SPECTRAL MULTIPLIERS FOR SELF-ADJOINT OPERATORS

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ABSTRACT. In this article, we give a survey of spectral multipliers and present (without proof) sharp Hörmander-type multiplier theorems for a self adjoint operator A under the assumption that A has Gaussian heat kernel bounds and satisfies appropriate estimates of the L^2 norm of the kernels of spectral multipliers. Our theorems imply several important, previously known results on spectral multipliers and give new results for sharp estimates for the critical exponent for the Riesz means summability.

1. INTRODUCTION

This paper contains discussion and survey of the topic of spectral multipliers and main results of [DOS] without giving proofs. Readers are referred to [DOS] for their proofs and more applications.

Suppose that A is a positive definite self-adjoint operator acting on $L^2(X)$, where X is a measure space. Such an operator admits a spectral decomposition $E_A(\lambda)$ and for any bounded Borel function $F\colon [0,\infty) \to \mathbf{C}$, we define the operator $F(A)$ by the formula

$$F(A) = \int_0^\infty F(\lambda)\, \mathrm{d}E_A(\lambda). \tag{1}$$

By the spectral theorem the operator $F(A)$ is continuous on $L^2(X)$. Spectral multiplier theorems investigate sufficient conditions on function F which ensure that the operator $F(A)$ extends to a bounded operator on L^q for some q, $1 \le q \le \infty$.

Spectral multiplier has been a very active topic of Harmonic analysis. Roughly, it is sufficient that F is differentiable to some order with appropriate bounds on its derivatives. The more information we know about the underlying space X and the operator A, the sharper multiplier results can be obtained, i.e. less derivatives on F are needed for $F(A)$ to be bounded. For example, when X is the Euclidean space $\mathbf{R}^d$ and A is the Laplacian $\Delta_d = \sum_{k=1}^d \partial_k^2$ a sufficient condition is that F possesses $[d/2]+1$ derivatives which satisfy certain size estimates where $[d/2]$ denotes the integral part of $d/2$. Recent results extend this to more general underlying spaces and more general self adjoint operators. For example, see, [He3, He2, DO, CS, Ale2] when A is an abstract positive self-adjoint operator which has heat kernel bounds (or finite propagation speed) and the underlying space X satisfies doubling volume property. (See Assumption 2.1).

Let us discuss two important examples of spectral multiplier theorems concerning group invariant Laplace operators acting on Lie groups of polynomial growth. Let $\mathbf{G}$ be a Lie group of polynomial growth and let $X_1, \dots, X_k$ be a system of left-invariant vector fields on a $\mathbf{G}$ satisfying the Hörmander condition. We define Laplace operator L acting on $L^2(\mathbf{G})$ by the formula

$$L = \sum_{i=1}^{k} X_i^2. \tag{2}$$

If $B(x,r)$ is a ball defined by the distance associated with system $X_1, \dots, X_k$ (see e.g. [VSC, §III.4]), then there exist natural numbers $d_0, d_\infty \geq 0$ such that $\mu(B(x,r)) \sim r^{d_0}$ for $r \leq 1$ and $\mu(B(x,r)) \sim r^{d_\infty}$ for $r > 1$ (see e.g. [VSC, §VIII.2]). We call $\mathbf{G}$ a homogeneous group if there exists a family of dilations on $\mathbf{G}$. A family of dilations on a Lie group $\mathbf{G}$ is a one-parameter group $(\tilde{\delta}_t)_{t>0}$ ($\tilde{\delta}_t \circ \tilde{\delta}_s = \tilde{\delta}_{ts}$) of automorphisms of $\mathbf{G}$ determined by

$$\tilde{\delta}_t Y_j = t^{d_j} Y_j, \tag{3}$$

where $Y_1, \dots, Y_l$ is a linear basis of Lie algebra of $\mathbf{G}$ and $d_j \geq 1$ for $1 \leq j \leq l$ (see [FS]). We say that an operator L defined by (2) is homogeneous if $\tilde{\delta}_t X_i = t X_i$ for $1 \leq i \leq k$. For a homogeneous Laplace operator $d_0 = d_\infty = \sum_{j=1}^{l} d_j$ (see [FS]).

Spectral multiplier theorems for homogeneous Laplace operators acting on homogeneous groups were investigated by Hulanicki and Stein [HS] (see also [FS, Theorem 6.25]), De Michele and Mauceri [dMM]. The following theorem was obtained independently by Christ [Ch2] and Mauceri and Meda [MM]. See also [Si3]. Its proof relies on heat kernel bounds, L^2 estimates from Plancherel theorems, translation and dilation invariant structures of homogeneous groups, Calderón Zygmund operator theory and interpolation theory.

Theorem 1. *Let L be a homogeneous operator defined by the formula (2) acting on a homogeneous group $\mathbf{G}$. Denote by $d = d_0 = d_\infty$ homogeneous dimension of the underlying group $\mathbf{G}$. Next suppose that $s > d/2$ and that $F\colon [0,\infty) \to \mathbf{C}$ is a bounded Borel function such that*

$$\sup_{t>0} \|\eta\, \delta_t F\|_{W_s^2} < \infty, \tag{4}$$

where $\delta_t F(\lambda) = F(t\lambda)$, $\|F\|_{W^p_s} = \|(I - \mathrm{d}^2/\mathrm{d}x^2)^{s/2}F\|_{L^p}$ *and* $\eta \in C^\infty_c(\mathbf{R}_+)$ *is a fixed function, not identically zero. Then* $F(L)$ *is of weak type* $(1,1)$ *and bounded on* L^q *when* $1 < q < \infty$.

Note that condition (4) is independent of the choice of $\eta \in C^\infty_c(\mathbf{R}_+)$.

The Hörmander multiplier theorem describes the Fourier multiplier on $\mathbf{R}^d$ (see [Hö1]). If we apply Theorem 1 to $\mathbf{R}^d$ we obtain a result equivalent to the Hörmander multiplier theorem restricted to radial Fourier multipliers. Therefore we call Theorem 1 the Hörmander-type multiplier theorem and condition (4) the Hörmander-type condition.

Theorem 1.1 is optimal for a general homogeneous group, see estimate 1.6. However, for specific groups such as Heisenberg and related groups, it is possible to obtain multiplier results where the number of derivatives needed is roughly half of the topological dimension n. Often the homogeneous dimension d is strictly greater than the topological dimension n, hence the number of derivatives needed could be less than $d/2$. See for example, [MS, He4, Du, CS].

In the setting of general Lie groups of polynomial growth spectral multipliers were investigated by Alexopoulos. Note that in this setting, the local dimension d_0 and dimension at infinity d_∞ are different in general and the group $\mathbf{G}$ does not have dilation invariants as in the case of a homogeneous group. Using finite propagation speed property, estimates on upper bounds of heat kernels and their space gradients, Alexopoulos proved the following (see [Ale1]).

Theorem 2. *Let* L *be a group invariant operator acting on a Lie group of polynomial growth defined by (2). Suppose that* $s > d/2 = \max(d_0, d_\infty)/2$ *and that* $F\colon [0,\infty) \to \mathbf{C}$ *is a bounded Borel function such that*

$$\sup_{t>0} \|\eta\, \delta_t F\|_{W^\infty_s} < \infty, \tag{5}$$

where $\delta_t F(\lambda) = F(t\lambda)$ *and* $\|F\|_{W^p_s} = \|(I - \mathrm{d}^2/\mathrm{d}x^2)^{s/2}F\|_{L^p}$. *Then* $F(L)$ *is of weak type* $(1,1)$ *and bounded on* L^q *when* $1 < q < \infty$.

Condition (5) is also independent of the choice of η. We note that Theorem 1.2 does not appear exactly the same but is essentially equivalent to the results of [Ale1].

In [He3] Hebisch extended Theorem 2 to a class of abstract operators acting on spaces satisfying "doubling condition" (see also [Ale2]). The order of differentiability in the Alexopoulos-Hebisch multiplier theorem is optimal. This means that for any $s < d/2$ we can find a function F such that F satisfies condition (5) but $F(A)$ is not of weak type $(1,1)$. Indeed, let A be a uniformly elliptic, self-adjoint second-order differential operator on $\mathbf{R}^d$, e.g. $A = \Delta_d$, where Δ_d is the standard Laplace operator. One can prove that

$$C_1(1+|\alpha|)^{d/2} \le \|A^{i\alpha}\|_{L^1\to L^{1,\infty}} \le C_2(1+|\alpha|)^{d/2} \tag{6}$$

(see [SW]). (See also [St1, pp. 52] and Christ [Ch2]). However, if we put $F_\alpha(\lambda) = |\lambda|^{i\alpha}$, then

$$C_1'(1+|\alpha|)^{s/2} \le \sup_{t>0} \|\eta\delta_t F_\alpha\|_{W^\infty_s} \le C_2'(1+|\alpha|)^{s/2}.$$

Therefore for any $s < d/2$ Theorem 2 does not hold.

If X is a manifold with exponential volume growth, i.e. $V(x,r) \le ce^{kr}$ and L is the Laplace-Beltrami operator, spectral multipliers was investigate by M. Taylor in [Tay] where it was shown that a sufficient condition is that F is holomorphic on a strip of width k for $F(\sqrt{L})$ to be bounded on L^p for $1 < p < \infty$. For more specific spaces such as certain Iwasawa AN groups, see [He5, CGHM] where it was shown that only a finite number of derivatives are required for $F(L)$ to be bounded on L^1 or to be of weak type (1,1). Note that the exponential volume growth is like the dimension $d = \infty$.

The theory of spectral multipliers is related to and motivated by the study of convergence of the Riesz means or convergence of other eigenfunction expansions of self-adjoint operators. To define the Riesz means of the operator A we put

$$\sigma_R^\alpha(\lambda) = \begin{cases} (1-\lambda/R)^\alpha & \text{for } \lambda \le R \\ 0 & \text{for } \lambda > R. \end{cases} \tag{7}$$

We then define the operator $\sigma_R^\alpha(A)$ using (1). We call $\sigma_R^\alpha(A)$ the Riesz or the Bochner-Riesz means of order α. The basic question in the theory of Riesz means is to establish the critical exponent for the continuity and convergence of the Riesz means. More precisely we want to study the optimal range of α for which the Riesz means $\sigma_R^\alpha(A)$ are uniformly bounded on $L^1(X)$ (or other $L^q(X)$ spaces).

Since the publication of Riesz paper [Ri] the summability of the Riesz means has been one of the most fundamental problems in Harmonic Analysis (see e.g. [St2, IX.2 and §IX.6B]). Despite the fact that the Riesz means have been extensively studied we do not have the full description of the optimal range of α even if we study only the space $L^1(X)$. On one hand we know that for the Laplace operator $\Delta_d = \sum_{k=1}^d \partial_k^2$ acting on $\mathbf{R}^d$ and the Laplace-Beltrami operator acting on compact d-dimensional Riemannian manifolds the critical exponent is equal $(d-1)/2$ (see [So1]). This means that Riesz means are uniformly continuous on $L^1(X)$ if and only if $\alpha > (d-1)/2$ (see also [ChS, Ta]). On the other hand, if we consider more general operators like e.g.

uniformly elliptic operators on R^d it is only known that Riesz means are uniformly continuous on $L^1(X)$ if $\alpha > d/2$ (see [He1]). One of the main points of our work is to investigate the summability of Riesz means for $d/2 \geq \alpha > (d-1)/2$.

The Alexopoulos-Hebisch multiplier theorem discussed above gives optimal value for the exponent $d/2$ of the number of derivatives needed in spectral multipliers, but it does not give the optimal range of the exponent α for the Riesz summability. Indeed, if $\|\sigma_1^\alpha\|_{W_s^\infty} < \infty$, then $\alpha \geq s$. However, $\|R_1^\alpha\|_{W_s^2} < \infty$ if and only if $\alpha > s - 1/2$. This means that in virtue of Theorem 2 one obtains uniform continuity of Riesz means on L^q for any $\alpha > d/2$ and for all $q \in (1, \infty)$, whereas Theorem 1 shows Riesz summability for $\alpha > (d-1)/2$ (see also [Ch2, pp. 74]). As we mentioned earlier $(d-1)/2$ is a critical index for Riesz summability for standard Laplace operator on R^d and Laplace-Beltrami operator on compact manifolds. To conclude we see that the optimal number of derivatives in multiplier theorems is $d/2$. However, in condition (5) we required $d/2$ derivatives in L^∞. In the Hörmander-type condition (4) we required $d/2$ derivatives in L^2. Note that functions $\eta\delta_t F$ are compactly supported so condition (5) is strictly stronger than (4).

We would like to investigate when it is possible to replace condition (5) in the Alexopoulos-Hebisch multiplier theorem by condition (4) from Theorem 1. As we investigate spectral multipliers in a general setting of abstract operators rather than in a specific setting of group invariant operators acting on Lie groups, we do not have certain estimates which are consequences of invariant structures of Lie groups. Also we only assume suitable bounds on heat kernels but not pointwise bounds on their space derivatives.

The subject of Bochner-Riesz means and spectral multipliers is so broad that it is impossible to provide comprehensive bibliography of it here. Hence we quote only papers directly related to our investigation and refer reader to [Ale1, Ch2, Ch1, ChS, CS, dMM, Du, He1, He3, Hö1, Hö3, HS, MM, SeSo, So1, So2, Si2, St1, St2, Ta] and their references.

2. Main results

In this section we first introduce some notation and describe the hypotheses of our operators and underlying spaces. We then state our main results.

Assumption 1. *Let X be an open subset of $\widetilde{X}$, where $\widetilde{X}$ is a topological space equipped with a Borel measure μ and a distance ρ. Let $B(x,r) = \{y \in \widetilde{X}, \rho(x,y) < r\}$ be the open ball (of $\widetilde{X}$) with centre at x and radius r. We suppose throughout that $\widetilde{X}$ satisfies the doubling* *property, i.e., there exists a constant C such that*

$$\mu(B(x,2r)) \le C\mu(B(x,r)) \ \forall x \in \widetilde{X}, \forall r > 0. \tag{8}$$

Note that (8) implies that there exist positive constants C and d such that

$$\mu(B(x,\gamma r)) \le C(1+\gamma)^d \mu(B(x,r)) \quad \forall \gamma > 0, x \in \widetilde{X}, r > 0. \tag{9}$$

In a sequel we always assume that (9) holds.

We state our results in terms of the value d in (9). Of course for any $d' \ge d$ (9) also holds. However, the smaller d the stronger multiplier theorem we will be able to obtain. Therefore we want to take d as small as possible. Note that in the case of the group of polynomial growth the smallest possible d in (9) is equal to $\max(d_0, d_\infty)$. Hence our notation is consistent with statements of Theorems 1 and 2.

Note that we do not assume that X satisfies doubling property. This poses certain difficulties which we overcome by using results of singular integral operators of [DM]. An example of such a space X is a domain of Euclidean space $\mathbf{R}^d$. If we do not assume any smoothness on its boundary, then doubling property fails in general.

Now we describe the notion of the kernel of the operator. Suppose that
$T\colon L^1(X,\mu) \to L^q(X,\mu)$ for $q > 1$. Then by $K_T(x,y)$ we denote the kernel of the operator T defined by the formula

$$\langle Tf_1, f_2 \rangle = \int_X Tf_1 \overline{f_2} \, \mathrm{d}\mu = \int_X K_T(x,y) f_1(y) \overline{f_2(x)} \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y). \tag{10}$$

for all $f_1, f_2 \in C_c(X)$. Note that

$$\|T\|_{L^1(X,\mu)\to L^q(X,\mu)} = \sup_{y \in X} \|K_T(\,\cdot\,,y)\|_{L^q(X,\mu)}.$$

Hence if $\|T\|_{L^1(X,\mu)\to L^q(X,\mu)} < \infty$, then its kernel K_T is a well defined measurable function. Vice versa, if $\sup_{y\in X} \|K_T(\,\cdot\,,y)\|_{L^q(X,\mu)} < \infty$, then K_T is a kernel of the bounded operator $T\colon L^1(X,\mu) \to L^q(X,\mu)$, even if $q = 1$.

Next we denote the weak type $(1,1)$ norm of an operator T on a measure space (X,μ) by $\|T\|_{L^1(X,\mu)\to L^{1,\infty}(X,\mu)} = \sup \lambda \ \mu(\{x \in X : |Tf(x)| > \lambda\})$, where the supremum is taken over $\lambda > 0$ and functions f with $L^1(X,\mu)$ norm less than one.

Assumption 2. *Let A be a self-adjoint positive definite operator. We suppose that the semigroup generated by A on L^2 has kernel*

$p_t(x,y) = K_{\exp(-tA)}(x,y)$ *which for all* $t>0$ *satisfies the following Gaussian upper bound*

$$|p_t(x,y)| \le C\mu(B(y,t))^{-1/m} \exp\Big(-b\frac{\rho(x,y)^{m/(m-1)}}{t^{1/(m-1)}}\Big) \tag{11}$$

where C, b *and* m *are positive constants and* $m \ge 2$.

Such estimates are typical Gaussian estimates for elliptic or subelliptic differential operators of order m (see e.g. [Da1, Ro, VSC]). We will call $p_t(x,y)$ the heat kernel associated with A. When order $m=2$, Gaussian estimates (2.4) is equivalent to finite propagation speed, see [Si1]. When $m \ne 2$, we can have (2.4) but finite propagation speed property does not hold.

In our following main results, we suppose that Assumptions 1 and 2 hold. The values d and m always refer to (9) and (11).

Theorem 3. *Suppose that* $s > d/2$ *and assume that for any* $R>0$ *and all Borel functions* F *such that* $\operatorname{supp} F \subseteq [0,R]$

$$\int_X |K_{F(\sqrt[m]{A})}(x,y)|^2 \, d\mu(x) \le C\mu(B(y,R^{-1}))^{-1} \|\delta_R F\|_{L^p}^2 \tag{12}$$

for some $p \in [2,\infty]$. *Then for any Borel bounded function* F *such that* $\sup_{t>0} \|\eta\delta_t F\|_{W^p_s} < \infty$ *the operator* $F(A)$ *is of weak type* $(1,1)$ *and is bounded on* $L^q(X)$ *for all* $1<q<\infty$. *In addition*

$$\|F(A)\|_{L^1(X,\mu)\to L^{1,\infty}(X,\mu)} \le C_s\Big(\sup_{t>0}\|\eta\delta_t F\|_{W^p_s} + |F(0)|\Big). \tag{13}$$

Note that if (12) holds for $p<\infty$, then the pointwise spectrum of A is empty. Indeed, for all $p<\infty$ and all $y\in X$

(14)

$$0 = C\|\chi_{\{1/2\}}\|_{L^p} = C\|\delta_{2a}\chi_{\{a\}}\|_{L^p} \ge \mu(B(y,\frac{1}{2a}))\|K_{\chi_{\{a\}}(\sqrt[m]{A})}(\,\cdot\,,y)\|_{L^2(X,\mu)}$$

so $\chi_{\{a\}}(\sqrt[m]{A}) = 0$. Hence for elliptic operators on compact manifolds, (12) cannot be true for any $p<\infty$. To be able to study these operators as well we introduce some variation of assumption (12). Following [CS] for a Borel function F such that $\operatorname{supp} F \subseteq [-1,2]$ we define the norm $\|F\|_{N,p}$ by the formula

$$\|F\|_{N,p} = \Big(\frac{1}{N}\sum_{l=1-N}^{2N} \sup_{\lambda\in[\frac{l-1}{N},\frac{l}{N})} |F(\lambda)|^p\Big)^{1/p},$$

where $p\in[1,\infty)$ and $N\in\mathbf{Z}_+$. For $p=\infty$ we put $\|F\|_{N,\infty} = \|F\|_{L^\infty}$. It is obvious that $\|F\|_{N,p}$ increases monotonically in p. The next theorem is a variation of Theorem 3. This variation can be used in the case of operators with nonempty pointwise spectrum (compare [CS, Theorem 3.6]).

Theorem 4. *Suppose that κ is a fixed natural number, $s > d/2$ and that for any $N \in \mathbf{Z}_+$ and for all Borel functions F such that* $\operatorname{supp} F \subseteq [-1, N+1]$

$$\int_X |K_{F(\sqrt[m]{A})}(\,\cdot\,, y)|^2 \, \mathrm{d}\mu(x) \le C\mu(B(y, 1/N))^{-1} \|\delta_N F\|^2_{N^\kappa, p} \tag{15}$$

for some $p \ge 2$. In addition we assume that for any $\varepsilon > 0$ there exists a constant C_ε such that for all $N \in Z_+$ and all Borel functions F such that $\operatorname{supp} F \subseteq [-1, N+1]$

$$\|F(\sqrt[m]{A})\|^2_{L^1(X,\mu)\to L^1(X,\mu)} \le C_\varepsilon N^{\kappa d + \varepsilon} \|\delta_N F\|^2_{N^\kappa, p}. \tag{16}$$

Then for any Borel bounded function F such that $\sup_{t>1} \|\eta \delta_t F\|_{W^p_s} < \infty$ *the operator $F(A)$ is of weak type $(1,1)$ and is bounded on $L^q(X)$ for all $q \in (1, \infty)$. In addition*

$$\|F(A)\|_{L^1(X,\mu)\to L^{1,\infty}(X,\mu)} \le C_s \Big(\sup_{t>1} \|\eta \delta_t F\|_{W^p_s} + \|F\|_{L^\infty} \Big). \tag{17}$$

Remarks 1. It is straightforward that (12) always holds with $p = \infty$ as a consequence of spectral theory. This means that Alexopoulos' multiplier theorem i.e. Theorem 2 follows from Theorem 3. Theorem 1 also follows from Theorem 3. Indeed, it is easy to check that for homogeneous operators (12) holds for $p = 2$ (see Section 6 [DOS] or [Ch2, Proposition 3]).

2. The main point of our theorems is that if one can obtain (12) or (15) then one can prove stronger multiplier results. If one shows (12) or (15) for $p = 2$, then this implies the sharp Hörmander-type multiplier result. Actually we believe that to obtain any sharp spectral multiplier theorem one has to investigate conditions of the same type as (12) or (15), i.e. conditions which allow us to estimate the norm $\|K_{F(\sqrt[m]{A})}(\,\cdot\,, y)\|_{L^2(X,\mu)}$ in terms of some kind of L^p norm of the function F.

3. We call hypotheses (12) or (15) the Plancherel estimates or the Plancherel conditions. In the proof of Theorems 3 and 4 one does not have to assume that $p \ge 2$ in estimates (12) or (15). However (12) or (15) for $p < 2$ would imply Riesz summability for $\alpha < (d-1)/2$ and we do not expect such a situation.

Note that (12) is weaker than (15) and we need additional hypothesis (16) in this case. However, in practice once (15) is proved, (16) is usually easy to check and we can often put $\varepsilon = 0$.

4. We conclude this paper with a theorem on Riesz summability for $d/2 \geq \alpha > (d-1)/2$. Theorem 3 with $p = 2$ implies Riesz summability for all $\alpha > (d-1)/2$ and that in addition it seems that Theorem 3 with $p = 2$ is essentially stronger than sharp Riesz summability. However, one can obtain only weak type $(1,1)$ estimates in virtue of Theorem 3 and formally Theorem 3 does not imply continuity and convergence of Riesz means on $L^1(X,\mu)$. However, Theorem 3 and 4 can be modified to prove that uniform continuity of Riesz means of order greater than $(d/2-1/p)$ on all spaces $L^q(X,\mu)$ for $q \in [1,\infty]$. We claim the following Theorem.

Theorem 5. *Suppose that operator A satisfies condition (12), or (15) and (16) for some $p \in [2,\infty]$. Then for any $\alpha > d/2 - 1/p$ and $q \in [1,\infty]$*

$$\sup_{R>0} \|\sigma_R^\alpha(A)\|_{L^q(X,\mu)\to L^q(X,\mu)} \leq C < \infty.$$

Hence for any $q \in [1,\infty)$ and $f \in L^q(X,\mu)$

$$\lim_{R\to\infty} \|\sigma_R^\alpha(A)f - f\|_{L^q(X,\mu)\to L^q(X,\mu)} = 0,$$

where σ_R^α is defined by (7).

For the proofs of our Theorems, we refer reader to [DOS]. Here let us only mention that the proofs of Theorems 5 and 3 are less complicated than most of earlier spectral multiplier results. Our strategy is to use the complex time heat kernel bounds (see [Da1, DO]) to show $W^2_{(d+1)/2}$ functional calculus for the considered operator A. Then we use Mauceri-Meda interpolation trick (see [MM]) and our Plancherel type assumption (12) to obtain $W^p_{d/2+\varepsilon}$ functional calculus. This is enough to show Riesz summability (i.e. Theorem 5. To prove Theorem 3 we need also some Calderón-Zygmund singular integral techniques. However in contrast to the standard Calderón-Zygmund singular integral estimates we do not use estimates for the gradient of the kernel of singular integral operators. Instead of that we follow the ideas of [DM, He3, He2].

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