L² HARMONIC FORMS ON NON-COMPACT Riemannian Manifolds

GILLES CARRON

First, I want to present some questions on L² harmonic forms on non-compact Riemannian manifolds. Second, I will present an answer to an old question of J. Dodziuk on L² harmonic forms on manifolds with flat ends. In fact some of the analytical tools presented here apply in other situations (see [C4]).

1. The space of harmonic forms

Let (Mⁿ, g) be a complete Riemannian manifold. We denote by \( \mathcal{H}^k(M, g) \) its space of L²-harmonic k-forms, that is to say the space of L² k-forms which are closed and coclosed:

\[
\mathcal{H}^k(M) = \{ \alpha \in L^2(\Lambda^k T^* M), \, d\alpha = \delta\alpha = 0 \},
\]

where

\[
d : C_0^\infty(\Lambda^k T^* M) \rightarrow C_0^\infty(\Lambda^{k+1} T^* M)
\]

is the exterior differentiation operator and

\[
\delta : C_0^\infty(\Lambda^{k+1} T^* M) \rightarrow C_0^\infty(\Lambda^k T^* M)
\]

its formal adjoint. The operator \( d \) does not depend on \( g \) but \( \delta \) does; \( \delta \) is defined with the formula:

\[
\forall \alpha \in C_0^\infty(\Lambda^k T^* M), \, \forall \beta \in C_0^\infty(\Lambda^{k+1} T^* M), \, \int_M (d\alpha, \beta) = \int_M (\alpha, \delta\beta).
\]

The operator \( (d + \delta) \) is elliptic hence the elements of \( \mathcal{H}^k(M) \) are smooth and the \( L^2 \) condition is only a decay condition at infinity.

2. If the manifold \( M \) is compact without boundary

If \( M \) is compact without boundary, then these spaces have finite dimension, and we have the theorem of Hodge-DeRham: the spaces \( \mathcal{H}^k(M) \) are isomorphic to the real cohomology groups of \( M \):

\[
\mathcal{H}^k(M) \simeq H^k(M, \mathbb{R}).
\]
Hence the dimension of $\mathcal{H}^k(M)$ is a homotopy invariant of $M$, i.e. it does not depend on $g$. A corollary of this and of the Chern-Gauss-Bonnet formula is:

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \dim \mathcal{H}^k(M) = \int_M \Omega^g,$$

where $\Omega^g$ is the Euler form of $(M^n, g)$; for instance if $\dim M = 2$ then $\Omega^g = \frac{K dA}{2\pi}$, where $K$ is the Gaussian curvature and $dA$ the area form.

3. **What is true on a non-compact manifold**

Almost nothing is true in general:

The space $\mathcal{H}^k(M, g)$ can have infinite dimension and the dimension, if finite, can depend on $g$. For instance, if $M$ is connected we have

$$\mathcal{H}^0(M) = \{ f \in L^2(M, d\text{vol}_g), f = \text{constant} \}.$$  

Hence $\mathcal{H}^0(M) = \mathbb{R}$ if $\text{vol} M < \infty$,
and $\mathcal{H}^0(M) = \{0\}$ if $\text{vol} M = \infty$.

For instance if $\mathbb{R}^2$ is equipped with the euclidean metric, we have $\mathcal{H}^0(\mathbb{R}^2, \text{eucl}) = \{0\}$, and if $\mathbb{R}^2$ is equipped with the metric $g = dr^2 + r^2 e^{-2r} d\theta^2$ in polar coordinates, then $\mathcal{H}^0(\mathbb{R}^2, g) = \mathbb{R}$. We have also that $\mathcal{H}^k(\mathbb{R}^n, \text{eucl}) = \{0\}$, for any $k \leq n$. But if we consider the unit disk in $\mathbb{R}^2$ equipped with the hyperbolic metric $4|dz|^2/(1-|z|^2)^2$ then it is isometric to the metric $g_1 = dr^2 + \sinh r^2 d\theta^2$ on $\mathbb{R}^2$. And then we have

$$\dim \mathcal{H}^1(\mathbb{R}^2, g_1) = \infty.$$  

As a matter of fact if $P(z) \in \mathbb{C}[z]$ is a polynomial, then $\alpha = P'(z)dz$ is a $L^2$ harmonic form on the unit disk for the hyperbolic metric (this comes from the conformal invariance, see 5.2).

So we get an injection $\mathbb{C}[z]/\mathbb{C} \rightarrow \mathcal{H}^1(\mathbb{R}^2, g_1)$. However, the spaces $\mathcal{H}^k(M, g)$ satisfy the following two properties:

- These spaces have a reduced $L^2$ cohomology interpretation:

Let $Z^k_2(M)$ be the kernel of the unbounded operator $d$ acting on $L^2(\Lambda^k T^* M)$, or equivalently

$$Z^k_2(M) = \{ \alpha \in L^2(\Lambda^k T^* M), \ d\alpha = 0 \},$$

where the equation $d\alpha = 0$ has to be understood in the distribution sense i.e. $\alpha \in Z^k_2(M)$ if and only if
∀β ∈ \( C_0^\infty(\Lambda^{k+1}T^*M) \), \( \int_M \langle \alpha, \delta \beta \rangle = 0 \).

That is to say \( Z_2^k(M) = (\delta C_0^\infty(\Lambda^{k+1}T^*M))^\perp \). The space \( L^2(\Lambda^kT^*M) \) has the following Hodge-DeRham-Kodaira orthogonal decomposition

\[
L^2(\Lambda^kT^*M) = H^k(M) \oplus dC_0^\infty(\Lambda^{k-1}T^*M) \oplus \delta C_0^\infty(\Lambda^{k+1}T^*M),
\]

where the closure is taken with respect to the \( L^2 \) topology. We also have

\[
Z_2^k(M) = H^k(M) \oplus dC_0^\infty(\Lambda^{k-1}T^*M),
\]

hence we have

\[
H^k(M) \simeq Z_2^k(M)/dC_0^\infty(\Lambda^{k-1}T^*M).
\]

A corollary of this identification is the following:

**Proposition 3.1.** The space \( H^k(M, g) \) are quasi-isometric invariant of \((M, g)\). That is to say if \( g_1 \) and \( g_2 \) are two complete Riemannian metrics such that for a \( C > 1 \) we have

\[
C^{-1}g_1 \leq g_2 \leq Cg_1,
\]

then \( H^k(M, g_1) \simeq H^k(M, g_2) \).

In fact, the spaces \( H^k(M, g) \) are biLipschitz-homotopy invariants of \((M, g)\).

- The finiteness of \( \dim H^k(M, g) \) depends only of the geometry of ends : 

**Theorem 3.2.** (J. Lott, [L]) The spaces of \( L^2 \)-harmonic forms of two complete Riemannian manifolds, which are isometric outside some compact set, have simultaneously finite or infinite dimension.

4. A general problem

In view of the Hodge-DeRham theorem and of J. Lott’s result, we can ask the following questions :

1. What geometrical condition on the ends of \( M \) insure the finiteness of the dimension of the spaces \( H^k(M) \)?

Within a class of Riemannian manifold having the same geometry at infinity:

2. What are the links of the spaces \( H^k(M) \) with the topology of \( M \) and with the geometry ‘at infinity’ of \((M, g)\)?
(3) And what kind of Chern-Gauss-Bonnet formula could we hope for the $L^2$-Euler characteristic

$$\chi_{L^2}(M) = \sum_{k=0}^{n} (-1)^k \dim \mathcal{H}^k(M) ?$$

There are many articles dealing with these questions. I mention only three of them:

(1) In the pioneering article of Atiyah-Patodi-Singer ([A-P-S]), the authors considered manifolds with cylindrical ends: that is to say there is a compact subset $K$ of $M$ such that $M \setminus K$ is isometric to the Riemannian product $\partial K \times [0, \infty]$. Then they show that the dimension of the space of $L^2$-harmonic forms is finite; and that these spaces are isomorphic to the image of the relative cohomology in the absolute cohomology. These results were used by Atiyah-Patodi-Singer in order to obtain a formula for the signature of compact manifolds with boundary.

(2) In [M, M-P], R. Mazzeo and R. Phillips give a cohomological interpretation of the space $\mathcal{H}^k(M)$ for geometrically finite real hyperbolic manifolds.

(3) The solution of the Zucker’s conjecture by Saper and Stern ([S-S]) shows that the spaces of $L^2$ harmonic forms on hermitian locally symmetric space with finite volume are isomorphic to the middle intersection cohomology of the Borel-Serre compactification of the manifold.

5. An example

I want now to discuss the $L^2$ Gauss-Bonnet formula through one example. The sort of $L^2$ Gauss-Bonnet formula one might expect is a formula of the type

$$\chi_{L^2}(M) = \int_K \Omega^g + \text{terms depending only on } (M - K, g),$$

where $K \subset M$ is a compact subset of $M$; i.e. $\chi_{L^2}(M)$ is the sum of a local term $\int_K \Omega^g$ and of a boundary (at infinity) term. Such a result will imply a relative index formula:

If $(M_1, g_1)$ and $(M_2, g_2)$ are isometric outside compact set $K_i \subset M_i$, $i = 1, 2$, then

$$\chi_{L^2}(M_1) - \chi_{L^2}(M_2) = \int_{K_1} \Omega^{g_1} - \int_{K_2} \Omega^{g_2}.$$
It had been shown by Gromov-Lawson and Donnelly that when zero is not in the essential spectrum of the Gauss-Bonnet operator $d + \delta$ then this relative index formula is true ([G-L, Do]). For instance, by the work of Borel and Casselman [BC], the Gauss-Bonnet operator is a Fredholm operator if $M$ is an even dimensional locally symmetric space of finite volume and negative curvature.

In fact such a relative formula is not true in general. The following counterexample is given in [C2]:

$(M_1, g_1)$ is the disjoint union of two copies of the euclidean plane and $(M_2, g_2)$ is two copies of the euclidean plane glued along a disk. As these surface are oriented with infinite volume, we have $i = 1, 2$:

$$\mathcal{H}^0(M_1, g_i) = \mathcal{H}^2(M_1, g_i) = \{0\}.$$  

And we also have $\mathcal{H}^1(M_1, g_1) = \{0\}$. Moreover

**Lemma 5.1.** $\mathcal{H}^1(M_2, g_2) = \{0\}$.

This comes from the conformal invariance of this space. Indeed, it is a general fact:

**Proposition 5.2.** If $(M^n, g)$ is a Riemannian manifold of dimension $n = 2k$, and if $f \in C^\infty(M)$ then

$$\mathcal{H}^k(M, g) = \mathcal{H}^k(M, e^{2fg}).$$

**Proof.** As a matter of fact the two Hilbert spaces $L^2(\Lambda^k T^*M, g)$ and $L^2(\Lambda^k T^*M, e^{2fg})$ are the same: if $\alpha \in \Lambda^k T^*_x M$, then

$$\|\alpha\|_{e^{2fg}}(x) = e^{-2kf(x)} \|\alpha\|_g(x)$$

and $d\text{vol}_{e^{2fg}} = e^{-2kf}d\text{vol}_g$.

We have

$$\mathcal{H}^k(M, e^{2fg}) = Z^k_1(M, e^{2fg}) \cap dC^\infty_0(\Lambda^{k-1} T^*M)$$

and

$$\mathcal{H}^k(M, g) = Z^k_2(M, g) \cap dC^\infty_0(\Lambda^{k-1} T^*M).$$

As the two Hilbert space $L^2(\Lambda^k T^*M, g)$ and $L^2(\Lambda^k T^*M, e^{2fg})$ are the same, these two spaces are the same.

But $(M_2, g_2)$ is conformally equivalent to the 2-sphere with two points removed.

A $L^2$ harmonic form on the 2-sphere with two points removed extends smoothly on the sphere.

The sphere has no non trivial $L^2$ harmonic 1-form, hence Lemma 5.1 follows.
The surfaces \((M_1, g_1)\) and \((M_2, g_2)\) are isometric outside some compact set but
\[
\chi_{L^2}(M_1) - \int_{M_1} \frac{K_{g_1} dA_{g_1}}{2\pi} = 0 - 0 = 0
\]
whereas
\[
\chi_{L^2}(M_2) - \int_{M_2} \frac{K_{g_2} dA_{g_2}}{2\pi} = -\int_{M_2} \frac{K_{g_2} dA_{g_2}}{2\pi} = -2.
\]
Hence the relative index formula is not true in general. A corollary of this argument is the following

**Corollary 5.3.** If \((S, g)\) is a complete surface with integrable Gaussian curvature, according to a theorem of A. Huber [H], we know that such a surface is conformally equivalent to a compact surface \(\bar{S}\) with a finite number of points removed. Then
\[
\dim \mathcal{H}^1(S, g) = b_1(\bar{S}).
\]

6. MANIFOLDS WITH FLAT ENDS

In (1982, [D]), J. Dodziuk asked the following question: according to Vesentini ([V]) if \(M\) is flat outside a compact set, the spaces \(\mathcal{H}^k(M)\) are finite dimensional. Do they admit a topological interpretation?

My aim is to present an answer to this question. For the detail, the reader may look at [C4]:

6.1. Visentini’s finiteness result.

**Theorem 6.1.** Let \((M, g)\) be a complete Riemannian manifold such that for a compact set \(K_0 \subset M\), the curvature of \((M, g)\) vanishes on \(M - K_0\). Then for every \(p\)
\[
\dim \mathcal{H}^p(M, g) < \infty.
\]

We give here a proof of this result; this proof will furnish some analytical tools to answer J. Dodziuk’s question.

We begin to define a Sobolev space adapted to our situation:

**Definition 6.2.** Let \(D\) be a bounded open set containing \(K_0\), and let \(W_D(\Lambda^* M)\) be the completion of \(C^\infty(\Lambda^* M)\) for the quadratic form
\[
\alpha \mapsto \int_D |\alpha|^2 + \int_M |(d + \delta)\alpha|^2 = N_D^2(\alpha).
\]
Proposition 6.3. The space $W_D$ doesn’t depend on $D$, that is to say if $D$ and $D'$ are two bounded open sets containing $K_0$, then the two norms $N_D$ and $N_{D'}$ are equivalent.

We write $W$ for $W_D$.

Proof.– The proof goes by contradiction. We notice that with the Bochner-Weitzenböck formula:

$$\forall \alpha \in C_0^\infty(\Lambda T^* M), \int_M |(d + \delta)\alpha|^2 = \int_M |\nabla \alpha|^2 + \int_{K_0} |\alpha|^2.$$ 

Hence, by standard elliptic estimates, the norm $N_D$ is equivalent to the norm

$$Q_D(\alpha) = \sqrt{\int_M |\nabla \alpha|^2 + \int_D |\alpha|^2}.$$ 

If $D$ and $D'$ are two connected bounded open set containing $K_0$, such that $D \subset D'$ then $Q_D \leq Q_{D'}$. Hence if $Q_D$ and $Q_{D'}$ are not equivalent there is a sequence $(\alpha_l)_{l \in \mathbb{N}} \in C_0^\infty(\Lambda T^* M)$, such that $Q_{D'}(\alpha_l) = 1$ whereas $\lim_{l \to \infty} Q_D(\alpha_l) = 0$.

This implies that the sequence $(\alpha_l)_{l \in \mathbb{N}}$ is bounded in $W^{1,2}(D')$ and $\lim_{l \to \infty} \|\nabla \alpha_l\|_{L^2(M)} = 0$. Hence we can extract a subsequence converging weakly in $W^{1,2}(D')$ and strongly in $L^2(\Lambda T^* D')$ to a $\alpha_\infty \in W^{1,2}(D')$. We can suppose this subsequence is $(\alpha_l)_{l}$. We must have $\nabla \alpha_\infty = 0$ and $\alpha_\infty = 0$ on $D$ and $\|\alpha_\infty\|_{L^2(D')} = 1$. This is impossible. Hence the two norms $Q_D$ and $Q_{D'}$ are equivalent.

Q.E.D

We have the corollary

Corollary 6.4. The inclusion $C_0^\infty \longrightarrow W^{1,2}_{loc}$ extends by continuity to a injection $W \longrightarrow W^{1,2}_{loc}$. 

We remark that the domain of the Gauss-Bonnet operator $D(d + \delta) = \{\alpha \in L^2, (d + \delta)\alpha \in L^2\}$ is in $W$. As a matter of fact, because $(M, g)$ is complete $D(d + \delta)$ is the completion of $C_0^\infty(\Lambda T^* M)$ equiped with the quadratic form

$$\alpha \mapsto \int_M |\alpha|^2 + \int_M |(d + \delta)\alpha|^2.$$ 

This norm is larger that the one used for defined $W$. Hence $D(d + \delta) \subset W$. As a corollary we get that a $L^2$ harmonic form is in $W$. The Visentini’s finiteness result will follow from:

Proposition 6.5. The operator $(d + \delta) : W \longrightarrow L^2$ is Fredholm. That is to say its kernel and its cokernel have finite dimension and its image is closed.
Proof. Let $A$ be the operator $(d + \delta)^2 + 1_D$, where

$$1_D(\alpha)(x) = \begin{cases} \alpha(x) & \text{if } x \in D \\ 0 & \text{if } x \notin D \end{cases}$$

We have

$$N_D(\alpha)^2 = \langle A\alpha, \alpha \rangle.$$ 

So the operator $A^{-1/2} = \int_0^\infty e^{-tA} \frac{dt}{\sqrt{\pi t}}$ realizes an isometry between $L^2$ and $W$. It is enough to show that the operator $(d + \delta)A^{-1/2} = B$ is Fredholm on $L^2$. But

$$B^*B = A^{-1/2}(d + \delta)^2A^{-1/2} = \text{Id} - A^{-1/2}1_D1_DA^{-1/2}.$$ 

The operator $1_DA^{-1/2}$ is the composition of the operator $A^{-1/2} : L^2 \rightarrow W$ then of the natural injection from $W$ to $W^{1,2}_{loc}$ and finally of the map $1_D$ from $W^{1,2}_{loc}$ to $L^2$. $D$ being a bounded set, this operator is a compact one by the Rellich compactness theorem. Hence $1_DA^{-1/2}$ is a compact operator. Hence, $B$ has a closed range and a finite dimensional kernel. So the operator $(d + \delta) : W \rightarrow L^2$ has a closed range and a finite kernel. But the cokernel of this operator is the orthogonal space to $(d + \delta)C_0^\infty(\Lambda T^*M)$ in $L^2$. Hence the cokernel of this operator is the $L^2$ kernel of the Gauss-Bonnet operator. We notice that this space is included in the space of the $W$ kernel of $(d + \delta)$. Hence it has finite dimension.

We also get the following corollary:

**Corollary 6.6.** There is a Green operator $G : W \rightarrow L^2$, such that

$$\text{on } L^2, \ (d + \delta)G = \text{Id} - P^{L^2}$$

where $P^{L^2}$ is the orthogonal projection on $\oplus H^k(M)$.

$$\text{On } W, \ G(d + \delta) = \text{Id} - P^W$$

where $P^W$ is the $W$ orthogonal projection on $\ker W(d + \delta)$.

Moreover, $\alpha \in Z_2^k(M)$ is $L^2$ cohomologous to zero if and only if there is a $\beta \in W(\Lambda^{k-1}T^*M)$ such that $\alpha = d\beta$.

6.2. **A long exact sequence.** In the DeRham cohomology, we have a long exact sequence linking the cohomology with compact support and the absolute cohomology. And this exact sequence is very useful to compute the DeRham cohomology groups. In $L^2$ cohomology, we can also define this sequence but generally it is not an exact sequence.

Let $\mathcal{O} \subset M$ be a bounded open subset, we can define the sequence:
Here \( H^k(M \setminus \mathcal{O}, \partial \mathcal{O}) = \{ h \in L^2(\Lambda^k T^* (M \setminus \mathcal{O})), \, dh = \delta h = 0 \, \text{and} \, i^* h = 0 \} \), where \( i : \partial \mathcal{O} \longrightarrow M \setminus \mathcal{O} \) is the inclusion map, and

- \( e \) is the extension by zero map: to \( h \in H^k(M \setminus \mathcal{O}, \partial \mathcal{O}) \) it associates the \( L^2 \) cohomology class of \( \hat{h} \), where \( \hat{h} = 0 \) on \( \mathcal{O} \) and \( \hat{h} = h \) on \( M \setminus \mathcal{O} \). It is well defined because of the Stokes formula:
  - if \( \beta \in C_\infty^0(\Lambda^{k+1} T^* M) \), then
  \[
  \langle \hat{h}, \delta \beta \rangle = \langle dh, \beta \rangle_{L^2(\mathcal{O})} - \int_{\partial \mathcal{O}} i^* h \wedge i^* \ast \beta = 0
  \]

- \( j^* \) is associated to the inclusion map \( j : \mathcal{O} \longrightarrow M \); to \( h \in H^k(M) \) it associates \([j^* h]\), the cohomology class of \( h|_\mathcal{O} \) in \( H^k(\mathcal{O}) \).
- \( b \) is the coboundary operator: if \([\alpha] \in H^k(\mathcal{O}) \), and if \( \bar{\alpha} \) is a smooth extension of \( \alpha \), with compact support, then \( b[\alpha] \) is the orthogonal projection of \( d\bar{\alpha} \) on \( H^{k+1}(M \setminus \mathcal{O}, \partial \mathcal{O}) \). The map \( b \) is well defined, that is to say, it does not depend on the choice of \( \alpha \) nor of its extension.

It is relatively easy to check that

\[ j^* \circ e = 0, \quad b \circ j^* = 0 \, \text{and} \, e \circ b = 0 ; \]

Hence we have the inclusion:

\[ \text{Im } e \subset \text{Ker } j^* , \text{ Im } j^* \subset \text{Ker } b \, \text{and} \, \text{Im } b \subset \text{Ker } e . \]

In [C1], we observed that

**Proposition 6.7.** The equality \( \text{ker } b = \text{Im } j^* \) always holds.

This comes from the long exact sequence in DeRham cohomology. Moreover, we have the following:

**Proposition 6.8.** On a manifold with flat ends, the equality \( \text{Im } b = \text{Ker } e \) always holds.

**Proof.** As a matter of fact, if \( h \in \text{Ker } e \) then by (6.6) we get a \( \beta \in W \), such that \( h = d\beta \) on \( M \). Hence \( h = b[\beta|_\mathcal{O}] \).

Q.E.D

The last fact requires more analysis:
Theorem 6.9. If \((M, g)\) is a complete manifold with flat ends and if for every end \(E\) of \(M\) we have

\[
\lim_{r \to \infty} \frac{\text{vol} E \cap B_x(r)}{r^2} = \infty,
\]

then the long sequence (6.1) is exact.

6.3. Hodge theorem for manifolds with flat ends. With the help of the geometric description of flat ends due to Eschenburg and Schroeder ([E-S], see also [G-P-Z]), we can compute the \(L^2\)-cohomology on flat ends. Then with the long sequence (6.1), we can give an answer to J. Dodziuk’s question; for sake of simplicity, we give here only the result for manifolds with one flat end.

Theorem 6.10. Let \((M^n, g)\) be a complete Riemannian manifold with one flat end \(E\). Then

1. If \((M^n, g)\) is parabolic, that is to say if the volume growth of geodesic ball is at most quadratic

\[
\lim_{r \to \infty} \frac{\text{vol} B_x(r)}{r^2} < \infty,
\]

then we have

\[
\mathcal{H}^k(M, g) \simeq \text{Im} \left( H^k_c(M) \to H^k(M) \right).
\]

2. If \((M^n, g)\) is non-parabolic (i.e. if \(\lim_{r \to \infty} \frac{\text{vol} B_x(r)}{r^2} = \infty\), then the boundary of \(E\) has a finite covering diffeomorphic to the product \(S^{n-1} \times T\) where \(T\) is a flat \((n-\nu)\)-torus. Let \(\pi : T \to \partial E\) the induced immersion, then

\[
\mathcal{H}^k(M, g) \simeq H^k(M \setminus E, \ker \pi^*),
\]

where \(H^k(M \setminus E, \ker \pi^*)\) is the cohomology associated to the subcomplex of differential forms on \(M \setminus E\) : \(\ker \pi^* = \{ \alpha \in C^\infty(\Lambda^\bullet T^\ast(M \setminus E)) \mid \pi^\ast \alpha = 0 \}\).

References


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Universite de Nantes, 44322 Nantes Cedex 02, FRANCE
E-mail address: Gilles.Carron@math.univ-nantes.fr