Walsh basis in not unconditional, except in the case $p = 2$, it was further proved by Paley that partitioning the Walsh system into dyadic blocks yields an unconditional Schauder decomposition. This paper provides an overview of recent work by the present authors (and collaborators) which develops the theory of orthogonal systems in the setting of reflexive non-commutative L_p-spaces of measurable operators affiliated with a semifinite von Neumann algebra. The work finds its roots in that of Paley and in the subsequent development of Paley's ideas in the more general setting of Vilenkin systems by Watari [Wa], Schipp [Sc], Simon [Si1,2] and Young [Y].

In the classical setting, the Walsh and Vilenkin systems arise as the system of characters of the familiar dyadic and Vilenkin groups. The present approach exhibits non-commutative orthogonal systems as eigenvectors corresponding to the action of an ergodic flow on the underlying non-commutative L_p-space. The classical results are recovered by specialisation to the case where the flow is given by the action of right translation on the space $L_p(G)$, with G an arbitrary Vilenkin group. In this case, the eigenspaces are one-dimensional and are spanned by the characters of G.

Research supported by the Australian Research Council (ARC).

2. Preliminaries

There is an extensive literature concerning harmonic analysis on compact Vilenkin groups that is related to the theme of present article. In particular we refer the reader to [AVDR], [BaR], [Gos], [SWS], [Sc], [Si1,2], [Vi], [W], [Wa], [Y] and the references contained therein.

We shall be concerned here with Vilenkin groups of the form $G_{\mathbf{m}} = \prod_{k=0}^{\infty} \mathbb{Z}_{m(k)}$, equipped with the product topology and normalized Haar measure. Here $\mathbf{m} = \{m(k) \mid k \in \mathbb{N} \cup \{0\}\}$ is a sequence of natural numbers greater than one and

$$\mathbb{Z}_{m(k)} = \{0, 1, 2 \ldots, m(k) - 1\}$$

is the discrete cyclic group of order $m(k)$. The dual group $\widehat{G_{\mathbf{m}}}$ of $G_{\mathbf{m}}$ can be identified with the sum

$$\coprod_{k=0}^{\infty} \widehat{\mathbb{Z}_{m(k)}} = \coprod_{k=0}^{\infty} \mathbb{Z}_{m(k)}$$

consisting of all sequences $\mathbf{n} = (n_0, n_1, \ldots)$ with $n_k \in \mathbb{Z}_{m(k)}$ for all k and $n_k \neq 0$ for at most finitely many k. See for example [SWS]. The pairing between $G_{\mathbf{m}}$ and $\widehat{G_{\mathbf{m}}}$ is given by

$$\langle \mathbf{t}, \mathbf{n} \rangle = \psi_{\mathbf{n}}(\mathbf{t}),$$

where

$$\psi_{\mathbf{n}}(\mathbf{t}) = \prod_{k=0}^{\infty} \phi_k^{n_k}(\mathbf{t}), \quad \forall \mathbf{n} = (n_0, n_1, \ldots) \in \widehat{G_{\mathbf{m}}}$$

and

$$\phi_k(\mathbf{t}) := \exp(2\pi i \frac{t_k}{m(k)}), \quad \forall \mathbf{t} = (t_0, t_1, \ldots) \in G_{\mathbf{m}}.$$

The dual group $\widehat{G_{\mathbf{m}}}$ is linearly ordered by the (reverse) lexicographical ordering: for $\mathbf{n}, \mathbf{p} \in \widehat{G_{\mathbf{m}}}$ we define $\mathbf{n} < \mathbf{p}$ if and only if there exists $k \in \mathbb{N} \cup \{0\}$ such that $n_j = p_j$ for all $j > k$ and $n_k < p_k$. We shall always consider the system of characters $\{\psi_{\mathbf{n}} : \ \mathbf{n} \in \widehat{G_{\mathbf{m}}}\}$ with the enumeration induced by the reverse lexicographical ordering of $\widehat{G_{\mathbf{m}}}$. For $k = 1, 2, \ldots$ and $1 \leq j \leq m(k) - 1$, define $\mathbf{d}_0,\ \mathbf{d}_k,\ \mathbf{d}_{(k,j)} \in \widehat{G_{\mathbf{m}}}$ via

$$\mathbf{d}_0 := 0, \quad \mathbf{d}_k := \{\delta_{jk}\}_{j=1}^{\infty}, \quad \mathbf{d}_{(k,j)} := \{j\delta_{ik}\}_{i=1}^{\infty}.$$

$\mathbf{d}_k,\ \mathbf{d}_{(k,j)}$ correspond to $\phi_k,\ \phi_k^j$ respectively.

The system of characters $\{\psi_{\mathbf{n}} : \ \mathbf{n} \in \widehat{G_{\mathbf{m}}}\}$ forms a complete orthonormal system in $L^2(G_{\mathbf{m}}, d\mathbf{t}_{\mathbf{m}})$, called the **Vilenkin** system corresponding to $\mathbf{m}$. Here $d\mathbf{t}_{\mathbf{m}}$ denotes normalised Haar measure on $G_{\mathbf{m}}$.

We set $M_0 := 1$ and $M_k := m(k-1)M_{k-1}$ and to each $\mathbf{n} \in \widehat{G_{\mathbf{m}}}$ we assign the natural number

$$n = \sum_{k=0}^{\infty} n_k M_k, \quad 0 \le n_k < m(k).$$

This defines an order preserving bijection between $\widehat{G_{\mathbf{m}}}$ and $\mathbb{N} \cup \{0\}$. If we denote the character $\psi_{\mathbf{n}}$ by ψ_n, with n corresponding to $\mathbf{n}$ under this bijection, then we may write the Vilenkin system as $\{\psi_n\}_{n=0}^{\infty}$.

There is a well-known and natural measure preserving identification between the Vilenkin group $G_{\mathbf{m}}$ and the closed interval $[0,1]$ given by the map $\mathbf{t} \to t(\mathbf{t}) \in [0,1]$ where

$$t(\mathbf{t}) := \sum_{k=0}^{\infty} \frac{t_k}{M_{k+1}}.$$

In the special case that $m(k) = 2$ for all $k \in \mathbb{N} \cup \{0\}$, the Vilenkin group $G_{\mathbf{m}}$ is the dyadic group $\mathbb{D}$, and the characters $\{\phi_k\}_{k=0}^{\infty}$ may be identified with the usual Rademacher system $\{r_k\}_{k=0}^{\infty}$ on $[0,1]$. In this case, the Vilenkin system $\{\psi_{\mathbf{n}}\}_{n=0}^{\infty}$ coincides with the familiar Walsh system $\{w_k\}_{k=0}^{\infty}$, taken in the Walsh-Paley ordering.

We begin with the following classical results concerning Vilenkin systems. If $f \in L_p(G_{\mathbf{m}}, d\mathbf{t}_{\mathbf{m}})$, we set

$$c_{\mathbf{k}}(f) := \int_{G_{\mathbf{m}}} f\overline{\psi_{\mathbf{k}}} d\mathbf{t}_{\mathbf{m}}, \quad \mathbf{k} \in \widehat{G_{\mathbf{m}}}.$$

Theorem 1 *Suppose* $1 < p < \infty$.

(i) $\exists K_p, \forall f \in L_p(G_{\mathbf{m}}, d\mathbf{t}_{\mathbf{m}}), \forall \mathbf{n} \in \widehat{G_{\mathbf{m}}}$

$$\left\| \sum_{\mathbf{k}<\mathbf{n}} c_{\mathbf{k}}(f)\psi_{\mathbf{k}} \right\|_p \le K_p \|f\|_p. \tag{1.1}$$

(ii) $\exists K_p, \forall \epsilon_k = \pm 1, \forall f \in L_p(G_{\mathbf{m}}, d\mathbf{t}_{\mathbf{m}})$

$$\left\| \sum_{k=0}^{\infty} \epsilon_k \left(\sum_{\mathbf{d}_k \le \mathbf{n} < \mathbf{d}_{k+1}} c_{\mathbf{n}}\psi_{\mathbf{n}} \right) \right\|_p \le K_p \|f\|_p. \tag{1.2}$$

(iii) *If* $\sup_k m(k) < \infty$, $\exists K_p, \forall \epsilon_{k,j} = \pm 1, \forall f \in L_p\left(G_{\mathbf{m}}, d\mathbf{t}_{\mathbf{m}}\right)$

$$\left\| \sum_{k=0}^{\infty} \sum_{j=1}^{m(k)-1} \epsilon_{k,j} \left(\sum_{\mathbf{d}_{k,j} \leq \mathbf{n} < \mathbf{d}_{k,j+1}} c_{\mathbf{n}} \psi_{\mathbf{n}} \right) \right\|_p \leq K_p \|f\|_p. \tag{1.3}$$

Remarks In the terminology of Banach spaces (see, for example [LT1,2]), the assertion (1.1) is equivalent to the statement that the Vilenkin system $\{\psi_{\mathbf{n}}, \mathbf{n} \in \widehat{G_{\mathbf{m}}}\}$ is a Schauder basis in $L_p\left(G_{\mathbf{m}}, d\mathbf{t}_{\mathbf{m}}\right)$. Similarly, the estimate (1.2) asserts that the system

$$X_k = \operatorname{clm}\{\psi_{\mathbf{n}} : \mathbf{d}_k \leq \mathbf{n} < \mathbf{d}_{k+1}\}, \quad k = 0, 1, 2, \ldots$$

is an unconditional Schauder decomposition of $L_p\left(G_{\mathbf{m}}, d\mathbf{t}_{\mathbf{m}}\right)$. The statement of (1.3) is that the system

$$X_{k,j} = \operatorname{clm}\{\psi_{\mathbf{n}} : \mathbf{d}_{k,j} \leq \mathbf{n} < \mathbf{d}_{k,j+1}\}, \quad 1 \leq j \leq m(k-1), \quad k = 0, 1, 2, \ldots$$

is a (finer) unconditional Schauder decomposition of $L_p\left(G_{\mathbf{m}}, d\mathbf{t}_{\mathbf{m}}\right)$, provided that the Vilenkin group $G_{\mathbf{m}}$ is bounded in the sense that $\sup_k m(k) < \infty$. Here, $\operatorname{clm}(A)$ denotes the closed linear manifold generated by A.

We remark further that if we set

$$\Psi_k := \sum_{\mathbf{d}_k \leq \mathbf{n} < \mathbf{d}_{k+1}} c_{\mathbf{n}} \psi_{\mathbf{n}}, \quad k = 0, 1, 2, \ldots,$$

then (1.2) equally asserts that the sequence $\{\Psi_k, \quad k = 0, 1, 2, \ldots\}$ is an unconditional martingale difference sequence in $L_p\left(G_{\mathbf{m}}, d\mathbf{t}_{\mathbf{m}}\right)$.

In the case of the classical Walsh-Paley system, that is, for $m(k) = 2,\ k = 0, 1, 2, \ldots,$ Theorem 1 is due to Paley [Pa, 1931], who showed further that statements (i), (ii) are equivalent. In the case of more general but bounded Vilenkin systems, Theorem 1 was established by Watari [Wa,1958]. In this case, (ii) is valid, but is not equivalent to (i). Watari showed that (i) is equivalent to (iii), but that (iii) is not equivalent to (i) if the Vilenkin group is not bounded. For general Vilenkin systems that are not necessarily bounded, Theorem 1(i) is due independently to Young, Schipp and Simon [Y,Sc,Si1, 1976].

3. Non-commutative Vilenkin systems

In order to establish a general framework, it will be convenient to recall some basic facts concerning representations of compact Abelian groups on Banach spaces.

Let X be a Banach space, let G be a compact Abelian group with dual group $\hat{G}$ and normalized Haar measure dm. Let $\alpha = \{\alpha_t\}_{t\in G}$ be an action (strongly continuous representation) of G by automorphisms of X. We set

$$c_\alpha := \sup_{t\in G} \|\alpha_t\|_{X\to X} < \infty. \tag{1.1}$$

For each $\gamma \in \hat{G}$, we define the eigenspace X_γ corresponding to γ and the representation α by setting

$$X_\gamma := \{x \in X : \ \alpha_t(x) = \overline{\langle t, \gamma\rangle} x, \ \forall t \in G\}.$$

It should be noted that the eigenspace X_γ may be $\{0\}$, that $X_\gamma \cap X_{\gamma'} = \{0\}$ if $\gamma \neq \gamma'$, and that $\mathrm{clm}\{X_\gamma : \ \gamma \in \hat{G}\} = X$.

It is important to note that if $X = L_p(G, dm)$ and if α is defined by setting

$$\alpha_t(x)(s) := x(s-t), \quad s, t \in G$$

then

$$X_\gamma = \{c\gamma : c \in \mathbb{C}\}.$$

In other words, the eigenspaces corresponding to the action of G on $L_p(G, dm)$ given by forward translation are precisely the one-dimensional eigenspaces spanned by the characters of G.

The non-commutative framework that we require is provided by the well-known theory of non-commutative integration with respect to a semi-finite trace, introduced by Segal [Se] and Dixmier[Di]. We suppose that $(\mathcal{M}, \tau)$ is a semi-finite von Neumann algebra equipped with a faithful, normal semi-finite trace τ, and unit $\mathbf{1}$. For relevant definitions see [Ta]. A closed densely defined operator x is said to be affiliated with $\mathcal{M}$ if $ux = xu$ for any unitary u in the commutant $\mathcal{M}'$ of $\mathcal{M}$. An operator x affiliated with $\mathcal{M}$ is said to be τ-measurable if $\tau(e_\lambda(|x|)) < \infty$ for some $\lambda > 0$, where $e_\lambda(|x|)$ denotes the spectral projection $\chi_{(-\infty,\lambda]}(|x|)$. If $1 \leq p < \infty$, we denote by $L_p(\mathcal{M}, \tau)$ the space of all τ-measurable operators x affiliated with $\mathcal{M}$ for which

$$\|x\|_p := \tau(|x|^p)^{\frac{1}{p}} < \infty$$

where $|x| = \sqrt{x^*x}$. The precise definitions and relevant properties may be found in [FK],[GK1]. Let α be an action of the Vilenkin group $G_{\mathbf{m}}$ on $L_p(\mathcal{M},\tau)$. The system of eigenspaces

$$\left\{L_p(\mathcal{M},\tau)_{\mathbf{n}} : \mathbf{n} \in \widehat{G_{\mathbf{m}}}\right\}$$

taken in the (reverse) lexicographical ordering will be called a *Vilenkin decomposition* of $L_p(\mathcal{M},\tau)$. An important special case is given when $\tau(1) < \infty$ and the representation α is given by an ergodic $G_{\mathbf{m}}$-flow, that is α is an ultraweakly continuous group of trace preserving $*$-automorphisms of $\mathcal{M}$, and which is ergodic in the sense that the fixed-point algebra consists of scalar multiples of the identity (see [OPT]). In this case, α extends to an isometric action on $L_p(\mathcal{M},\tau)$ and each eigenspace $L_p(\mathcal{M},\tau)_{\mathbf{n}}$ for the exended action coincides the corresponding eigenspace $\mathcal{M}_{\mathbf{n}}$ for the original action of α on $\mathcal{M}$. Further, each eigenspace $\mathcal{M}_{\mathbf{n}}$ is the one-dimensional span of some unitary operator $W_{\mathbf{n}} \in \mathcal{M}$ and the sequence $\{W_{\mathbf{n}}\}$ is an orthonormal basis (in the natural sense) in the space $L_2(\mathcal{M},\tau)$.

We consider the special case obtained by taking $\mathcal{M}$ to be $L^\infty(G_{\mathbf{m}}, d\mathbf{t})$ acting by multiplication on $L_2(G_{\mathbf{m}}, d\mathbf{t})$, with trace given by integration with respect to Lebesgue measure. If α is given by right translation, then the Vilenkin decomposition $\{\mathcal{M}_{\mathbf{n}}\} = \{\psi_{\mathbf{n}} : \mathbf{n} \in \widehat{G_{\mathbf{m}}}\}$ is simply that given by the classical Vilenkin basis. This observation shows that Theorem 1 is actually a special case of the following Theorem.

Theorem 2 [DFPS] (i) *If $1 < p < \infty$, then each Vilenkin decomposition $\{L_p(\mathcal{M},\tau)_{\mathbf{n}}, \mathbf{n} \in \widehat{G_{\mathbf{m}}}\}$ is a Schauder decomposition that is, for all $x \in L_p(\mathcal{M},\tau)$ there exists a unique sequence $x_{\mathbf{n}} \in L_p(\mathcal{M},\tau)_{\mathbf{n}}$ such that $x = \sum_{\mathbf{n}\in\widehat{G_{\mathbf{m}}}} x_{\mathbf{n}}$ and there exists a constant K_p, depending only on p and the bound c_α of the representation α such that*

$$\left\|\sum_{\mathbf{k}<\mathbf{n}} x_{\mathbf{k}}\right\|_p \leq K_p\|x\|_p \tag{2.1}$$

(ii) **[SF1,2, FS]** There exists a constant K_p depending only on p such that for all $\epsilon_k = \pm 1$,

$$\left\|\sum_{k=0}^{\infty} \epsilon_k \left(\sum_{\mathbf{d}_k \leq \mathbf{n} < \mathbf{d}_{k+1}} x_{\mathbf{n}}\right)\right\|_p \leq K_p\|x\|_p. \tag{2.2}$$

(iii)[**DS**] If $\sup m(k) < \infty$, there exists a constant K_p depending only on p such that for all $\epsilon_{k,j} = \pm 1$,

$$\left\| \sum_{k=0}^{\infty} \sum_{j=1}^{m(k)-1} \epsilon_{k,j} \left(\sum_{\mathbf{d}_{k,j} \le \mathbf{n} < \mathbf{d}_{k,j+1}} x_{\mathbf{n}} \right) \right\|_p \le K_p \|x\|_p. \tag{2.3}$$

We remark first that the starting point for the proof of Theorem 2 may be taken to be the assertion of (ii). This is a consequence of the fact that reflexive non-commutative L_p-spaces have the (so-called) UMD-property combined with an application of the well- known transference principle. Details may be found in [SF1,2]. In the case of bounded Vilenkin groups and finite von Neumann algebras, Theorem 2 was established on [DS]. The case that the von Neumann algebra has infinite trace follows from [DS] and [CPSW]. The general theorem is proved in [DFPS].

As observed above, Theorem 2 contains Theorem 1 as a special case. However, the proof of Theorem 2 is new even in its commutative specialisation and is based on essentially non-commutative techniques. To illustrate some of the ideas on which the proof of Theorem 2 is based, we will consider some very special cases.

We begin with the commutative example given by taking $\mathbf{m} = (n, 0, 0, \ldots)$ for some natural number $n \ge 2$ so that $G_{\mathbf{m}}$ may be identified with $\mathbb{Z}_n$. We take $\mathcal{M}$ to be $L^\infty(\mathbb{Z}_n) = l_n^\infty$, where $\mathbb{Z}_n$ is equipped with counting measure and acting on $L_2(\mathbb{Z}_n) = l_n^2$ by pointwise multiplication. We identify $k \in \widehat{\mathbb{Z}_n}$ with the character $\phi_k \in L^\infty(\mathbb{Z}_n)$ given by

$$\phi_k(j) = (\varepsilon^k)^j, \quad \varepsilon = e^{\frac{2\pi i}{n}}, \quad j = 0, 1, \ldots, n-1$$

with $\varepsilon = e^{\frac{2\pi i}{n}}$. The system $\{\phi_k\}_{k \in \widehat{\mathbb{Z}_n}}$ is a vector basis of l_n^p. The assertion that the system $\{\phi_k\}_{k \in \widehat{\mathbb{Z}_n}}$ is a Schauder basis of l_n^p with basis constant independent of n is the assertion that there exists a constant K_p such that

$$\| \sum_{k=0}^{l} c_k \phi_k \|_p \le K_p \| \sum_{k=0}^{n-1} c_k \phi_k \|_p, \quad l = 0, 1, \ldots, n-1, \tag{2.4}$$

for all scalars $c_0, c_1, \ldots c_{n-1}$. To see what is involved in the proof of (2.4), we consider the representations π, λ of $\widehat{\mathbb{Z}_n}, \mathbb{Z}_n$ respectively on l_n^2 given by

$$\pi(k) = \sum_{i=0}^{n-1} (\varepsilon^k)^i e_{i,i}, \quad \lambda(j) = \sum_{i=0}^{n-1} e_{i,(i+j) \bmod \mathrm{n}}, \quad k \in \widehat{\mathbb{Z}_n}, \quad j \in \mathbb{Z}_n$$

Here $\{e_{i,l}\}$ denotes the usual system of matrix units in $M_n(\mathbb{C})$. Note that the linear span of the system $\pi(k)$, $k \in \widehat{\mathbb{Z}_n}$ is precisely the image of $L^\infty(\mathbb{Z}_n)$, identified as acting by multiplication on $L_2(\mathbb{Z}_n)$. At the same time, the mapping

$$\sum_{k\in\mathbb{Z}_n} c_k\pi(k) \to \sum_{k\in\widehat{\mathbb{Z}_n}} c_k\lambda(k)$$

is an algebraic, trace-preserving $*$-isomorphism from $L^\infty(\mathbb{Z}_n)$ to $M_n(\mathbb{C})$. This implies that

$$\left\|\sum_{k=0}^{l} c_k\phi_k\right\|_{L_p(\mathbb{Z}_n)} = \left\|\sum_{k=0}^{l} c_k\pi(k)\right\|_{C_p} = \left\|\sum_{k=0}^{l} c_k\lambda(k)\right\|_{C_p}$$

for all $0 \le l \le n-1$. Here $\|\cdot\|_{C_p}$ denotes the usual Schatten p-norm on $M_n(\mathbb{C})$ given by

$$\|x\|_{C_p}^p = \mathrm{Tr}(|x|^p), \quad x \in M_n(\mathbb{C}).$$

Accordingly to establish (2.4), it suffices to show the existence of a constant K_p, independent of n such that

$$\left\|\sum_{k=0}^{l} c_k\lambda(k)\right\|_{C_p} \le K_p \left\|\sum_{k=0}^{n-1} c_k\lambda(k)\right\|_{C_p}, \quad 0 \le l \le n-1, \tag{2.5}$$

A simple proof of the estimate (2.5) may be based on the well-known theorem of Macaev: if $1 < p < \infty$, then there exists a constant K_p, independent of n such that

$$\|T(x)\|_{C_p} \le C_p\|x\|_{C_p}$$

where T is the operator of triangular truncation given by

$$T(\sum_{i,j=0}^{n-1} x_{ij}e_{i,j}) = \sum_{i<j} x_{ij}e_{i,j},$$

for all $x = \sum_{i,j=0}^{n-1} x_{ij}e_{i,j} \in M_n(\mathbb{C})$ (see [GK1,2]). In the above argument, the use of the operator of triangular truncation introduces ideas that are essentially non-commutative in order to prove an estimate in a setting that is commutative.

The general counterpart to the estimates (2.4), (2.5) given in the proof of Theorem 2(i) [DFPS] is based on the preceding ideas but implemented via the introduction of a generalised operator of triangular truncation. Boundedness of this generalised truncation operator is proved by applying a very general version of the Macaev theorem due to Zsido [Zs 80] (see also [DDPS]), which contains as a special case the classical Riesz projection theorem, and which further implies the fact that replexive non-commutative L_p-spaces have the UMD -property. A further essential ingredient in the proof of Theorem 2 is the application of general non-commutative Khintchine inequalities [LP 86], [LPP 91].

4. Non-commutative Examples

We will indicate here how the ideas of the preceding section yield an explicit Schauder basis in the non-commutative L_p-spaces associated with the hyperfinite von Neumann II_1 factor $\mathcal{R}$. In a certain sense, this factor may be regarded in a natural manner as a non-commutative extension of the commutative von Neumann algebra $L^\infty([0,1])$.

To set ideas, we consider a simple non-commutative example. We take $\mathbf{m} = (n, n, 0, 0, \dots)$ so that $G_{\mathbf{m}}$ may be identified with $\mathbb{Z}_n \times \mathbb{Z}_n$. We let $(\mathcal{M}, \tau)$ be $M_n(\mathbb{C})$, equipped with normalised trace tr. If $k \in \widehat{\mathbb{Z}_n}$, then the equality

$$\phi_k(s - j) = \overline{\langle j, k \rangle} \phi_k(s), \quad s \in \mathbb{Z}_n \tag{3.1}$$

identifies $\{\phi_k\}_{k \in \widehat{\mathbb{Z}_n}}$ as a Vilenkin system for the action of $\mathbb{Z}_n$ on $L^\infty(\mathbb{Z}_n)$ by right translation. If we now identify $L^\infty(\mathbb{Z}_n)$ as the commutative diagonal subalgebra in $M_n(\mathbb{C})$, then the relation (3.1) in matrix form becomes

$$\lambda(j)^* \pi(k) \lambda(j) = \overline{\langle j, k \rangle} \pi(k) \tag{3.2}$$

for all $j \in \mathbb{Z}_n, k \in \widehat{\mathbb{Z}_n}$. These identities are of course easily checked by direct calculation and are a special case of the (so-called) Weyl-Heisenberg relations. The equalities (3.2) identify the matrices $\{\pi(k) : k \in \widehat{\mathbb{Z}_n}\}$ as a system of eigenvectors for the action

$$x \to \lambda(j)^* x \lambda(j), \quad x \in M_n(\mathbb{C}).$$

A corresponding Vilenkin decomposition for this action is easily seen to be

$$(M_n(\mathbb{C}))_k = \operatorname{span}\{\pi(k)\lambda(m) : 0 \le m \le n-1\}.$$

The equalities given in (3.2) may be written equivalently in the form

$$\pi(k)\lambda(j)\pi(k)^* = \overline{\langle j, k \rangle} \lambda(j) \tag{3.3}$$

The equalities (3.3) identify the matrices $\{\lambda(j) : j \in \mathbb{Z}_n\}$ as a system of eigenvectors for the action

$$x \to \pi(k) x \pi(k)^*, \quad x \in M_n(\mathbb{C}).$$

This yields a corresponding Vilenkin decomposition

$$(M_n(\mathbb{C}))_j = \operatorname{span}\{\pi(l)\lambda(j) : 0 \le l \le n-1\}, \quad j \in \widehat{\mathbb{Z}_n}$$

If we now successively apply (3.2),(3.3), we obtain that

$$\pi(m)\lambda(l)^* \left(\pi(k)\lambda(j)\right) \lambda(l)\pi(m)^* = \overline{\langle l,k\rangle\langle m,j\rangle}\pi(k)\lambda(j) \tag{3.4}$$

for all $(l,m) \in \mathbb{Z}_n \times \mathbb{Z}_n$ and $(k,j) \in \widehat{\mathbb{Z}_n} \times \widehat{\mathbb{Z}_n} = \widehat{\mathbb{Z}_n \times \mathbb{Z}_n}$ The equalities (3.4) identify the matrices $\{\pi(k)\lambda(j) : (k,j) \in \widehat{\mathbb{Z}_n \times \mathbb{Z}_n}\}$ as a Vilenkin basis in $L^p(M_n(\mathbb{C}), \mathrm{tr})$ for the action

$$x \to \pi(m)\lambda(l)^* x \lambda(l)\pi(m)^*, \quad x \in M_n(\mathbb{C}).$$

We remark that span $\{\pi(k)\lambda(0) : 0 \le k \le n-1\}$ may be identified in the obvious way with $L_p(\mathbb{Z}_n)$. We summarise as follows.

Corollary 1 *For every $n \in \mathbb{N}$, the system $\{\pi(k)\lambda(j)\}_{(k,j)\in\widehat{\mathbb{Z}_n\times\mathbb{Z}_n}}$, taken in the reverse lexicographical order, forms a Schauder basis of $L_p(M_n(\mathbb{C}), \mathrm{tr})$ with basis constants independent of n.*

To proceed further, identify $L^\infty[0,1]$ with the space $L^\infty(G_{\mathbf{m}}, d\mathbf{t}_{\mathbf{m}})$ in the sense of measure spaces. We observe that the equality

$$L^\infty(G_{\mathbf{m}}, d\mathbf{t}_{\mathbf{m}}) = L^\infty(\mathbb{Z}_{m(0)}) \otimes L^\infty(\mathbb{Z}_{m(1)}) \otimes \cdots$$

holds in the sense of infinite product of measure spaces. The hyperfinite von Neumann II_1 factor $(\mathcal{R}, \tau)$ is given by the tensor product

$$(\mathcal{R}, \tau) = (M_{m(0)}(\mathbb{C}), tr) \otimes (M_{m(1)}(\mathbb{C}), tr) \otimes \cdots$$

with product trace τ given by

$$\tau(x_0 \otimes x_1 \otimes \cdots) = tr(x_0)tr(x_1)\cdots$$

Details of this construction may be found in [Sa]. Observe that $L^\infty(G_{\mathbf{m}}, d\mathbf{t}_{\mathbf{m}})$ may be identified with the commutative “diagonal ”subalgebra

$$(M^d_{m(0)}(\mathbb{C}), tr) \otimes (M^d_{m(1)}(\mathbb{C}), tr) \otimes \cdots$$

where $M^d_n(\mathbb{C})$ denotes the subalgebra of $M_n(\mathbb{C})$ consisting of all diagonal $n \times n$ matrices.

If $\mathbf{l} = (l_0, l_1, \dots)$, $\mathbf{n} = (n_0, n_1, \dots) \in \widehat{G_{\mathbf{m}}}$, we set

$$\pi(\mathbf{l}) = \pi(l_0) \otimes \pi(l_1) \otimes \dots, \quad \lambda(\mathbf{n}) = \lambda(n_0) \otimes \lambda(n_1) \otimes \dots$$

It may be shown that the mapping

$$x \to \pi(\mathbf{l})\lambda(\mathbf{n})^* x \lambda(\mathbf{n})\pi(\mathbf{l})^*$$

extends to an ergodic $G_{\mathbf{m}} \times G_{\mathbf{m}}$-flow on $\mathcal{R}$, with eigenspace $(L_p(\mathcal{R}, \tau))_{(\mathbf{k},\mathbf{j})}$ given by the one-dimensional span of the unitary operator $\pi(\mathbf{k})\lambda(\mathbf{j})$ for each $(\mathbf{k}, \mathbf{j}) \in \widehat{G_{\mathbf{m}} \times G_{\mathbf{m}}} = \widehat{G_{\mathbf{m}}} \times \widehat{G_{\mathbf{m}}}$. We obtain the following

Corollary 2 *The system of unitary operators* $\{\pi(\mathbf{k})\lambda(\mathbf{j})\}_{(\mathbf{k},\mathbf{j})\in\widehat{G_{\mathbf{m}}\times G_{\mathbf{m}}}}$ *taken in the reverse lexicographic ordering, is a Schauder basis in* $L_p(\mathcal{R},\tau),\ 1<p<\infty$.

We note finally that the basis exhibited in the preceding Corollary contains the classical Vilenkin systems via the "diagonal"subalgebra $\pi(\mathbf{k})\lambda(0), \mathbf{k}\in\widehat{G_{\mathbf{m}}}$.

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