PRINCIPAL SERIES AND WAVELETS

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ABSTRACT. Recently Antoine and Vandergeynst [1, 2] have produced continuous wavelet transforms on the $n$-sphere based on a principal series representation of $SO(n,1)$. We present some of their calculations in a more general setting, from the point of view of Fourier analysis on compact groups and spherical function expansions.

1. COHERENT STATES

We begin with Antoine and Vandergeynst’s definition of a coherent state, as presented in [1, 2]. Here $G$ is a locally compact group.

- Suppose that $X$ is a homogeneous space of $G$, $X = G/H$, equipped with a $G$-invariant measure.
- Let $(U, L^2(Y))$ be a unitary representation of $G$ on some Lebesgue space $L^2(Y)$.
- Assume there is a Borel cross section
  \[ \sigma : X \rightarrow G, \quad \sigma(x)H = x, \quad \forall x \in X. \]
- Say that $\eta \in L^2(Y)$ is admissible mod$(H,\sigma)$ when
  \[ \int_X |(U(\sigma(x))\eta|\varphi)|^2 \, dx < \infty, \quad \forall \varphi \in L^2(Y). \]
- The orbit of an admissible vector $\eta$ under $\sigma(X)$,
  \[ \{ U(\sigma(x))\eta : x \in X \} \]
  is called a coherent state.

Note that there are other variations on the theme of “restricted square integrability”, such as the case described in [3].
2. Frames

Suppose now that \( \eta \) is an admissible vector in \( L^2(Y) \). Define a linear operator

\[
A_{\sigma, \eta} : L^2(Y) \longrightarrow L^2(Y)
\]

by

\[
\langle A_{\sigma, \eta} \varphi_1 | \varphi_2 \rangle = \int_X \langle \varphi_1 | U(\sigma(x)) \eta \rangle \langle U(\sigma(x)) \eta | \varphi_2 \rangle \, dx, \quad \forall \varphi_1, \varphi_2 \in L^2(Y).
\]

When this has a bounded inverse, say that the coherent state is a frame.

When the orbit of \( \eta \) under \( \sigma(X) \) is a frame of \( L^2(Y) \) there is the continuous wavelet transform,

\[
W_\eta : L^2(Y) \longrightarrow L^2(X)
\]

defined by

\[
W_\eta \varphi(x) = \langle \varphi | U(\sigma(x)) \eta \rangle, \quad \forall \varphi \in L^2(Y).
\]

This operator is one-to-one and its range \( \mathcal{H}_\eta \) is complete with respect to the inner-product:

\[
\langle W_\eta \varphi | W_\eta \psi \rangle_{\mathcal{H}_\eta} = \langle W_\eta \varphi | W_\eta A_{\sigma, \eta}^{-1} \psi \rangle_{L^2(X)}, \quad \psi, \varphi \in L^2(Y).
\]

Hence there is a unitary isomorphism \( W_\eta : L^2(Y) \longrightarrow \mathcal{H}_\eta \).

3. The setting

For the calculations which we will describe here, the ingredients are:

- \( G \) is a noncompact connected semisimple Lie group with finite centre and Cartan involution \( \theta \).
- \( K \) is the corresponding maximal compact subgroup.
- \( G = KAN \) is an Iwasawa decomposition.
- \( M \) is the centralizer of \( A \) in \( K \).
- \( X = G/N \).
- \( Y = K/M \).
- \( U \) is a certain principal series action of \( G \) on \( L^2(K/M) \), to be defined below.
- Assume that \( (K, M) \) is a Gel'fand pair.

See Knapp’s book for details [5, page 119].
4. Decompositions

There are Iwasawa projections $K : G \to K$, $A : G \to A$, $N : G \to N$, for which

$$g = k(g) a(g) n(g), \quad \forall g \in G.$$ 

The Haar measure on $G$ is given in terms of that of $K$ and right Haar measure of $AN$, [5, page 139] with

$$dg = dk \, d_r(an).$$

The measure on $K$ is normalized so that

$$\int_K dk = 1.$$ 

There is a mapping $\log : A \to a$ with

$$\exp(\log(a)) = a, \quad \forall a \in A.$$ 

For each $\nu \in a^*$ let

$$a^\nu = e^{\nu(\log(a))}, \quad \forall a \in A.$$ 

5. Invariant Integration

There is the special functional $\rho \in a^*$ determined by the structure of the group $G$. For $f \in C_c(G)$ the integral formula for Haar measure on $G$ is

$$\int_G f(x) \, dx = \int_K \int_A \int_N f(kan) a^{2\rho} \, dndadk.$$ 

See [6, Prop. 7.6.4] for details.

We can use $KA$ to parametrize $G/N$ and the $G$-invariant integral on $G/N$ is given by

$$\int_{G/N} F(y) \, dy = \int_K \int_A F(kaN) a^{2\rho} \, dndk$$

for $F \in C_c(G/N)$. Hence, we take the Borel section $\sigma : G/N \to G$ to be

$$\sigma(kaN) = ka, \quad \forall a \in A, k \in K.$$
6. Induced Representations

Consider the space of continuous covariant functions:
\[ I(G) = \left\{ f : G \to \mathbb{C} \text{ continuous} \right\}. \]
Left translation by elements of \( G \) preserves the property of covariance:
\[(U(g)f)(x) = f(g^{-1}x), \quad \forall g, x \in G, f \in I(G). \]
\[ U(g) : I(G) \to I(G), \quad \forall g \in G. \]
For a covariant function \( f \in I(G) \),
\[ f(x) = f(K(x)A(x)M(x)) = A(x)^{-\rho}f(K(x)), \quad \forall x \in G. \]
Equip \( I(G) \) with the inner product
\[ \langle f_1|f_2 \rangle = \int_K f_1(k)\overline{f_2(k)} \, dk \]
and norm
\[ \|f\| = \left(\int_K |f(k)|^2 \, dk \right)^{1/2}. \]
The completion of \( I(G) \) is
\[ \mathcal{H}_U \cong L^2(K/M). \]
The action of \( G \) on \( \mathcal{H}_U \) is an example of a principal series representation, see section 8.3 of Wallach’s book [6]. For our purposes, the essential fact is that \( U|_K \) is the regular representation of \( K \) on a subspace of \( L^2(K) \). If \( f \in L^2(K/M) \), extend it to be an element of \( \mathcal{H}_U \) by assigning
\[ f(kan) = a^{-\rho}f(k). \]
Notice that if \( f \in L^2(K/M) \),
\[ U(g)f(k) = A(g^{-1}k)^{-\rho}f(K(g^{-1})k), \quad k \in K, g \in G. \]
For each \( g \in G \) the action of \( U(g) \) extends to a continuous linear operator on \( \mathcal{H}_U \). It is a unitary representation:
\[ \langle U(g)f_1|U(g)f_2 \rangle = \int_K (U(g)f_1)(k)\overline{(U(g)f_2)(k)} \, dk \]
\[ = \int_K A(g^{-1}k)^{-2\rho}f_1(K(g^{-1}k))\overline{f_2(K(g^{-1}k))} \, dk = \langle f_1|f_2 \rangle \]

Lemma 1. The representation \((U, \mathcal{H}_U)\) is unitary. When restricted to \( K \), it is the action of \( K \) by left translation on \( L^2(K/M) \).
7. Fourier analysis on the compact group $K$

We review some basic facts about analysis on compact groups. Let $\hat{K}$ be the dual object of $K$, consisting of a maximal set of inequivalent irreducible unitary representations $(\gamma, V_\gamma)$ of $K$.

For each integrable function $f$ on $K$ there is the Fourier series:

$$f(x) = \sum_{\gamma \in \hat{K}} d_\gamma f \ast \chi_\gamma(x).$$

Convolution with a character is

$$f \ast \chi_\gamma(x) = \int_K f(y) \text{tr}(\gamma(y^{-1})\gamma(x)) \, dy = \text{tr}(\hat{f}(\gamma)\gamma(x))$$

where the Fourier coefficient is

$$\hat{f}(\gamma) = \int_K f(x)\gamma(x^{-1}) \, dx = \int_K f(x)\gamma(x)^* \, dx.$$

The Fourier coefficients are linear transformations

$$\hat{f}(\gamma) \in \text{Hom}_\mathbb{C}(V_\gamma, V_\gamma).$$

Fourier coefficients of convolutions are products of Fourier coefficients:

$$(f \ast g)^\wedge(\gamma) = \int_K \int_K f(x)g(x^{-1}y)\gamma(y^{-1}) \, dx \, dy = \int_K \int_K f(x)g(x^{-1}y)\gamma(y^{-1}xx^{-1}) \, dx \, dy = \hat{g}(\gamma)\hat{f}(\gamma).$$

Define left translation on $K$ by

$$xf(y) = f(x^{-1}y), \quad \forall x, y \in K,$$

and the composition with inversion

$$f^\vee(x) = f(x^{-1}), \quad \forall x \in K.$$

Fourier coefficients of left translates satisfy

$$(xf)^\wedge(\gamma) = \int_K f(x^{-1}y)\gamma(y^{-1}xx^{-1}) \, dy = \hat{f}(\gamma)\gamma(x^{-1})$$

Fourier coefficients of adjoints satisfy

$$(g^\vee)^\wedge(\gamma) = \hat{g}(\gamma)^*. $$

The $L^2(K)$ inner product can be viewed as a convolution:

$$\int_K f(x)\overline{g(x)} \, dx = \int_K f(x)\overline{g^\vee(x^{-1})} \, dx = f \ast \overline{g^\vee}(1).$$
For $f, g \in L^2(K)$, the Fourier series of their convolution is absolutely convergent, see [4],

$$f * g(x) = \sum_{\gamma \in \hat{K}} d_\gamma f * g * \chi_\gamma(x)$$

$f$ and $g$ in $L^2(K)$:

$$f * g(x) = \sum_{\gamma \in \hat{K}} d_\gamma \text{tr} \left( \hat{g}(\gamma) \hat{f}(\gamma) \gamma(x) \right),$$

$$\int_K f(x) \overline{g(x)} \, dx = \sum_{\gamma \in \hat{K}} d_\gamma \text{tr} \left( \hat{f}(\gamma) \hat{g}(\gamma)^* \right),$$

$$\|f\|_2^2 = \sum_{\gamma \in \hat{K}} d_\gamma \left\| \hat{f}(\gamma) \right\|_{\phi_2}^2.$$  

In particular, for each $\gamma \in \hat{K}$,

$$\left\| \hat{f}(\gamma) \right\|_{\phi_2}^2 = d_\gamma \|f * \chi_\gamma\|_2^2.$$  

See Appendix D of Hewitt and Ross [4] for details about the norms

$$\| \cdot \|_{\phi_p}, \quad 1 \leq p \leq \infty.$$  

If $h \in L^1(K)$ then

$$f \mapsto f * h, \quad L^2(K) \longrightarrow L^2(K),$$

is a bounded linear operator which commutes with left translation. Similarly,

$$f \mapsto h * f, \quad L^2(K) \longrightarrow L^2(K),$$

is a bounded linear operator which commutes with right translation. The norm of both of these operators is

$$\sup_{\gamma \in \hat{K}} \left\| \hat{h}(\gamma) \right\|_{\phi_\infty}.$$  

8. Homogeneous Spaces

Now we return to dealing with functions on $K/M$, which we identify with right-$M$-invariant functions on $K$.

For each $\gamma \in \hat{K}$, let

$$V_\gamma^M = \{ v \in V_\gamma : \gamma(m)v = v, \quad \forall m \in M \}$$

and $P_\gamma : V_\gamma \longrightarrow V_\gamma^M$, the orthogonal projection on to this subspace.

Let $\mu$ be the normalized Haar measure on $M$. Its Fourier coefficients are

$$\hat{\mu}(\gamma) = P_\gamma, \quad \forall \gamma \in \hat{K}.$$
If \( f \in L^1(K/M) \) then
\[
f = f \ast \mu, \quad \implies \quad \hat{f}(\gamma) = P_\gamma \hat{f}(\gamma), \quad \forall \gamma \in \hat{K}.
\]

We are restricting our attention to the case where \((K, M)\) is a Gel’fand pair, which means that
\[
\dim (V^M_\gamma) \leq 1, \quad \forall \gamma \in \hat{K}.
\]

**Lemma 2.** If \((K, M)\) is a Gel’fand pair and \( f \in L^1(K/M) \), then for all \( \gamma \in \hat{K} \),
\[
\text{rank}(\hat{f}(\gamma)) \leq 1 \quad \text{and} \quad (V^M_\gamma) \subseteq \ker(\hat{f}(\gamma)^*).
\]

**Lemma 3.** If \((K, M)\) is a Gel’fand pair and \( f \in L^1(K/M) \), then for all \( \gamma \in \hat{K} \),
\[
\hat{f}(\gamma) \hat{f}(\gamma)^* = \|\hat{f}(\gamma)\|_{\phi_2}^2 P_\gamma.
\]

**Lemma 4.** If \((K, M)\) is a Gel’fand pair and \( f \in L^1(K/M) \), then for all \( \gamma \in \hat{K} \),
\[
\|\hat{f}(\gamma)\|_{\phi_p} = \|\hat{f}(\gamma)\|_{\phi_2}, \quad 1 \leq p \leq \infty.
\]

**Lemma 5.** If \((K, M)\) is a Gel’fand pair and \( h \in L^1(K/M) \), then the norm of the operator
\[
f \mapsto f \ast h, \quad L^2(K) \longrightarrow L^2(K/M),
\]
is
\[
\sup \left\{ \|\hat{h}(\gamma)\|_{\phi_2} : \gamma \in \hat{K} \right\} = \sup \left\{ \sqrt{d_\gamma} \|h \ast \chi_\gamma\|_2 : \gamma \in \hat{K} \right\}.
\]

In this lemma, if \( \dim (V^M_\gamma) = 0 \) then \( \hat{h}(\gamma) = 0 \) and so we need only take the supremum over those \( \gamma \) for which \( \dim (V^M_\gamma) = 1 \).

**9. Admissible Vectors**

In [2] the unitary representation \((U, \mathcal{H}_U)\) of \( G \) is said to be *square-integrable modulo \( N \)* if there is a non-zero vector \( \eta \) for which
\[
\int_K \int_A |(U(k a)\eta, \xi)|^2 a^{2p} \, d\mu < \infty
\]
for all \( \xi \in \mathcal{H}_U \). Such an \( \eta \) is called *admissible*.

Notice that this can be rearranged to say
\[
\int_K \int_A |\langle U(a)\eta, U(k^{-1})\xi \rangle|^2 a^{2p} \, d\mu < \infty
\]
for all \( \xi \in \mathcal{H}_U \). Recall that \( U|_K \) is left translation.
We then find that
\[
\int_K \left| \langle U(k)a \eta \mid \xi \rangle \right|^2 dk = \int_K \left| \int_K (U(a)\eta)(x) \xi(kx) dx \right|^2 dk = \int_K \left| (U(a)\eta) \ast \xi'(k) \right|^2 dk = \left\| (U(a)\eta) \ast \xi' \right\|_2^2
\]
Using the Plancherel formula for this,
\[
\left\| (U(a)\eta) \ast \xi' \right\|_2^2 = \sum_\gamma d_\gamma \text{tr} \left( (U(a)\eta)^\wedge(\gamma)^\ast \hat{\xi}(\gamma) \hat{\xi}(\gamma)^\ast (U(a)\eta)^\wedge(\gamma) \right) = \sum_\gamma d_\gamma \left\| (U(a)\eta)^\wedge(\gamma) \right\|_{\ell_2^A}^2 \left\| \hat{\xi}(\gamma) \right\|_{\ell_2^A}^2
\]
We arrive at the general version of Antoine and Vanderheynst’s criterion for admissibility.

**Theorem 1.** If $\eta \in \mathcal{H}_U = L^2(K/M)$ has the property that
\[
\sup_{\gamma \in \hat{K}} \int_A \left\| (U(a)\eta)^\wedge(\gamma) \right\|_{\ell_2^A}^2 a^{2\rho} da < \infty
\]
then $\eta$ is admissible.

Since the functions here are right-$M$-invariant, the only non-zero parts of the Fourier series correspond to those $\gamma$ for which $P_\gamma \neq 0$.

**Theorem 2.** If $\eta \in \mathcal{H}_U = L^2(K/M)$ is admissible and there are constants $0 < c_1 \leq c_2$ for which
\[
c_1 \leq \int_A \left\| (U(a)\eta)^\wedge(\gamma) \right\|_{\ell_2^A}^2 a^{2\rho} da \leq c_2
\]
for all $\gamma \in \hat{K}$ with $P_\gamma \neq 0$, then the corresponding coherent state is a frame.

We can reword this to see that the criterion for $\eta$ to give rise to a frame for $L^2(K/M)$ is that there are constants $0 < c_1 \leq c_2$ for which
\[
c_1 \leq d_\gamma \int_A \left\| (U(a)\eta) \ast \chi_\gamma \right\|_2^2 a^{2\rho} da \leq c_2,
\]
for all $\gamma \in \hat{K}$ with $P_\gamma \neq 0$. 

10. Spherical Functions

Let \( \hat{K}_M \) denote the set of those \( \gamma \in \hat{K} \) with \( P_\gamma \neq 0 \). For each \( \gamma \in \hat{K}_M \) define the spherical function

\[ \varphi_\gamma = \chi_\gamma * \mu = \mu * \chi_\gamma. \]

If \( f \in L^1(K/M) \) its Fourier series is

\[ \sum_{\gamma \in \hat{K}_M} d_\gamma f * \varphi_\gamma. \]

When \( K/M = S^n \), this is the usual spherical harmonic expansion.

To use the criterion for a frame, we need estimates on

\[ d_\gamma \int_A \| (U(a)\eta) * \varphi_\gamma \|^2 a^{2p} da, \]

uniformly in \( \gamma \in \hat{K}_M \).

11. Zonal Functions

A special case occurs when \( \eta \) is bi-\( M \)-invariant, since it is then expanded in a series

\[ \eta = \sum_{\gamma \in \hat{K}_M} d_\gamma c_\gamma \varphi_\gamma \quad \text{with} \quad c_\gamma = \langle \eta | \varphi_\gamma \rangle. \]

But \( U(a)\eta \) is also bi-\( M \)-invariant and its expansion is

\[ U(a)\eta = \sum_{\gamma \in \hat{K}_M} d_\gamma c_\gamma(a) \varphi_\gamma \]

with

\[ c_\gamma(a) = \langle U(a)\eta | \varphi_\gamma \rangle = \langle \eta | U(a^{-1})\varphi_\gamma \rangle. \]

Since the spherical functions \( \varphi_\gamma \) are matrix entries of irreducible representations,

\[ \varphi_\gamma * \varphi_\gamma' = \begin{cases} \varphi_\gamma / d_\gamma & \text{if } \gamma = \gamma' \\ 0 & \text{if } \gamma \neq \gamma', \end{cases} \]

and \( \| \varphi_\gamma \|^2 = 1/d_\gamma \). Hence, Theorem 2 says that a bi-\( M \)-invariant function \( \eta \) produces a frame for \( L^2(K/M) \) when there are positive constants \( c_1 \leq c_2 \) for which

\[ 0 < c_1 \leq \int_A |c_\gamma(a)|^2 a^{2p} da \leq c_2 \]

for all \( \gamma \in \hat{K}_M \).
12. Antoine and Vanderghynst

The results in [2] are concerned with the case where:

- $G = SO_0(1, 3), K \cong SO(3), M \cong SO(2), \text{ and } K/M \cong S^2$.
- $A \cong (0, \infty)$ with multiplication, $X \cong SO(3) \times A, \rho = 1$.
- $\tilde{K}_M = \{0, 1, 2, 3, \ldots\}, d_n = 2n + 1$, and the spherical functions $\varphi_n$ are normalized ultraspherical polynomials.

Suppose we use spherical coordinates $(\theta, \phi)$ to parametrize $S^2$. Proposition 3.4 of [2] states that if $\eta \in L^2(S^2)$ is admissible and

$$\int_0^{2\pi} \eta(\theta, \phi) d\phi \neq 0$$

then $\eta$ gives rise to a frame. This is achieved using the spherical harmonic expansion of $U(a)\eta$ and the asymptotics of the zonal spherical functions, to get the inequality in Theorem 2 above.

In [2] there is presented a sufficient condition on a function $\eta \in L^2(S^2)$ so that it satisfies the hypotheses of Theorem 1. These are similar to the moment conditions in the Euclidean space setting, see Proposition 7 in [3]. Proposition 3.6 [2] states that if $\eta \in L^2(S^2)$ satisfies

$$\int_0^\pi \int_0^{2\pi} \frac{\eta(\theta, \phi)}{1 + \cos(\theta)} \sin(\theta) d\theta d\phi = 0$$

then it is admissible.

References


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