L^2 HARMONIC FORMS ON NON-COMPACT RIEMANNIAN MANIFOLDS

GILLES CARRON

First, I want to present some questions on L^2- harmonic forms on non-compact Riemannian manifolds. Second, I will present an answer to an old question of J. Dodziuk on L^2- harmonic forms on manifolds with flat ends. In fact some of the analytical tools presented here apply in other situations (see [C4]).

1. The space of harmonic forms

Let (M^n, g) be a complete Riemannian manifold. We denote by $\mathcal{H}^k(M, g)$ or $\mathcal{H}^k(M)$ its space of L^2-harmonic k-forms, that is to say the space of L^2 $k-$forms which are closed and coclosed:

$$\mathcal{H}^k(M) = \{\alpha \in L^2(\Lambda^k T^*M),\ d\alpha = \delta\alpha = 0\},$$

where

$$d\ :\ C_0^\infty(\Lambda^k T^*M) \longrightarrow C_0^\infty(\Lambda^{k+1}T^*M)$$

is the exterior differentiation operator and

$$\delta\ :\ C_0^\infty(\Lambda^{k+1}T^*M) \longrightarrow C_0^\infty(\Lambda^k T^*M)$$

its formal adjoint. The operator d does not depend on g but δ does; δ is defined with the formula:

$$\forall \alpha \in C_0^\infty(\Lambda^k T^*M),\ \forall \beta \in C_0^\infty(\Lambda^{k+1}T^*M),\ \int_M \langle d\alpha, \beta\rangle = \int_M \langle \alpha, \delta\beta\rangle.$$

The operator $(d + \delta)$ is elliptic hence the elements of $\mathcal{H}^k(M)$ are smooth and the L^2 condition is only a decay condition at infinity.

2. If the manifold M is compact without boundary

If M is compact without boundary, then these spaces have finite dimension, and we have the theorem of Hodge-DeRham : the spaces $\mathcal{H}^k(M)$ are isomorphic to the real cohomology groups of M:

$$\mathcal{H}^k(M) \simeq H^k(M, \mathbb{R}).$$

Hence the dimension of $\mathcal{H}^k(M)$ is a homotopy invariant of M, i.e. it does not depend on g. A corollary of this and of the Chern-Gauss-Bonnet formula is :

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \dim \mathcal{H}^k(M) = \int_M \Omega^g \,,$$

where Ω^g is the Euler form of (M^n, g) ; for instance if $\dim M = 2$ then $\Omega^g = \frac{KdA}{2\pi}$, where K is the Gaussian curvature and dA the area form.

3. What is true on a non-compact manifold

Almost nothing is true in general:

The space $\mathcal{H}^k(M, g)$ can have infinite dimension and the dimension, if finite, can depend on g. For instance, if M is connected we have

$$\mathcal{H}^0(M) = \{f \in L^2(M, d\mathrm{vol}_g), f = \text{constant } \}.$$

$$\begin{aligned} \text{Hence} \quad & \mathcal{H}^0(M) = \mathbb{R} \quad && \text{if } \mathrm{vol}\, M < \infty, \\ \text{and} \quad & \mathcal{H}^0(M) = \{0\} \quad && \text{if } \mathrm{vol}\, M = \infty. \end{aligned}$$

For instance if $\mathbb{R}^2$ is equipped with the euclidean metric, we have $\mathcal{H}^0(\mathbb{R}^2, \text{eucl}) = \{0\}$, and if $\mathbb{R}^2$ is equiped with the metric $g = dr^2 + r^2 e^{-2r} d\theta^2$ in polar coordinates, then $\mathcal{H}^0(\mathbb{R}^2, g) = \mathbb{R}$. We have also that $\mathcal{H}^k(\mathbb{R}^n, \text{eucl}) = \{0\}$, for any $k \leq n$. But if we consider the unit disk in $\mathbb{R}^2$ equipped with the hyperbolic metric $4|dz|^2/(1 - |z|^2)^2$ then it is isometric to the metric $g_1 = dr^2 + \sinh r^2 d\theta^2$ on $\mathbb{R}^2$. And then we have $\dim \mathcal{H}^1(\mathbb{R}^2, g_1) = \infty$.

As a matter of fact if $P(z) \in \mathbb{C}[z]$ is a polynomial, then $\alpha = P'(z)dz$ is a L^2 harmonic form on the unit disk for the hyperbolic metric (this comes from the conformal invariance, see 5.2).

So we get an injection $\mathbb{C}[z]/\mathbb{C} \to \mathcal{H}^1(\mathbb{R}^2, g_1)$. However, the spaces $\mathcal{H}^k(M, g)$ satisfy the following two properties:

- These spaces have a reduced L^2 cohomology interpretation:

Let $Z_2^k(M)$ be the kernel of the unbounded operator d acting on $L^2(\Lambda^k T^* M)$, or equivalently

$$Z_2^k(M) = \{\alpha \in L^2(\Lambda^k T^* M),\ d\alpha = 0\},$$

where the equation $d\alpha = 0$ has to be understood in the distribution sense i.e. $\alpha \in Z_2^k(M)$ if and only if

$$\forall \beta \in C_0^\infty(\Lambda^{k+1}T^*M),\ \int_M \langle \alpha, \delta\beta\rangle = 0\ .$$

That is to say $Z_2^k(M) = \left(\delta C_0^\infty(\Lambda^{k+1}T^*M)\right)^\perp$. The space $L^2(\Lambda^k T^*M)$ has the following Hodge-DeRham-Kodaira orthogonal decomposition

$$L^2(\Lambda^k T^*M) = \mathcal{H}^k(M) \oplus \overline{dC_0^\infty(\Lambda^{k-1}T^*M)} \oplus \overline{\delta C_0^\infty(\Lambda^{k+1}T^*M)},$$

where the closure is taken with respect to the L^2 topology. We also have

$$Z_2^k(M) = \mathcal{H}^k(M) \oplus \overline{dC_0^\infty(\Lambda^{k-1}T^*M)},$$

hence we have

$$\mathcal{H}^k(M) \simeq Z_2^k(M)/\overline{dC_0^\infty(\Lambda^{k-1}T^*M)}.$$

A corollary of this identification is the following:

Proposition 3.1. *The space $\mathcal{H}^k(M,g)$ are quasi-isometric invariant of (M,g). That is to say if g_1 and g_2 are two complete Riemannian metrics such that for a $C > 1$ we have*

$$C^{-1}g_1 \le g_2 \le Cg_1,$$

then $\mathcal{H}^k(M,g_1) \simeq \mathcal{H}^k(M,g_2)$.

In fact, the spaces $\mathcal{H}^k(M,g)$ are biLipschitz-homotopy invariants of (M,g).

• The finiteness of $\dim \mathcal{H}^k(M,g)$ depends only of the geometry of ends :

Theorem 3.2. *(J. Lott,* [L]*) The spaces of L^2-harmonic forms of two complete Riemannian manifolds, which are isometric outside some compact set, have simultaneously finite or infinite dimension.*

4. A general problem

In view of the Hodge-DeRham theorem and of J. Lott's result, we can ask the following questions :

(1) What geometrical condition on the ends of M insure the finiteness of the dimension of the spaces $\mathcal{H}^k(M)$?

Within a class of Riemannian manifold having the same geometry at infinity:

(2) What are the links of the spaces $\mathcal{H}^k(M)$ with the topology of M and with the geometry 'at infinity' of (M,g) ?

(3) And what kind of Chern-Gauss-Bonnet formula could we hope for the L^2- Euler characteristic

$$\chi_{L^2}(M)=\sum_{k=0}^{n}(-1)^k\dim\mathcal{H}^k(M)\quad ?$$

There are many articles dealing with these questions. I mention only three of them :

(1) In the pioneering article of Atiyah-Patodi-Singer ([A-P-S]), the authors considered manifolds with cylindrical ends : that is to say there is a compact subset K of M such that $M\setminus K$ is isometric to the Riemannian product $\partial K\times]0,\infty[$. Then they show that the dimension of the space of L^2-harmonic forms is finite ; and that these spaces are isomorphic to the image of the relative cohomology in the absolute cohomology. These results were used by Atiyah-Patodi-Singer in order to obtain a formula for the signature of compact manifolds with boundary.
(2) In [M, M-P], R. Mazzeo and R. Phillips give a cohomological interpretation of the space $\mathcal{H}^k(M)$ for geometrically finite real hyperbolic manifolds.
(3) The solution of the Zucker's conjecture by Saper and Stern ([S-S]) shows that the spaces of L^2 harmonic forms on hermitian locally symmetric space with finite volume are isomorphic to the middle intersection cohomology of the Borel-Serre compactification of the manifold.

5. An example

I want now to discuss the L^2 Gauss-Bonnet formula through one example. The sort of L^2 Gauss-Bonnet formula one might expect is a formula of the type

$$\chi_{L^2}(M)=\int_K\Omega^g+\text{terms depending only on }(M-K,g),$$

where $K\subset M$ is a compact subset of M; i.e. $\chi_{L^2}(M)$ is the sum of a local term $\int_K\Omega^g$ and of a boundary (at infinity) term. Such a result will imply a relative index formula :

If (M_1,g_1) and (M_2,g_2) are isometric outside compact set $K_i\subset M_i$, $i=1,2$, then

$$\chi_{L^2}(M_1)-\chi_{L^2}(M_2)=\int_{K_1}\Omega^{g_1}-\int_{K_2}\Omega^{g_2}.$$

It had been shown by Gromov-Lawson and Donnelly that when zero is not in the essential spectrum of the Gauss-Bonnet operator $d+\delta$ then this relative index formula is true ([G-L, Do]). For instance, by the work of Borel and Casselman [BC], the Gauss-Bonnet operator is a Fredholm operator if M is an even dimensional locally symmetric space of finite volume and negative curvature.

In fact such a relative formula is not true in general. The following counterexample is given in [C2]:

(M_1, g_1) is the disjoint union of two copies of the euclidean plane and (M_2, g_2) is two copies of the euclidean plane glued along a disk. As these surface are oriented with infinite volume, we have $i = 1, 2$:

$$\mathcal{H}^0(M_i, g_i) = \mathcal{H}^2(M_i, g_i) = \{0\}.$$

And we also have $\mathcal{H}^1(M_1, g_1) = \{0\}$. Moreover

Lemma 5.1. $\mathcal{H}^1(M_2, g_2) = \{0\}$.

This comes from the conformal invariance of this space. Indeed, it is a general fact:

Proposition 5.2. *If (M^n, g) is a Riemannian manifold of dimension $n = 2k$, and if $f \in C^\infty(M)$ then*

$$\mathcal{H}^k(M, g) = \mathcal{H}^k(M, e^{2f} g).$$

Proof.– As a matter of fact the two Hilbert spaces $L^2(\Lambda^k T^*M, g)$ and $L^2(\Lambda^k T^*M, e^{2f} g)$ are the same: if $\alpha \in \Lambda^k T^*_x M$, then

$$\|\alpha\|^2_{e^{2f}g}(x) = e^{-2kf(x)} \|\alpha\|^2_g(x)$$

and $d\mathrm{vol}_{e^{2f}g} = e^{-2kf} d\mathrm{vol}_g$.

We have

$$\mathcal{H}^k(M, e^{2f} g) = Z_2^k(M, e^{2f} g) \cap \overline{dC_0^\infty(\Lambda^{k-1} T^*M)}$$

$$\text{and } \mathcal{H}^k(M, g) = Z_2^k(M, g) \cap \overline{dC_0^\infty(\Lambda^{k-1} T^*M)}.$$

As the two Hilbert space $L^2(\Lambda^k T^*M, g)$ and $L^2(\Lambda^k T^*M, e^{2f} g)$ are the same, these two spaces are the same. **Q.E.D**

But (M_2, g_2) is conformally equivalent to the 2-sphere with two points removed.

A L^2 harmonic form on the 2-sphere with two points removed extends smoothly on the sphere.

The sphere has no non trivial L^2 harmonic 1-form, hence Lemma 5.1 follows.

The surfaces (M_1, g_1) and (M_2, g_2) are isometric outside some compact set but

$$\chi_{L^2}(M_1) - \int_{M_1} \frac{K_{g_1} dA_{g_1}}{2\pi} = 0 - 0 = 0$$

whereas

$$\chi_{L^2}(M_2) - \int_{M_2} \frac{K_{g_2} dA_{g_2}}{2\pi} = -\int_{M_2} \frac{K_{g_2} dA_{g_2}}{2\pi} = -2.$$

Hence the relative index formula is not true in general. A corollary of this argument is the following

Corollary 5.3. *If (S, g) is a complete surface with integrable Gaussian curvature, according to a theorem of A. Huber* [H], *we know that such a surface is conformally equivalent to a compact surface $\bar{S}$ with a finite number of points removed. Then*

$$\dim \mathcal{H}^1(S, g) = b_1(\bar{S}).$$

6. MANIFOLDS WITH FLAT ENDS

In (1982, [D]), J. Dodziuk asked the following question: according to Vesentini ([V]) if M is flat outside a compact set, the spaces $\mathcal{H}^k(M)$ are finite dimensional. Do they admit a topological interpretation ?

My aim is to present an answer to this question. For the detail, the reader may look at [C4]:

6.1. **Visentini's finiteness result.**

Theorem 6.1. *Let (M, g) be a complete Riemannian manifold such that for a compact set $K_0 \subset M$, the curvature of (M, g) vanishes on $M - K_0$. Then for every p*

$$\dim \mathcal{H}^p(M, g) < \infty.$$

We give here a proof of this result; this proof will furnish some analytical tools to answer J. Dodziuk's question.

We begin to define a Sobolev space adapted to our situation:

Definition 6.2. *Let D be a bounded open set containing K_0, and let $W_D(\Lambda T^*M)$ be the completion of $C_0^\infty(\Lambda T^*M)$ for the quadratic form*

$$\alpha \mapsto \int_D |\alpha|^2 + \int_M |(d+\delta)\alpha|^2 = N_D^2(\alpha).$$

Proposition 6.3. *The space W_D doesn't depend on D, that is to say if D and D' are two bounded open sets containing K_0, then the two norms N_D and $N_{D'}$ are equivalent.*

We write W for W_D.

Proof.– The proof goes by contradiction. We notice that with the Bochner-Weitzenböck formula:

$$\forall \alpha \in C_0^\infty(\Lambda T^*M),\ \int_M |(d+\delta)\alpha|^2 = \int_M |\nabla\alpha|^2 + \int_{K_0} |\alpha|^2.$$

Hence, by standard elliptic estimates, the norm N_D is equivalent to the norm

$$Q_D(\alpha) = \sqrt{\int_M |\nabla\alpha|^2 + \int_D |\alpha|^2}.$$

If D and D' are two connected bounded open set containing K_0, such that $D \subset D'$ then $Q_D \le Q_{D'}$. Hence if Q_D and $Q_{D'}$ are not equivalent there is a sequence $(\alpha_l)_{l\in\mathbb{N}} \in C_0^\infty(\Lambda T^*M)$, such that $Q_{D'}(\alpha_l) = 1$ whereas $\lim_{l\to\infty} Q_D(\alpha_l) = 0$.

This implies that the sequence $(\alpha_l)_{l\in\mathbb{N}}$ is bounded in $W^{1,2}(D')$ and $\lim_{l\to\infty} \|\nabla\alpha_l\|_{L^2(M)} = 0$. Hence we can extract a subsequence converging weakly in $W^{1,2}(D')$ and strongly in $L^2(\Lambda T^*D')$ to a $\alpha_\infty \in W^{1,2}(D')$. We can suppose this subsequence is $(\alpha_l)_l$.

We must have $\nabla\alpha_\infty = 0$ and $\alpha_\infty = 0$ on D and $\|\alpha_\infty\|_{L^2(D')} = 1$. This is impossible. Hence the two norms Q_D and $Q_{D'}$ are equivalent. **Q.E.D**

We have the corollary

Corollary 6.4. *The inclusion $C_0^\infty \longrightarrow W^{1,2}_{loc}$ extends by continuity to a injection $W \longrightarrow W^{1,2}_{loc}$.*

We remark that the domain of the Gauss-Bonnet operator $\mathcal{D}(d+\delta) = \{\alpha \in L^2,\ (d+\delta)\alpha \in L^2\}$ is in W. As a matter of fact, because (M,g) is complete $\mathcal{D}(d+\delta)$ is the completion of $C_0^\infty(\Lambda T^*M)$ equiped with the quadratic form

$$\alpha \mapsto \int_M |\alpha|^2 + \int_M |(d+\delta)\alpha|^2.$$

This norm is larger that the one used for defined W. Hence $\mathcal{D}(d+\delta) \subset W$. As a corollary we get that a L^2 harmonic form is in W. The Visentini's finiteness result will follow from:

Proposition 6.5. *The operator $(d+\delta) : W \longrightarrow L^2$ is Fredholm. That is to say its kernel and its cokernel have finite dimension and its image is closed.*

Proof.– Let A be the operator $(d+\delta)^2+1_D$, where

$$1_D(\alpha)(x)=\begin{cases}\alpha(x) & \text{if } x\in D\\ 0 & \text{if } x\notin D\end{cases}$$

We have

$$N_D(\alpha)^2=\langle A\alpha,\alpha\rangle.$$

So the operator $A^{-1/2}=\int_0^\infty e^{-tA}\frac{dt}{\sqrt{\pi t}}$ realizes an isometry between L^2 and W. It is enough to show that the operator $(d+\delta)A^{-1/2}=B$ is Fredholm on L^2. But

$$B^*B=A^{-1/2}(d+\delta)^2A^{-1/2}=\mathrm{Id}-A^{-1/2}1_D1_DA^{-1/2}.$$

The operator $1_DA^{-1/2}$ is the composition of the operator $A^{-1/2}$: $L^2\longrightarrow W$ then of the natural injection from W to $W^{1,2}_{loc}$ and finally of the map 1_D from $W^{1,2}_{loc}$ to L^2. D being a bounded set, this operator is a compact one by the Rellich compactness theorem. Hence $1_DA^{-1/2}$ is a compact operator. Hence, B has a closed range and a finite dimensional kernel. So the operator $(d+\delta):W\longrightarrow L^2$ has a closed range and a finite kernel. But the cokernel of this operator is the orthogonal space to $(d+\delta)C_0^\infty(\Lambda T^*M)$ in L^2. Hence the cokernel of this operator is the L^2 kernel of the Gauss-Bonnet operator. We notice that this space is included in the space of the W kernel of $(d+\delta)$. Hence it has finite dimension. **Q.E.D**

We also get the following corollary :

Corollary 6.6. *There is a Green operator $G:W\longrightarrow L^2$, such that*

$$\text{on } L^2,\ (d+\delta)G=\mathrm{Id}\,-P^{L^2}$$

where P^{L^2} is the orthogonal projection on $\oplus\mathcal{H}^k(M)$.

$$\text{On } W,\ G(d+\delta)=\mathrm{Id}\,-P^{W}$$

where P^W is the W orthogonal projection on $\ker_W(d+\delta)$.

Moreover, $\alpha\in Z_2^k(M)$ is L^2 cohomologous to zero if and only if there is a $\beta\in W(\Lambda^{k-1}T^*M)$ such that $\alpha=d\beta$.

6.2. **A long exact sequence.** In the DeRham cohomology, we have a long exact sequence linking the cohomology with compact support and the absolute cohomology. And this exact sequence is very useful to compute the DeRham cohomology groups. In L^2 cohomology, we can also define this sequence but generally it is not an exact sequence.

Let $\mathcal{O}\subset M$ be a bounded open subset, we can define the sequence:

(6.1)
$$\longrightarrow \mathcal{H}^k(M \setminus \mathcal{O}, \partial\mathcal{O}) \xrightarrow{\mathrm{e}} \mathcal{H}^k(M) \xrightarrow{\mathrm{j}^*} H^k(\mathcal{O}) \xrightarrow{\mathrm{b}} \mathcal{H}^{k+1}(M \setminus \mathcal{O}, \partial\mathcal{O}) \longrightarrow$$

Here

$$\mathcal{H}^k(M \setminus \mathcal{O}, \partial\mathcal{O}) = \{h \in L^2(\Lambda^k T^*(M \setminus \mathcal{O}),\ dh = \delta h = 0 \text{ and } \mathrm{i}^* h = 0\},$$

where $\mathrm{i} : \partial\mathcal{O} \longrightarrow M \setminus \mathcal{O}$ is the inclusion map, and

- e is the extension by zero map:
  to $h \in \mathcal{H}^k(M \setminus \mathcal{O}, \partial\mathcal{O})$ it associates the L^2 cohomology class of $\bar{h}$, where $\bar{h} = 0$ on $\mathcal{O}$ and $\bar{h} = h$ on $M \setminus \mathcal{O}$. It is well defined because of the Stokes formula:
  if $\beta \in C_0^\infty(\Lambda^{k+1} T^* M)$, then

$$\langle \bar{h}, \delta\beta \rangle = \langle dh, \beta \rangle_{L^2(\mathcal{O})} - \int_{\partial\mathcal{O}} i^* h \wedge i^* * \beta = 0$$

- j^* is associated to the inclusion map $j : \mathcal{O} \longrightarrow M$; to $h \in \mathcal{H}^k(M)$ it associates $[\mathrm{j}^* h]$, the cohomology class of $h|_{\mathcal{O}}$ in $H^k(\mathcal{O})$.
- b is the coboundary operator: if $[\alpha] \in H^k(\mathcal{O})$, and if $\bar{\alpha}$ is a smooth extension of α, with compact support, then $\mathrm{b}[\alpha]$ is the orthogonal projection of $d\bar{\alpha}$ on $\mathcal{H}^{k+1}(M \setminus \mathcal{O}, \partial\mathcal{O})$. The map b is well defined, that is to say, it does not depend on the choice of α nor of its extension.

It is relatively easy to check that

$$\mathrm{j}^* \circ \mathrm{e} = 0,\ \mathrm{b} \circ \mathrm{j}^* = 0 \text{ and } \mathrm{e} \circ \mathrm{b} = 0\ ;$$

Hence we have the inclusion:

$$\mathrm{Im}\ \mathrm{e} \subset \mathrm{Ker}\ \mathrm{j}^*,\ \mathrm{Im}\ \mathrm{j}^* \subset \mathrm{Ker}\ \mathrm{b} \text{ and } \mathrm{Im}\ \mathrm{b} \subset \mathrm{Ker}\ \mathrm{e}.$$

In [C1], we observed that

Proposition 6.7. *The equality* $\ker \mathrm{b} = \mathrm{Im}\ \mathrm{j}^*$ *always holds.*

This comes from the the long exact sequence in DeRham cohomology. Moreover, we have the following:

Proposition 6.8. *On a manifold with flat ends, the equality* $\mathrm{Im}\ \mathrm{b} = \mathrm{Ker}\ \mathrm{e}$ *always holds.*

Proof.– As a matter of fact, if $h \in \mathrm{Ker}\,\mathrm{e}$ then by (6.6) we get a $\beta \in W$, such that $h = d\beta$ on M. Hence $h = \mathrm{b}\,[\beta|_{\mathcal{O}}]$. **Q.E.D**

The last fact requires more analysis:

Theorem 6.9. *If (M,g) is a complete manifold with flat ends and if for every end E of M we have*

$$\lim_{r\to\infty} \frac{\operatorname{vol} E\cap B_x(r)}{r^2}=\infty,$$

then the long sequence (6.1) is exact.

6.3. **Hodge theorem for manifolds with flat ends.** With the help of the geometric description of flat ends due to Eschenburg and Schroeder ([E-S], see also [G-P-Z]), we can compute the L^2-cohomology on flat ends. Then with the long sequence (6.1), we can give an answer to J. Dodziuk's question; for sake of simplicity, we give here only the result for manifolds with one flat end.

Theorem 6.10. *Let (M^n,g) be a complete Riemannian manifold with one flat end E. Then*

(1) *If (M^n,g) is parabolic, that is to say if the volume growth of geodesic ball is at most quadratic*

$$\lim_{r\to\infty} \frac{\operatorname{vol} B_x(r)}{r^2}<\infty,$$

then we have

$$\mathcal{H}^k(M,g)\simeq \operatorname{Im}\left(H^k_c(M)\longrightarrow H^k(M)\right).$$

(2) *If (M^n,g) is non-parabolic (i.e. if $\lim_{r\to\infty}\frac{\operatorname{vol} B_x(r)}{r^2}=\infty$), then the boundary of E has a finite covering diffeomorphic to the product $S^{\nu-1}\times T$ where T is a flat $(n-\nu)$-torus. Let $\pi : T\longrightarrow \partial E$ the induced immersion, then*

$$\mathcal{H}^k(M,g)\simeq H^k(M\setminus E,\ker\pi^*),$$

where $H^k(M\setminus E,\ker\pi^*)$ is the cohomology associated to the subcomplex of differential forms on $M\setminus E$: $\ker\pi^*=\{\alpha\in C^\infty(\Lambda^\bullet T^*(M\setminus E)),\ \pi^*\alpha=0\}$.

References

[A-P-S] M.F. Atiyah, V.K. Patodi, I.M. Singer. Spectral asymmetry and Riemannian geometry I. *Math. Proc. Camb. Phil. Soc.***77** (1975), 43–69.

[BC] A. Borel and W. Casselman. L^2–Cohomology of Locally Symmetric manifolds of Finite Volume. *Duke Math. J. 50* (1983), 625–647.

[C1] G. Carron. Une suite exacte en L^2-cohomologie. *Duke Math. J.* **95**, n^{o}2 (1998), 343-372.

[C2] G. Carron. Un théorème de l'indice relatif. *Pacific J. of Math.* **198**, n^{o}1, (2001), 81–107.

[C3] G. Carron. Théorèmes de l'indice sur les variétés non-compactes. *J. Reine Angew. Math.*, **541**, (2001), 81–115.

[C4] G. Carron. L^2 cohomology of manifolds with flat ends. Preprint available at http://www.math.univ-nantes.fr/~carron

[D] J. Dodziuk. L^2-harmonic forms on complete manifolds. *Semin. differential geometry, Ann. Math. Stud.* **102** (1982), 291-302.

[Do] H. Donnelly, Essential spectrum and heat kernel. *J. Funct. Anal.*, **75** (1987), 362–381.

[E-S] J.H. Eschenburg, V. Schroeder. Riemannian manifolds with flat ends. *Math. Z.* **196** (1987), 573–589.

[G-P-Z] R.E. Greene, P. Petersen, S. Zhu. Riemannian manifolds of faster than quadratic curvature decay. *I.M.R.N.* **9** (1994), 363–377.

[G-L] M. Gromov, H.B. Lawson. Jr, Positive scalar curvature and the Dirac operator on a complete Riemannian manifold, *Publ. Math. I.H.E.S.* **58** (1983), 83–196.

[H] A. Huber, On subharmonic functions and differential geometry in the large, *Comment. Math. Helv.* **32** (1957), 13–72.

[L] J. Lott. L^2-cohomology of geometrically infinite hyperbolic 3-manifold. *Geom. Funct. Anal.* **7** (1997), 81–119.

[M] R. Mazzeo, The Hodge cohomology of a conformally compact metric, *J. Differential Geom.* **28** (1988) pp 309–339.

[M-P] R. Mazzeo, R.S. Phillips, Hodge theory on hyperbolic manifolds. *Duke Math. J.* **60** (1990) No.2, pp 509-559.

[S-S] L. Saper, M. Stern, L_2-cohomology of arithmetic varieties. *Ann. Math.* **132** (1990) No.1, pp 1-69.

[V] E. Visentini. Lectures on Levi convexity of complex manifolds and cohomology vanishing theorems. *Tata Institute of Fundamental Research*, Bombay (1967).

Universite de Nantes, 44322 Nantes Cedex 02, FRANCE

E-mail address: `Gilles.Carron@math.univ-nantes.fr`