FROM XY TO ADE

DAVID E. EVANS

ABSTRACT. We survey the role of non-commutative operator algebras in statistical mechanics and the relation between the classification of modular invariant partition functions in conformal field theories and braided subfactors.

As exposed in the treatises of Bratteli and Robinson [21] non-commutative operator algebras have a long tradition of providing a framework for understanding quantum statistical mechanics. For example, the one-dimensional XY-model is studied in the Pauli or Fermion algebra $\bigotimes_{\mathbb{Z}} M_2$ with the Hamiltonian

$$H = -\sum_{j\in\mathbb{Z}}\{(1+\gamma)\sigma_x^j\sigma_x^{j+1} + (1-\gamma)\sigma_y^j\sigma_y^{j+1} + 2\lambda\sigma_z^j\}.$$

Here $\sigma_\alpha^j, \alpha = x, y, z$ are the usual Pauli matrices placed at the jth site of the tensor product. Typically one studies time evolution on the Pauli algebra via the one-parameter group of *-automorphisms $\alpha_t = e^{iHt}(\cdot)e^{-iHt}$ suitably defined. In such lattice models one is interested in determining the set of equilibrium states, using the Gibbs conditions, KMS condition or a variational principle, minimizing the thermodynamic quantity (energy - temperature.entropy), as well as the return to equilibrium of locally perturbed models. Robinson played a seminal role in this theory, which is described in detail in [21]. Amongst other things, this led to the development of the theory of derivations on operator algebras, the infinitesimal generators of time evolution, which is still relevant today with the Powers-Sakai conjecture [58] a particularly challenging open problem. This led Robinson to working on the infinitesimal generators of (completely) positive semigroups on operator algebras and subsequently his most recent work on heat kernel methods. The Powers-Sakai conjecture asks whether every one-parameter dynamics on a UHF algebra (an infinite tensor product of matrix algebras) or more generally on a simple AF algebra (an inductive limit of finite dimensional algebras) is approximately inner, obtained as a limit of inner one-parameter groups as in the above XY-example. Kishimoto [49] has recently shown that a stronger form of the conjecture is false, namely the core problem of whether the infinitesimal generator has an AF-core in a suitable sense. If α is a strongly continuous one-parameter group of *-automorphisms of a simple AF algebra, where the domain of the generator is AF, then α is approximately inner. However, [49] constructs on any non type I simple AF C^*-algebra examples of approximately inner one-parameter groups of *-automorphisms where the domain of the generator is not AF. These can be regarded as one-parameter continuous analogues of the exotic examples of compact group actions on AF algebras whose fixed points are not AF (first shown by Blackadar [5] for $\mathbb{Z}/2$ on the Pauli algebra, and latter by Bratteli et al [18] for finite groups and Evans and Kishimoto [33] for compact groups).

Returning to our original starting point of this paper, the XY-model, notice that it degenerates at certain values of (λ, γ), namely at $(0, \pm 1)$, to the Ising nearest neighbour model. This is a classical Hamiltonian, and it would therefore appear to be artificial to study it via a non-commutative framework, the Pauli algebra. Nevertheless, there is a natural role for non-commutative operator algebras in the study of such classical statistical mechanical models which is the point of this present survey.

This connection begins with the transfer matrix method. Let us take a two dimensional nearest neighbour Ising model on a square lattice $\mathbb{Z}^2$ with Hamiltonian

$$H = -\sum_{\alpha,\beta \ nn} J\sigma^\alpha \sigma^\beta$$

with the summation over the vertices or sites α, β on $\mathbb{Z}^2$ which are nearest neighbours (nn). We switch from one to two dimensions because the one dimensional version does not have a phase transition at a non zero temperature. At each site α or vertex point of the lattice we have a variable, a spin or magnetization σ^α with either a positive or negative orientation or value represented by $+1$ or -1. Then a state $\sigma = (\sigma^\alpha)$ of the Ising model is a distribution of pluses and minuses over the square lattice $L = \mathbb{Z}^2$, so any configuration is represented by a point in configuration space the compact Hausdorff space $P = \{\pm 1\}^L$. Thus the natural home to study this Ising model is the space $C(P) = \bigotimes_{\mathbb{Z}^2} \mathbb{C}^2$

the commutative C^*-algebra of all continuous functions on the compact configuration space. At each inverse temperature β we may be interested in the simplex K_β of equilibrium states, given say by solutions to the equations of Dobrushin, Lanford and Ruelle [30][50] or the variational principle: minimize(energy - temperature.entropy). In the algebraic approach, one uses the transfer matrix formalism to transform the setting to that of a one dimensional quantum model, represented by a non commutative "one dimensional" C^*-algebra and time evolution α_t. The transfer matrix T is obtained for the partition function of a strip of finite length M and width length one. With boundary conditions ξ, η along the two lengths the corresponding partition function $T_{\xi\eta}$ defines us the transfer matrix T. The partition function Z of a finite rectangular lattice of length M and width N is then obtained by multiplying the strip partition functions, namely transfer matrix entries and summing over internal edges. For periodic boundary conditions we obtain

$$Z = \sum \exp(-\beta H(\sigma)) = \sum T_{\xi_1\xi_2} T_{\xi_2\xi_3} \ldots T_{\xi_N\xi_1} = \text{trace } T^N. \tag{1}$$

In this way we move from the commutative algebra $C(P) = \bigotimes_{\mathbb{Z}^2} \mathbb{C}^2$ to the non-commutative Pauli algebra $\mathcal{A} = \bigotimes_{\mathbb{Z}} M_2$ where the local transfer matrices T generate the even part $\mathcal{A}_+$. Time evolution can be formally written as $\alpha_t = T^{it}(\cdot)T^{-it}$, i.e. we consider $T = e^{-H}$ where H is now a quantum Hamiltonian which is no longer a (one dimensional) Ising Hamiltonian. Spatial translation by $\mathbb{Z}^2$ in the classical model $\bigotimes_{\mathbb{Z}^2} \mathbb{C}^2$ corresponds to spatial translation in $\mathcal{A}^P = \bigotimes_{\mathbb{Z}} M_2$ together with an evolution $\{T^n(\cdot)T^{-n} : n \in \mathbb{Z}\}$ in the orthogonal transfer direction.

For each inverse temperature β one looks for a map $F \to F_\beta$ from (local) classical observables in $C(P)$ to the quantum algebra $\mathcal{A}$, and a map $\mu \to \varphi_\mu$ from states on $C(P)$ (or measures on P) to linear functionals on the local observables in $\mathcal{A}^P$ such that one can recover the classical expectation values or correlation functions from a knowledge of the quantum ones alone: $\mu(F) = \varphi_\mu(F_\beta)$. Fixing some boundary conditions, then for each inverse temperature β, let φ_β denote the corresponding state on $\mathcal{A}$. [In general positivity of φ_μ is not automatic but follows from reflection positivity of μ]. Then if β_c denotes the inverse critical temperature of Onsager, there exist automorphisms

$\{\nu_\beta : \beta \neq \beta_c\}$ of $\mathcal{A}$ [34] which do not depend on boundary conditions, and real analytic in $\beta \neq \beta_c$ such that

$$\varphi_\beta = \begin{cases} \varphi_\infty \circ \nu_\beta & \beta > \beta_c \\ \varphi_0 \circ \nu_\beta & \beta < \beta_c. \end{cases}$$

Here $\varphi_0 = \bigotimes_{\mathbb{Z}} \omega_\Omega$, where $\Omega = \binom{1}{1}\sqrt{2}$ is the disordered state, and for + or - boundary conditions, $\varphi_\infty^\pm = \bigotimes_{\mathbb{Z}} \omega_{\Omega^\pm}$ where $\Omega^+ = \binom{1}{0}, \Omega^- = \binom{0}{1}$ respectively and $\varphi_\infty = (\varphi_\infty^+ + \varphi_\infty^-)/2$ for free or periodic boundary conditions. Thus with free or periodic boundary conditions, we conclude that φ_β is pure for $0 \leq \beta < \beta_c$ (also for $\beta = \beta_c$ by different methods [2]) and is a mixture of two inequivalent pure states for $\varphi_\beta^\pm$ for $\beta > \beta_c$. If $\langle\ \rangle^\pm$ denote the classical states corresponding to + and - boundary conditions respectively, then we can deduce that $\langle F \rangle_\beta^\pm$ is real analytic in $\beta > \beta_c$ when F is a local classical observable, as $\langle F \rangle_\beta^\pm = \varphi_\infty^\pm \nu_\beta(F_\beta)$ using analyticity of ν_β. A dynamical system α_t on $\mathcal{A}^P$ is formally given as $\alpha_t = T^{it}(\cdot)T^{-it}$ which has a unique ground state for $\beta < \beta_c$ and two extremal ground states $\varphi_\beta^\pm$ for $\beta > \beta_c$.

The Ising model can be generalized to the possibility of having more than two values or spins possible at any lattice site, and moreover some constraints or rules to determine allowable configurations. A particular value at one site may force only restricted choices at nearest neighbours. This would be achieved by distributing values of a fixed graph G at sites of the lattice L in such a way that if α and β are nearest neighbours in L, then the corresponding values σ_α and σ_β are joined in the graph. The state space P can be defined for any graph, but if G contains some multiple edges, we consider distributions of edges of G on edges of L. For the Dynkin diagram A_3 with vertices labelled by $\{\bullet, \pm\}$ and square lattice L one obtains two copies of the Ising model as in Fig. 1 by placing the frozen spin $\bullet$ on the even or odd sublattices of L. This graph may be generalised to the Dynkin diagrams of Fig. 1 for the models of Andrews, Baxter and Forrester [1]. These in turn can be generalised by considering the Weyl alcove $\mathcal{A}^{(n,k)}$ of the level k integrable representations of the Kač-Moody algebra $SU(n)^\wedge$. Boltzmann weights associated to a local configuration around minimal squares of the lattice can be chosen to satisfy the integrable Yang-Baxter equation [25].

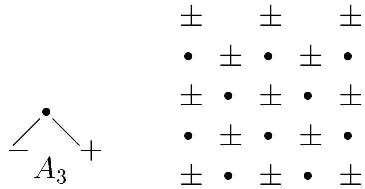

FIGURE 1. Ising model: Dynkin diagram A_3 and configuration space

The centre of $SU(n)$, the abelian $\mathbb{Z}/n$, acts on $\mathcal{A}^{(n,k)}$, e.g. $\mathbb{Z}/2$ on the Dynkin diagram A_{k+1} by a flip $i \to k-i$ which may or may not have a fixed point depending on the parity of k. The interesting case is when there is a fixed point. In any case, the Boltzmann weights are preserved under the symmetry, and yield new Boltzmann weights on the orbifold graphs $\mathcal{A}^{(n,k)}/(\mathbb{Z}/p)$, whenever p divides n, satisfying the integrable Yang Baxter equation [28] [35]. For example, when $n = 2, k = 2m$ we blow up the fixed point m to a pair (a copy of $\mathbb{Z}/2$) and replace each distinct pair $i, k-i$ $(i \neq m)$ interchanged by the symmetry with a singleton yielding the graph D_{m+2}. The case A_3 is self dual in that $A_3/(\mathbb{Z}/2) = A_3$. Nevertheless, the situation here is not entirely trivial. This is Kramers-Wannier high temperature–low temperature duality. This duality replaces the Boltzmann weights at a temperature t with ones at dual temperature t^*. Again the fixed point of the symmetry $t \to t^*$ is what provides the interesting structure — at the critical temperature t_c of Onsager.

We have mentioned the phenomena of AF algebras with non-AF fixed point algebras under symmetries. Such examples were first found using similar orbifold constructions. As a continuous version of the flip on a Dynkin diagram which yields symmetries on AF algebras, consider instead the flip on the interval around its midpoint or a flip on a circle around an axis in its plane through its centre. The orbifold space is best described by taking the cross product. For a pair of points interchanged by the symmetry, the local crossed product is simply a two by two matrix algebra. The diagonal elements represent the continuous functions on the pair, and the off-diagonal elements come from the transition between the two points. Each fixed point is replaced by a pair arising from the transitions only generating a copy of $\mathbb{C}^2$ as the continuous functions on the group (dual). Thus gluing together, we represent the action of the flip on the interval $[0,1]$ by the C^*-algebra of continuous functions on a half-interval $[0,1/2]$, which on the half-open half-interval $[0,1/2)$ (the part of the interval which has a non-trivial orbit) are two by two matrix valued but at other end point $1/2$ are diagonal. So the spectrum of this algebra is a continuous version of the D-Dynkin diagram. Topologically it is an interval with two non Hausdorff points at one end. The analogous action of the flip on the two torus which conjugates each variable has four fixed points, yielding a sphere with four singular points. The corresponding cross product is the space of two by two matrix valued functions on the sphere which are diagonal at four distinguished points. Replacing the two torus by a non-commutative torus generated by two unitaries U and V satisfying the commutation relation $VU = qUV$ where $q = \exp 2\pi i\theta$ we obtain a non commutative toroidal orbifold when taking the symmetry which inverts the generators U and V. It is Morita equivalent to the algebra of a singular flow on a sphere obtained as the quotient of the Kronecker flow on the torus as illustrated in [36], page 137 or http://www.cf.ac.uk/maths/opalg/ncto/. Remarkably these algebras are AF (when θ is irrational)([20][19] or [32],[65]) although the corresponding irrational rotation algebras and algebras of the Kronecker flow are not. The non-commutative torus has a representation on $L^2(\mathbb{T}^2)$ where U and V are represented as multiplication and translation operators. In this representation, or at least if one takes the Fourier transform, the Hamiltonian $H = U + U^{-1} + \lambda(V + V^{-1})$ are the Mathieu or discrete Schrödinger operators with almost periodic potentials. The natural home to study these operators is the fixed point algebra under our flip because when θ is irrational then $U+U^{-1}$ and $V+V^{-1}$ generate the fixed point algebra. It is still a tantalizing mystery as to whether there is a relation between the AF property of the fixed point algebras, a strong form of non-commutative disconnectedness, and the Cantor spectra of such almost Mathieu operators - which are at least known to be Cantor for generic coupling constant λ and rotation number θ. Symmetries on such algebras, where there are underlying fixed points can produce algebras with totally different properties. Similarly, symmetries on subfactors, statistical mechanical models, conformal field theories can produce totally different subfactors etc from what one started with.

From a lattice model one may obtain a field theory by taking a continuum or scaling limit, letting the lattice spacing go to zero whilst simultaneously approaching the critical temperature. As the scale or correlation length becomes infinite, one obtains a scale invariant or conformal invariant theory. Belavin et al [4] suggested that the scale invariance at a critical point is enhanced to conformal invariance. One of the invariants of the conformal field theory is the central charge a multiplier in projective representations L_m of the vector fields $-z^{m+1}d/dz$ on the circle, the Virasoro algebra. However the central charge can already raise its head in the statistical mechanical model. Going back to the partition function Z of Eq. (1) the free energy $f = -\log Z/NM$ is independent of boundary conditions as $N, M \to \infty$. However the asymptotics depend on boundary conditions; if $1 << N << M$, then $Z \approx \exp(-NMf + M\pi c/N6)$, where c is the central charge. (See e.g. [26] (or [36], Chapter 8) for explicit computation in the case of the Ising model).

Let us however proceed to the conformal field theory at criticality. It is argued on physical grounds that the partition function $Z(\tau)$ in a conformal field theory on the torus should be invariant under reparametrization of the torus by $SL(2,\mathbb{Z})$: $Z(\tau) = Z((a\tau+b)/(c\tau+d))$ [23]. In the string theory formulation, modular invariance is essentially built into the definition of the partition function (although Nahm [53] has argued the case for modular invariance in terms of the chiral algebra and its representations rather than a functional integral setting). In the transfer matrix picture of the statistical mechanical picture, we wrote the partition function as an average over $e^{-\beta H}$, where H is the Hamiltonian. The Hamiltonian is now $L_0 + \bar{L}_0 - c/12$ where $L_0, \bar{L}_0$ are commuting generators of rotation groups, c the central charge and the shift by $c/24$ comes from mapping the Virasoro algebra on the plane to a cylinder. We also have a momentum $P = L_0 - \bar{L}_0$ describing evolution along the closed string, so taking both evolutions into account we first compute

$$Z(\tau) = \mathrm{tr}\ e^{-\beta H} e^{i\eta P} = \mathrm{tr}\ e^{2\pi i\tau(L_0 - c/24)}\ e^{-2\pi i\bar{\tau}(\bar{L}_0 - c/24)}.$$

Here $2\pi i\tau = -\beta + i\eta$ parametrizes the metric of the torus, and we then have to average over τ. If we choose one τ from each orbit under the action of $SL(2,\mathbb{Z})$ and integrate we implicitly assume that $Z(\tau)$

is modular invariant. In our $SU(n)$ setting the Hilbert space decomposes according to the associated loop group representations. The loop group LG is the group of smooth maps from S^1 into a compact Lie group G under pointwise multiplication. We are interested in projective representations of $LG \rtimes Rot(S^1)$ where $Rot(S^1)$ is the rotation group, which are highest weight representations in that the generator L_0 of the rotation group is bounded below. Such representations are called positive energy representations and are classified by irreducible representations of G (by restriction to the constant loops) and a level k describing the multiplier in the projective representation. For unitary irreducible positive energy representations, the possibilities are severely restricted. Indeed k must be integral and for a given value of the level, there are only a finite number of admissible (vacuum vector) irreducible representations of G. For $G = SU(n)$ the admissible ones are precisely the vertices of $\mathcal{A}^{(n,k)}$, the same labeling set as used to construct our statistical mechanical model.

The partition function then decomposes as

$$Z(\tau) = \sum_{\lambda\mu} Z_{\lambda,\mu}\chi_\lambda(\tau)\chi_\mu(\tau)^*$$

where

$$Z_{\lambda\mu} = 0, 1, 2, ..., Z_{00} = 1 \tag{2}$$

and characters $\chi_\lambda(\tau) = \mathrm{tr}_\lambda e^{2\pi i\tau(L_0 - c/24)}$, $\mathrm{Im}\ \tau > 0$.

Here the label 0 refers to the vacuum representation, and the condition $Z_{00} = 1$ refers to the physical concept of uniqueness of the vacuum state. The matrix Z arising in this way is called a modular invariant mass matrix. (More precisely, for current algebras the characters depend also on variables other than τ, corresponding to Cartan subalgebra generators which are omitted here for simplicity. These extra variables mean we are dealing with $SL(2, \mathbb{Z})$ rather than $PSL(2, \mathbb{Z})$). From the canonical generators

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \text{ and } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

of $SL(2; \mathbb{Z})$ we obtain the unitary Kač-Petersen matrices $S = [S_{\lambda\mu}]$, $T = [T_{\lambda\mu}]$ transforming characters, where S is symmetric as well as $S_{\lambda 0} \geq$

$S_{00} > 0$ and T is diagonal:

$$\chi_\lambda(-1/\tau) = \sum_\mu S_{\lambda\mu}\chi_\mu(\tau), \;\; \chi_\lambda(\tau+1) = \sum_\mu T_{\lambda\mu}\chi_\mu(\tau).$$

Then the classification of modular invariant partition functions can be reformulated as a matrix problem. Find all matrices Z subject to Eq. (2) commuting with S and T. This is a rather concrete problem. For $SU(2)$ at level k, $SU(2)_k$, the admissible weights are the spins $\lambda = 0, 1, .., k$ and the Kač-Peterson matrices are given explicitly as

$$S_{\lambda\mu} = \sqrt{\frac{2}{k+2}}\ \sin\frac{\pi(\lambda+1)(\mu+1)}{k+2}$$

$$T_{\lambda\mu} = \delta_{\lambda\mu}\ \exp\left(\pi i\frac{(\lambda+1)^2}{2k+4}\ -\pi\frac{i}{4}\right)$$

with $\lambda, \mu = 0, 1, ..., k$, and the characters as

$$\chi_\lambda(q) = \frac{q^{(\lambda+1)^2/4(k+2)}}{\eta(q)^3}\sum_{n\in\mathbb{Z}}(2n(k+2)+\lambda+1)q^n(n(k+2)+\lambda+1)$$

if $q = e^{2\pi i\tau}$, and the Dedekind function $\eta(q) = q^{1/24}\prod_{n=1}^\infty(1-q^n)$.

For the Ising model, the characters are (in the notation of Fig. 1),

$$\chi_\bullet = [\vartheta_2/2\eta]^{3/2}, \chi_\pm = ([\vartheta_3/\eta]^{3/2} \pm [\vartheta_4/n]^{3/2})/2$$

with the ϑ-functions:

$$\sqrt{\vartheta_3/\eta} = q^{-1/48}\ \prod_{n=0}^\infty\left(1+q^{n+1/2}\right)$$

$$\sqrt{\vartheta_4/\eta} = q^{-1/48}\ \prod_{n=0}^\infty\left(1-q^{n+1/2}\right)$$

$$\sqrt{\vartheta_2/\eta} =\ q^{1/24}\prod_{n=1}^\infty(1+q^n).$$

Here the Kač-Petersen matrices are simply

$$S = \tfrac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}, T = e^{i\pi/24}\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\pi 3/8} & 0 \\ 0 & 0 & e^{i\pi} \end{pmatrix}$$

so that there is only one modular invariant the diagonal mass matrix or:

$$Z = \left(|\vartheta_2|^3 + |\vartheta_3|^3 + |\vartheta_4|^3\right)/2|\eta|^3.$$

Whilst the mass matrix is trivial, the partition function itself has some structure. The following is also a modular invariant particular function

$$Z = (|\vartheta_2| + |\vartheta_3| + |\vartheta_4|)/2|\eta|$$

this time for the coset model $su(2)_1 \bigoplus su(2)_1/su(2)_2$, which also exhibits Ising fusion rules (as does $(E_8)_1 \oplus (E_8)_1/(E_8)_2$) and $so(5)_1$).

At first sight, it might appear that generally there may be an infinite number of solutions to this modular invariant problem. However, there is a following estimate [12]: $Z_{\lambda\mu} \leq d_\lambda d_\mu$ which is a strengthening of the inequality of Gannon [42]: $\sum Z_{\lambda\mu} \leq 1/S_{00}^2$ if $d_\lambda = S_{0\lambda}/S_{00}$. Thus since $Z_{\lambda\mu}$ is positive and integral there are at most finitely many solutions, for a fixed representation of $SL(2,\mathbb{Z})$. In the case of $SU(2)$, there are at most three solutions for a fixed level k. This is the ADE classification of Capelli, Itzykson and Zuber [22]. A Dynkin diagram is associated to each invariant through the identification of diagonal terms of the invariant $\{\lambda : Z_{\lambda\lambda} \neq 0\} = \mathcal{I}$ with eigenvalues $\{S_{f\lambda}/S_{00} : \lambda \in \mathcal{I}\}$ of the corresponding Dynkin diagram if $f = 1$ the fundamental representation of $SU(2)$. The A refers to the diagonal invariant, D to orbifold invariants and E to the three E_6, E_7, E_8 exceptional invariants. For $SU(3)$ there is an anologous ADE classification due to Gannon [43]; di Francesco and Zuber [28] sought to show systematically the existence of graphs with spectra matching the modular invariant, give a meaning to these graphs themselves and compute them in a number of examples.

As we have said there are at most finitely many solutions to the modular invariant conditions. There is always one solution the trivial diagonal invariant:

$$\sum_{\lambda\in\mathcal{A}} |\chi_\lambda|^2$$

where the corresponding mass matrix is diagonal $Z_{\mu\lambda} = \delta_{\mu\lambda}$. In some sense, [52] [29] [11] 'every' modular invariant is diagonal if looked at properly. If we can extend the $\mathcal{A}$ system to a $\mathcal{B}$ system so that the characters decompose

$$\chi_\tau = \sum_{\lambda\in\mathcal{A}} b_{\tau\lambda}\ \chi_\lambda\ , \tau\ \in\ \mathcal{B}$$

according to some branching rules, then the diagonal $\mathcal{B}$-modular invariant will give an $\mathcal{A}$-modular invariant

$$\sum_{\tau\in\mathcal{B}}|\chi_\tau|^2 = \sum_{\tau\in\mathcal{B}}|\sum_{\lambda\in\mathcal{A}} b_{\tau\lambda}\chi_\lambda|^2.$$

In some sense, every modular invariant should look like this or with a possible twist $\Sigma_\tau\chi_\tau\chi^*_{w(\tau)}$, for a permutation w of the extended fusion rules, preserving the vacuum. The problem in general is then to find such extensions. When there is no twist present we have what are sometimes called type I invariants:

$$Z_{\lambda\mu} \; = \; \sum_\tau b_{\tau\lambda,} b_{\tau\mu}.$$

These are automatically symmetric: $Z_{\lambda\mu} = Z_{\mu\lambda}$. In the presence of a non-trivial twist, we have the type II invariants

$$Z_{\lambda\mu} \; = \; \sum_\tau b_{\tau\lambda} b_{w(\tau)\mu}.$$

These are not necessarily symmetric, but at least there is symmetric vacuum coupling $Z_{0\lambda} \; = \; Z_{\lambda 0}$. Not every modular invariant is even symmetric in this sense, (e.g. for $SO(16n)_1$) but every known $SU(n)$ modular invariant is even symmetric in the usual sense.

Our aim is to study or even construct modular invariants from subfactors. The framework can be summarised as follows. We have a hyperfinite III_1 factor N on which there is a system of endomorphisms $\{\lambda \in \mathcal{A}\}$ labelled by our positive energy representations or our original states in the original statistical mechanical setting. We induce these endomorphisms to endomorphisms $\alpha^\pm_\lambda$ on a larger ambient factor M—there will be two natural ways to do this labelled $\pm$.

The modular invariant will then be constructed or recovered as

$$Z_{\lambda\mu} \; = \; \langle \alpha^+_\lambda, \alpha^-_\mu \rangle$$

where the right hand side will be computed as dimensions of intertwiner spaces or the number of common sectors when we decompose $\alpha^\pm_\lambda$ into irreducibles. The original endomorphism $\lambda \in \mathcal{A}$ will be irreducible but $\alpha^\pm_\lambda$ may not be. The factor N will carry the modular data for S and T matrices, varying the inclusion may change the modular invariant but somehow the inclusion will have to be related to the original $\mathcal{A}$-system.

The system $\mathcal{A}$ on the factor N can be constructed via the method of Jones-Wassermann. First for any positive energy representation

$\lambda \in \mathcal{A}$, the objects $\lambda L_I SU(n)$ and $\lambda L_{I'} SU(n)$, if $L_I, L_{I'}$ denote loops on $SU(n)$ concentrated on complementary non-trivial intervals I and I′ in the circle. We can thus form the inclusion

$$\lambda L_I SU(n)'' \subset \lambda L_{I'} SU(n)'. \tag{3}$$

For $\lambda = 0$, the vacuum representation, there is Haag duality and this inclusion is not proper but gives us a single hyperfinite III_1 factor N (more precisely a net $N(I)$ of such factors). The inclusion Eq. (3) then determines a system of endomorphisms $\lambda \in \mathcal{A}$, so that the inclusion Eq. (3) is isomorphic to $\lambda N \subset N$, with index $[N, \lambda N] = d_\lambda^2$. Wassermann [66] has shown that the fusion rules of such endomorphisms are precisely the same as that of $SU(n)$ at a root of unity $\left(q = e^{2\pi i/(k+n)}\right)$. Moreover, rotation through 180^0 on the circle, interchanges the role of I and I′. This has the effect that the system $\mathcal{A}$ is naturally braided, i.e. not only is the system commutative $\lambda\mu = \mu\lambda$ as sectors if $\lambda, \mu \in \mathcal{A} \subset End(N)$, but there is a choice $\varepsilon(\lambda, \mu)$ of unitaries taking $\lambda\mu$ to $\mu\lambda$ satisfying the Yang Baxter equation, braiding fusion equation etc.

Thus we have commutative matrices $N_\lambda = [N_{\lambda\mu}^\nu]_{\mu,\nu}, \lambda \in \mathcal{A}$, determining the fusion

$$\lambda\mu = \sum_\nu N_{\lambda\mu}^\nu \ \nu$$

with composition of endomorphisms, or rather sectors, their unitary equivalence classes and a natural notion of addition. Fusion by the endomorphism of the conjugate $\bar\lambda$ of λ is given by $N_{\bar\lambda} = N_\lambda^{\mathrm{tr}}$, the transpose. Thus $\{N_\lambda : \lambda \in \mathcal{A}\}$ is a family of commuting normal matrices and so simultaneously diagonalisable. By the Verlinde theory the unitary matrix which performs this diagonalisation is the S matrix itself. Inverting the consistency condition or the regular representation

$$N_\lambda N_\mu = \sum N_{\lambda\mu}^\nu N_\nu \tag{4}$$

we obtain

$$N_{\lambda\mu}^\nu = \sum_\rho \frac{S_{\lambda\rho}}{S_{0\rho}} S_{\lambda\rho} \ \bar S_{\nu\rho}$$

or

$$N_\lambda = \sum_\rho \frac{S_{\lambda\rho}}{S_{0\rho}} |S_\rho\rangle\langle S_\rho|.$$

The modular invariants will provide representations other than the regular representation Eq. (4), and pick out subsets $\{S_{\lambda\rho}/S_{0\rho} : \rho \in \mathcal{I}\}$

where $\mathcal{I} = \{\lambda : Z_{\lambda\lambda} \neq 0\}$, the diagonal part of the modular invariant. At the same time these representations will replace N_λ by other families G_λ of graphs — to be associated with the modular invariant or at least the subfactor $N \subset M$ which yields that particular invariant.

As we have said the inclusion, which is meant to duplicate the modular invariant, should be related to the original $\mathcal{A}$-system. This is achieved as follows. There is a conjugation on endomorphisms of N, (extending for groups the notion of inverting automorphisms or conjugating a representation in the dual) compatible with the conjugation on $\mathcal{A}$. Similarly one can conjugate endomorphisms or sectors of M, or those between N and M, M and N. In particular, we can take the inclusion $\iota = N \to M$, its conjugate $\bar{\iota} = M \to N$ and compose to get endomorphisms $\gamma = \iota\bar{\iota}$ on M and $\theta = \gamma|N = \bar{\iota}\iota$ on N called the canonical and dual canonical endomorphisms respectively. What we need is θ lies in the system generated by $\mathcal{A}$, i.e. decomposes as a sum of sectors from $\mathcal{A}$. Note that we do not need to specify M when we ask whether a particular endomorphism θ of N is a dual canonical endomorphism. It may not be particularly clear in a given situation whether a certain endomorphism is a dual canonical endomorphism or what M may be. When Z is a modular invariant typical candidates for dual canonical endomorphisms will be $\sum_\lambda Z_{0\lambda}\lambda; \sum_\lambda Z_{\lambda 0}\lambda$ on N and $\sum Z_{\lambda\mu}\lambda \bigotimes \mu^{opp}$ on $N \bigotimes N^{opp}$ where N^{opp} is the opposite algebra, etc.

The first non trivial (i.e. exceptional) invariant for $SU(2)$ occurs at level 10:

$$Z_{E_6} = |\chi_0 + \chi_6|^2 + |\chi_4 + \chi_{10}|^2 + |\chi_3 + \chi_7|^2. \tag{5}$$

The diagonal part of the invariant $\mathcal{I} = \{\lambda : Z_{\lambda\lambda} \neq 0\}$ matches the spectrum of the Dynkin diagram E_6, namely $\{S_{1\lambda}/S_{0\lambda} = 2\cos\pi(\lambda+1)/12 : \lambda = 0, 6, 4, 10, 3, 7\}$. For this reason Capelli, Itzykson and Zuber labelled the invariant by the graph E_6. In the subfactor setting we can derive this graph as follows. First, we turn to the conformal embedding description of this invariant due to Bouwknegt and Nahm [17] which provides the extended system $\mathcal{B}$ which diagonalises the invariant. The embedding $SU(2)_{10} \subset SO(5)_1$ means there is a mapping of $SU(2)$ in $SO(5)$ such that the three level 1 representations $\mathcal{B}$ of $SO(5)$ decompose into level 10 representations of $SU(2)$ with finite multiplicity. The system $SO(5)_1$ has three inequivalent representations, b,v,s basic, vector and spinor which reproduce the Ising fusion rules. They decompose (cf Eq. (5)) as

$$\chi_b = \chi_0 + \chi_6, \chi_v = \chi_4 + \chi_{10}, \chi_s = \chi_3 + \chi_7 \tag{6}$$

so that the E_6 modular invariant for $SU(2)_{10}$ arises from the diagonal invariant for $SO(5)_1$:

$$Z_{E_6} = |\chi_b|^2 + |\chi_v|^2 + |\chi_s|^2.$$

Moving now to the loop group factors the conformal embedding gives us an inclusion of factors:

$$L_\pm SU(2) \subset L_\pm SO(5)$$

using the vacuum representation on $LSO(5)$, a net of subfactors $N(I) \subset M(I)$. Fixing I, we have subfactor $N \subset M$ on which there are systems $\mathcal{A} = SU(2)_{10}$ and $\mathcal{B} = SO(5)_1$ of endomorphisms acting respectively.

These two systems can be related by a form of Mackey induction-restriction which in the subfactor setting goes back to Longo-Rehren [51]. Using the braiding ε^+ or its opposite braiding ε^-, we can lift endomorphisms λ in $\mathcal{A}$ to those of M: $\alpha_\lambda^\pm = \gamma^{-1} Ad\varepsilon^\pm(\lambda, \theta)\lambda\gamma$. The maps $[\lambda] \to [\alpha_\lambda^\pm]$ preserve all the operations of conjugation, addition and multiplication of sectors [67][8][9][10]. However, they are not injective, and $\alpha_\lambda^\pm$ may be reducible. We find that $\{\alpha_\lambda^+ : \lambda \in \mathcal{A}\}$ decomposes into six irreducible sectors such that the graph E_6 is multiplication by α_1^+ [67] [9]. In fact $[\alpha_1^+] = G$, is part of a system of matrices with non-negative entries $\{G_\lambda : \lambda \in \mathcal{A}\}$ which represents the original $\mathcal{A}$-fusion rules. This had been noticed empirically in e.g. [28] [55] which now gets a subfactor explanation.

To bring the $\mathcal{B}$ system into the game we use σ-restriction, $\sigma_\beta = \bar{\iota}\beta\iota$ to take M-sectors to N-sectors. This map σ is not multiplicative, but in the type I situation there is a reciprocity: $\langle \alpha_\lambda^\pm, \beta \rangle = \langle \lambda, \sigma_\beta \rangle$ (with inequality on the type II setting) as long as say β is a subsector of the induced system $E_6^\pm$ respectively. Since σ restriction takes the $E_6^\pm$ systems into the $\mathcal{A}$-system by Eq. (6), namely

$$\sigma_b = \theta = \lambda_0 + \lambda_6, \sigma_v = \lambda_4 + \lambda_{10}, \sigma_s = \lambda_3 + \lambda_7$$

the reciprocity means that the $\mathcal{B}$ system, coming from the three level 1 representations of $SO(5)$ must lie in the induced systems $E_6^\pm$, i.e.

$A_3 = \mathcal{B} \subset E_6^+ \cap E_6^-$. In fact we can identify as sectors $b = \alpha_0, v = \alpha_{10}, s = \alpha_3 - \alpha_9$, and $E_6^+ \cap E_6^- = A_3$ precisely. The dual canonical endomorphism θ lies in $\mathcal{A}$, but a priori we do not have much information about its Fourier transform γ. In fact as sectors: $\gamma = id_M + \alpha_1^+ \alpha_1^-$ so that $\gamma \in E_6^+ \vee E_6^-$, the full induced system. Indeed $\mathcal{A}^\alpha = E_6^+ \vee E_6^-$, the system generated by $E_6^\pm$ is precisely all subsectors of $\{\iota\lambda\bar{\iota} : \lambda \in \mathcal{A}\}$ in fact the latter has global dimension $w = \sum_{\lambda \epsilon \mathcal{A}} d_\lambda^2$, whilst if $w_\alpha = \sum_{\beta \in \mathcal{A}^\alpha} d_\beta^2$ denotes the global dimension of the induced system then [10]:

$$w / w_\alpha = \sum Z_{0\lambda} d_\lambda$$

with the sum over only the degenerate sectors in $\mathcal{A}$ — which have trivial monodromy with all other sectors. In this case the $\mathcal{A}$-system, as far $SU(n)_k$ is non-degenerate, the vacuum is the only degenerate sector. Moreover we can recover the modular invariant as

$$Z_{\lambda\mu} = \langle \alpha_\lambda^+, \alpha_\mu^- \rangle, \lambda, \mu \in \mathcal{A}.$$

In this case the $E_6^\pm$ systems are commutative (but not braided) as is the $E_6^+ \vee E_6^-$ system — but this is not always the case. The neutral system $\mathcal{A}^0 = \mathcal{A}^+ \cap \mathcal{A}^-$, if $\mathcal{A}^\pm$ are the induced chiral systems, is braided, with the braiding non-degenerate if that of $\mathcal{A}$ is. Complexifying the finite dimensional algebras $\mathcal{A}^\pm$ we can decompose them in the non-degenerate case as [15]:

$$\mathcal{A}^\pm = \bigoplus_{\tau \in \mathcal{A}^0} \bigoplus_{\lambda \in \mathcal{A}} \mathrm{Mat}(b_{\tau\lambda}^\pm). \tag{7}$$

Here $b_{\tau,\lambda}^\pm$ are the chiral branching coefficients $\langle \tau, \alpha_\lambda^\pm \rangle, \tau \in \mathcal{A}^\pm, \lambda \in \mathcal{A}$. (In the case of chiral locality where the extended net $M(I)$, is local, i.e. observables associated with disjoint intervals commute, then $b_{\tau,\lambda}^\pm = \langle \tau, \alpha_\lambda^\pm \rangle = \langle \sigma_\tau, \lambda \rangle, \tau \in \mathcal{A}^0, \lambda \in \mathcal{A}$.) In particular the extended systems are commutative only when $b_{\tau,\lambda}^\pm \leq 1, \tau \in \mathcal{A}^0, \lambda \in \mathcal{A}$ Thus the informal inclusions $SU(n)_n \subset SU(n^2-1)_1$ give non-commutative chiral systems when $n \geq 4$; and it explains the computations of Feng Xu [67] who found non-commutativity in case $n = 4$ by a direct computation.

Thus we can decompose the modular invariant as

$$Z_{\lambda\mu} = \langle \alpha_\lambda^+, \alpha_\mu^- \rangle = \sum_{\tau \in \mathcal{A}^0} b_{\tau,\lambda}^\pm b_{\tau,\mu}^\pm$$

and from Eq. (7) by counting dimension we see that

$$|\mathcal{A}^{\pm}| = \sum (b^{\pm}_{\tau,\lambda})^2 = \mathrm{tr} b^{\pm t} b^{\pm}.$$

In the case of chiral locality where $b^+ = b^-$, so that the invariant is type I, we see that $|\mathcal{A}^{\pm}| = \mathrm{tr} Z^t Z$, more generally $|\mathcal{A}^{\pm}|$ only sees the type I parent of a type II invariant. Thus with

$$Z_{D_{10}} = |\chi_0 + \chi_{16}|^2 + |\chi_4 + \chi_{12}|^2 + |\chi_6 + \chi_{10}|^2 + 2|\chi_8|^2 + |\chi_2 + \chi_{14}|^2$$

$$Z_{E_7} = |\chi_0+\chi_{16}|^2+|\chi_4+\chi_{12}|^2+|\chi_6+\chi_{10}|^2+|\chi_2|^2+(\chi_2+\chi_{14})\chi_8^- +\chi_8(\chi_2+\chi_{14})^-$$

then in either case $\mathrm{tr} b^{\pm} b^{\pm} = 10$ so that multiplication by $[\alpha_1^{\pm}]$ gives the graph D_{10} in either case so we do not get the graph E_7 for Z_{E_7} (where we can use the dual canonical endomorphisms $\lambda_0 + \lambda_{16}$, $\lambda_0 + \lambda_{16} + \lambda_8$ respectively).

In general (and this will work when either chiral locality holds or fails) we look at the action of $\mathcal{A}^{\alpha}$ on the M-N sectors ${}_M\mathcal{A}_N$ which are the irreducible sectors of $\iota\lambda = \alpha^{\pm}_{\lambda}\iota$ (which can be identified with $\mathcal{A}^{\pm}$ when chiral locality holds). This action decomposes as: $\bigoplus_{\lambda} \mathrm{Mat}(Z_{\lambda\lambda})$, with

$$\alpha^{\pm}_{\mu} = \bigoplus_{\lambda} \frac{S_{\mu\lambda}}{S_{0\lambda}} 1_{Z_{\lambda\lambda}}.$$

Thus we get the desired representation with spectrum matching the diagonal part of the modular invariant, and counting dimension then $|{}_M\mathcal{A}_N| = \mathrm{tr} Z$, e.g. $\mathrm{tr} Z_{E_7} = 7$ so that we do indeed now recover the correct graph.

The subfactor framework is rich enough to produce a Moore-Seiberg type decomposition of modular invariants as well as handle possibly non-symmetric modular invariants. As we have already observed, in the case of chiral locality, $b^+_{\tau,\lambda} = b^-_{\tau,\lambda} (= \langle \lambda, \sigma_\tau \rangle)$ for $\lambda \in \mathcal{A}, \tau \in \mathcal{A}^0$. So the question arises as to how far we can identify b^+ and b_-, say $b^-_{\tau,\lambda} = b^+_{w(\tau),\lambda}$ for a permutation w of the extended neutral system $\mathcal{B}$ or if we need different labellings $\mathcal{B}^+$ or $\mathcal{B}^-$ to handle possibly non-symmetric modular invariants. Now locality holds if and only if $\theta = \sum Z_{\lambda 0}\lambda = \sum Z_{0\lambda}\lambda$. In general we define $\theta_+ = \sum Z_{\lambda 0}\lambda, \theta_- = \sum Z_{0\lambda}\lambda$. Using the theory of intermediate subfactors of [45], we can show [16] that both $\theta_{\pm}$ are dual canonical endomorphisms for inclusions $N \subset M_{\pm}$ which satisfy chiral locality and $M_{\pm} \subset M$. This means we can use α-induction on both inclusions $N \subset M_\pm$, to obtain type I modular invariants $Z^\pm$, such that

$$Z^+_{\lambda 0} = Z^+_{0\lambda} = Z_{\lambda 0}, Z^-_{\lambda 0} = Z^-_{0\lambda} = Z_{0\lambda},$$

where we can identify both neutral systems ${}_{M_\pm}\mathcal{A}^0{}_{M_\pm}$ with ${}_M\mathcal{A}^0_M$. If $M_+ = M_-$, then we can write

$$Z^\pm_{\lambda\mu} = \sum_{\tau\in\mathcal{A}^0} b_{\tau\lambda} b_{\tau\mu}$$

in particular $(Z^+ = Z^-)$ and using the identifications ${}_{M_\pm}\mathcal{A}^0_{M_\pm}$ with ${}_M\mathcal{A}^0_M$ to produce an automorphism ω of the neutral elements ${}_M\mathcal{A}^0_M$ we have:

$$Z_{\lambda\mu} = \sum b_{\tau\lambda} b_{w(\tau)\mu}.$$

In the case of E_7 invariant we have $N \subset M_\pm \subset M$ where $M_+ = M_-$ and the dual canonical endomorphism for $N \subset M_\pm, N \subset M$ are $\lambda_0 + \lambda_{16}$, $\lambda_0 + \lambda_{16} + \lambda_8$ as we have said before.

It may happen that $M_+ \neq M_-$ and this does occur for $SO(16n)_1$ where there are non-symmetric modular invariants where we must use different labelling ${}_{M_+}\mathcal{A}^0_{M_+}, {}_{M_-}\mathcal{A}^0_{M_-}$ on the left and right to decompose $Z^{ext}_{\tau_+,\tau_-}$ as $\delta_{\tau_-,\vartheta(\tau_+)}$, where $\vartheta = \vartheta_-\vartheta_+^{-1}$ is the identification. The situation is summarised [10] using recent work of Rehren [62] on canonical tensor product subfactors as a pair of inclusions:

$$N \bigotimes N^{opp} \subset M_+ \bigotimes M_-^{opp} \subset B$$

where the dual canonical endomorphisms for $N \bigotimes N^{opp} \subset B$ and $M_+ \bigotimes M_-^{opp} \subset B$ as $\sum Z_{\lambda\mu}\ \lambda \bigotimes \mu^{opp}, \sum_{\tau\in\mathcal{A}^0} \vartheta_+(\tau) \bigotimes \vartheta_-(\tau)^{opp}$ respectively.

There is a connection between the two chiral inductions and the picture of left- and right-chiral algebras in conformal field theory. Suppose that our factor N is obtained as a local factor $N = N(I_\circ)$ of a quantum field theoretical net of factors $\{N(I)\}$ indexed by proper intervals $I \subset \mathbb{R}$ on the real line, and that the system ${}_N\mathcal{X}_N$ is obtained as restrictions of DHR-morphisms (cf. [44]) to N. This is in fact the case in our examples arising from conformal field theory where the net is defined in terms of local loop groups in the vacuum representation. Taking two copies of such a net and placing the real axes on the light cone, then this defines a local net $\{A(\mathcal{O})\}$, indexed by double cones $\mathcal{O}$ on two-dimensional Minkowski space (cf. [61] for such constructions). Given a subfactor $N \subset M$, determining in turn two subfactors $N \subset M_\pm$ obeying chiral locality, will provide two local nets of subfactors $\{N(I) \subset M_\pm(I)\}$ as a local subfactor basically encodes the entire information about the net of subfactors [51]. Arranging $M_+(I)$ and $M_-(J)$ on the two light cone axes defines a local net of subfactors $\{A(\mathcal{O}) \subset A_{\text{ext}}(\mathcal{O})\}$ in Minkowski space. The embedding $M_+ \otimes M_-^{\text{opp}} \subset B$ gives rise to another net of subfactors $\{A_{\text{ext}}(\mathcal{O}) \subset B(\mathcal{O})\}$, where the net $\{B(\mathcal{O})\}$ obeys local commutation relations. The existence of the local net was already proven in [62], and now the decomposition of $[\Theta_{\text{ext}}]$ tells us that the chiral extensions $N(I) \subset M_+(I)$ and $N(I) \subset M_-(I)$ for left and right chiral nets are indeed maximal (in the sense of [61]), following from the fact that the coupling matrix for $\{A_{\text{ext}}(\mathcal{O}) \subset B(\mathcal{O})\}$ is a bijection. This shows that the inclusions $N \subset M_\pm$ should in fact be regarded as the subfactor version of left- and right maximal extensions of the chiral algebra.

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School of Mathematics, University of Wales Cardiff, PO Box 926, Senghennydd Road, Cardiff CF24 4YH, Wales, U.K., *EvansDE@cardiff.ac.uk*