RESONANCE QUANTUM SWITCH: MATCHING DOMAINS

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Abstract. Quantum Switch is one of most important elements of quantum networks, see, for instance [53, 54, 55]. In the present paper mathematical modeling of the Resonance Quantum Switch is reduced to the solution of the corresponding Scattering Problem on a network consisting of a quantum well and few quantum wires attached to it. In a simplest practical problem the roles of quantum wires are played by straight rectangular channels and the role of the quantum well is played by a circular domain, with the boundary having no flat pieces to match the rectangular channels. We suggest a procedure of matching the rectangular channels and a general convex domain with piecewise smooth boundary. The matching problem is reduced to some finite-dimensional equation with matrix-functions. Important spectral characteristics of the above scattering problem such as positions and life-times of resonances may be found numerically from the corresponding determinant condition or via asymptotic analysis — for sufficiently narrow channels.

1. Introduction

Central problems of mathematical design of quantum electronic devices were formulated, based on Landauer formula [5], in terms of quantum scattering on networks in the beginning of nineties, see [12, 11]. Interference of wave-functions and resonance phenomena on networks were studied in [36]. Nevertheless practical design of quantum electronic devices, beginning from the classical Esaki diode up to modern devices, see for instance [51], [52] was based on the resonance of energy levels rather than on resonance properties of the corresponding wave functions. At the same time modern experimental technique already permits to observe resonance effects caused by details of the shape of the resonance wave functions, see [48], [49], [50].

In actual paper we develop the mathematical idea of the resonance manipulation of the quantum current which is based on an observation from [13]: Scattering Matrix of a resonator with an opening depends on values of the relevant eigenfunctions of the inner problem on the opening. This idea was already used to construct solvable models of a three-terminal Quantum Switch RQS-3 (for triadic logic) - the task formulated by Professor G. Metakides and Doctor R.Compano (Industrial Department of the European Comission) as a work-package of EU ESPRIT Project 28890 NTCONGS, see the papers and reports [16, 17, 18, 19, 20] published in this connection. Main idea of the Resonance Quantum Switch presented in [17] is based on using of the resonance eigenfunction of the Quantum Well as a main working detail of the switch, see also the preliminary paper [16],where the solvable model of the Switch is discussed. In [21] more realistic scattering problem is considered for the Resonance Quantum Switch based on a deep quantum well, modeled by a circular domain with the Dirichlet boundary condition on the boundary. The corresponding scattering matrix describing the switching process is presented in [21] via the DN-map of the corresponding *modified domain* which already has flat pieces on the boundary matching the rectangular channels.

In actual paper we describe the procedure of construction of the Green function of the modified domain based on the spectral characteristics of the convex original domain, thus accomplishing the analysis developed in [21]. Similarly the Poisson kernel, the DN-map and other spectral characteristics of the Schrödinger operator on the modified domain may be constructed.

Actual paper is supplied with two Appendices. In the first Appendix the general properties of the DN-map are described. In the second Appendix an approximate calculation of the basic characteristics of the Resonance Quantum Switch is done following the previous joint paper [21]. The text of the Appendix 2 contains corrections of some essential details of the text [21] .The approximate estimation of the position of the resonance arising from the resonance eigenvalue was done thanks to an essential help from A. Mikhailova, who actually supplied the author with a draft of her own paper (in preparation).

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2. Basic Scattering Problem and DN-map.

Consider a network formed on a base of a quantum well - a convex domain Ω_0 with a piecewise smooth boundary and a few [1] straight rectangular semi-infinite quantum wires $\Omega_s,\ s = 1, 2, \dots n$, width δ_s, $s = 1, 2, \dots$ attached to Ω_0. We assume here that the effective potential on the wires lies below the Fermi level E_f and there exists the *resonance energy level* $E_0 = E_f$ on the well embedded into the continuous spectrum of the Schrödinger operator on the whole network [2]. We consider a single act of an electron's transmission from the incoming wire to one of terminals as a scattering process in this quantum network. Mathematically the corresponding scattering problem may be studied in course of construction of spectral characteristics of the Schrödinger operator on the whole $2 - d$ space :

$$-\frac{\hbar^2}{2m} \triangle u + V(x)u = Eu$$

with the potential $V(x)$ equal to zero outside the network and asymptotically (at infinity) equal to the constant V_2 which lies below the Fermi level E_f in the wires, [6]. The potential on the well may be specially selected to ensure a possibility of manipulation quantum current across the quantum well, see for instance [17, 21] where the potential on the well Ω_0 is chosen as $V_0(x) = \mathcal{E}e\langle x, \nu\rangle + V_0 < 0$. This assumption corresponds to the constant (macroscopic) electric field applied to the device. The direction ν of the field (and hence the transmission through the device) is manipulated by the rotation of the unit vector ν in the plane parallel to the device. In actual paper we assume that the De-Broghlie wavelength in the wires is greater than the radius of the well, which minimizes de-coherence in the corresponding scattering process. We assume also that the Fermi level in the wires lies deep enough so that we may replace the original spectral problem on the whole space by the spectral problem on the network $\Omega = \Omega_0 \cup \Omega_1 \cup \dots$ with homogeneous Dirichlet boundary condition on the border $\Gamma = \partial\Omega$. We consider further the non-dimensional spectral problem. For the macroscopic electric field the basic domain may be reduced by retraction $x = R\xi$,to the unit disc,with proportional transformation of the wires

[1] If no special indications are given, we consider below the most interesting case of a triadic switch with one input wire and three terminals, that is $n = 4$

[2] We use here the term suggested in [42].

$\Omega_s \to \Omega_s^\xi,\ s = 0,\, 1,\, \ldots$ and the equation

$$-\triangle_\xi \psi + \epsilon\langle \xi,\, \nu\rangle \psi + v_0 \psi = \lambda \psi,\ \xi \in \Omega_0^\xi,$$
$$-\triangle_\xi \psi + v_s \psi = \lambda\psi,\ \xi \in \Omega_s^\xi,\ s = 1, 2, \ldots,$$
$$\psi\big|_{\partial\Omega^\xi} = 0.$$

Here we use the notations $v_0 = \frac{2mR^2}{\hbar^2}(V_0 - V_2),\ \lambda = \frac{2mR^2}{\hbar^2}(E - V_2),\ \epsilon = \mathcal{E} e \frac{2mR^3}{\hbar^2}$. The re-normalized width of each wire is equal to $\delta = \frac{\delta_s}{R},\ s = 1, 2, \ldots$. From now on we assume that there are four wires : the wire Ω_1 which will play the role of the input wire and the wires $\Omega_2,\, \Omega_3,\, \Omega_4$ which will play roles of terminals. The re-normalized potential V on the whole network is a real piece-wise continuous function and the potential on the wires is equal to zero, $v_s(\xi) = 0$. We are aimed now to the corresponding scattering problem, for the differential operator in the retracted composite domain $\Omega_0,\, \Omega_1,\, \ldots$ [3]:

$$-\triangle u + V(x) u = \lambda u,\ x \in \Omega,$$

with zero boundary conditions on the boundary $\Gamma = \partial\Omega$.

$$u|_\Gamma = 0.$$

The transmission through the switch and, generally, the corresponding scattering matrix may be calculated in terms of the Dirichlet-to-Neumann map of the corresponding Quantum Well. The Dirichlet-to-Neumann map (DN-map) Λ or Lyapunoff map, see [1, 2, 31, 7, 10] and the appendix below, for the boundary problem in a compact domain $\Omega \subset R_2$ with the piecewise C_2-smooth boundary Γ and no inner angles :

$$-\triangle u + V(x) u = \lambda u,$$
$$u|_\Gamma := u_\Gamma$$

is defined as a (linear) transformation of boundary values of the solution into the boundary values of the normal component of the corresponding current $\bigtriangledown_n u|_\Gamma$ (in the direction of the outer normal n).

For generalized solutions of the above boundary problem from the Sobolev class $W_2^2(\Omega_0)$ with the smooth boundary Γ_0 the corresponding DN-map Λ is a linear operator acting from the Sobolev class $W_2^{3/2}(\Gamma_0)$ onto $W_2^{1/2}(\Gamma_0)$, see [35] and for the generalized solutions from $W_2^1(\Omega_0)$ it is a linear operator from the Sobolev class $W_2^{1/2}(\Gamma_0)$ onto $W_2^{-1/2}(\Gamma_0)$. Generally due to the M. Riesz interpolation theorem for solutions from $W_2^{1+\beta}(\Omega_0)$ it is a linear operator from $W_2^{1/2+\beta}(\Gamma_0)$ onto $W_2^{\beta-1/2}(\Gamma_0)$.

[3] From this moment we rename the retracted variable $\xi \to x$ and denote Ω^ξ by Ω again.

The DN-map exists at any regular point λ of the corresponding operator L_0 in $L_2(\Omega_0)$ with zero boundary condition on the boundary and may be considered as an operator from $W_2^{\beta+1/2}$ onto $W_2^{\beta-1/2}$, $0 \leq \beta \leq 1$. In particular the bi-linear form $\langle \Lambda u_{\Gamma_0}, v_{\Gamma_0} \rangle$ on each pair of elements $u_{\Gamma_0}, v_{\Gamma_0} \in W_2^{\beta+1/2}$, is a meromorphic function of the spectral parameter λ on any compact domain of the complex plane with possible poles at the eigenvalues of the Dirichlet problem for the operator L_0

$$Lu = - \triangle u + V(x)u = \lambda u, \ x \in \Omega_0,$$

$$u|_{\Gamma_0} = 0.$$

Our analysis is based on an explicit formula connecting the scattering matrix of the operator L in the composite domain $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2 \dots$ with the Dirichlet-to-Neumann map of the basic domain. On the first stage, in this section, we consider the *modified* basic domain $\hat{\Omega}_0$

$$\hat{\Omega}_0 = \Omega_0 \backslash \left\{ \cup_{i=1}^4 \Omega_i \right\}$$

which has flat pieces on the boundary matching the bottom sections δ_i of the wires Ω_i, $i = 1, 2, 3, 4$. We aim first on the formula connecting the Scattering Matrix on the composite domain with the DN map $\hat{\Lambda}$ of the modified domain.

The spectrum of the above Schrödinger operator L in the space of all square integrable function on the composite domain $L_2(\Omega)$, see for instance [39], consists of a countable set of absolutely-continuous branches $[(\frac{l\pi}{\delta})^2, \infty]$, $l = 1, 2, \dots$, each of them multiplicity $N = 4$, and possibly a finite number of eigenvalues λ_r of a finite multiplicity below the first threshold $\frac{\pi^2}{\delta^2}$. There may be also some embedded eigenvalues. The eigenvalues below the first threshold are not involved into the process of quantum conductivity, but the embedded eigenvalues, see for instance [37] and [38], may be transformed to resonances by small perturbations. We shall consider this transformation in the next section in connection with the resonance eigenvalue on the basic domain. The total multiplicity of the absolutely-continuous spectrum is (countably) infinite. The eigenfunctions of the absolutely-continuous spectrum are so-called scattered waves, see [33]. These are solutions Ψ of the homogeneous equation $L\Psi = \lambda\Psi$, which fulfill some (asymptotic) conditions.Generally, the scattered wave *incident with the plane wave incoming from the channel* Ω_t has the components Ψ_{st}, $s = 1, 2, 3, 4$, in channels Ω_s. They are represented on the first spectral band $(\frac{\pi}{\delta})^2 < \lambda < (\frac{2\pi}{\delta})^2$ as

$$\Psi_{tt}(x,y) = \sqrt{\frac{2}{|\delta|}}\sin\frac{\pi y}{|\delta|}e^{-i\sqrt{\lambda-(\frac{\pi}{|\delta|})^2}\,x} + S^1_{tt}\sqrt{\frac{2}{|\delta|}}\sin\frac{\pi y}{|\delta|}e^{i\sqrt{\lambda-(\frac{\pi}{|\delta|})^2}\,x} +$$

$$+\sum_{l>1} s^l_{tt}\sqrt{\frac{2}{|\delta|}}\sin\frac{l\pi y}{|\delta|}e^{-\sqrt{(\frac{l\pi}{|\delta|})^2-\lambda}\,x}, x,y \in \Omega_s \ ,$$

$$\Psi_{st}(x,y) = S^1_{st}\sqrt{\frac{2}{|\delta|}}\sin\frac{\pi y}{|\delta|}e^{i\sqrt{\lambda-(\frac{\pi}{|\delta|})^2}\,x}$$

$$+\sum_{l>1} s^l_{st}\sqrt{\frac{2}{|\delta_s|}}\sin\frac{l\pi y}{|\delta|}e^{-\sqrt{(\frac{l\pi}{|\delta_s|})^2-\lambda}\,x}, \ \ (x,y)\in\Omega_s, \ s\neq t \ . \tag{1}$$

The coefficients S^1_{st}, $s \neq t$, $x,y \in \Omega_s, x,y \in \Omega_s$, $s \neq t$ in front of the oscillating exponentials are elements of the *Scattering Matrix* on the first spectral band below the second threshold $(\frac{2\pi}{\delta})^2$, and s^l_{st} are just the amplitudes of exponentially decreasing modes. If λ sits on the upper spectral band, then higher oscillating modes are present. If not specified, we assume below that the incident wave is incoming from the first wire, $t = 1$, and use the notations Ψ_s instead of Ψ_{s1} for components of the scattered wave in the wires.

To construct the scattered wave on the first spectral band one must find the solution Ψ_0 of the Schrödinger equation on the modified basic domain $\hat{\Omega}_0$ which satisfies the matching conditions on the bottom chords δ_s of the wave-guides Ω_t with scattered waves Ψ_s on the wires:

$$\Psi_0 - \Psi_s|_{\delta_s} = 0, \ \frac{\partial}{\partial n}(\Psi_0 - \Psi_s)|_{\delta_s} = 0. \tag{2}$$

Based on results of the next section we will suggest a convenient integral equation which is equivalent to this boundary problem. Now we assume that all DN-maps on the modified domain $\hat{\Omega}_0$ and the channels Ω_s, $s = 1,2,3,4$ are already constructed. Then we derive an important formula for the 4×4 Scattering matrix on the first spectral band (below the second threshold $\lambda = \frac{4\pi^2}{|\delta|^2}$) and similar formulae for higher spectral bands, connecting it with the DN - map $\hat{\Lambda}_0$ of the Schrödinger operator $\hat{L}$ in $L_2(\hat{\Omega}_0)$ on the modified basic domain $\hat{\Omega}_0$.

Note that the restriction onto δ_s of the DN-map $\hat{\Lambda}_0$ is connected to the restriction $\hat{\mathcal{P}}_{\delta_s\ \gamma_s}$ of the Poisson map $\hat{\mathcal{P}}$ in $\hat{\Omega}_0$ onto δ_s:

$$\hat{\Lambda}_0(x,y)\big|_{x\in\delta_s,\,y\in\hat{\Gamma}} = \frac{\partial}{\partial n_{\delta_s}}\hat{\mathcal{P}}_\lambda(x,y)\big|_{x\in\delta_s,y\in\hat{\Gamma}} = -\frac{\partial}{\partial n_x \partial n_y}\hat{G}_\lambda(x,y)\big|_{x\in\delta_s,\,y\in\hat{\Gamma}},$$

see Appendix 1, and hence is uniquely defined for all values of the spectral parameter outside of the discrete spectrum of the operator $\hat{L}_0$.

In fact the scattering matrix is a *sub-matrix* of an infinite *parent S-matrix* $\mathbf{S}$ which is defined by the matrix elements $S_{st},\ s_{st}$ introduced above[4]. Consider the normalized eigenfunctions $e_s^l = \sqrt{\frac{2}{\delta}}\sin\frac{\pi l y}{\delta}$ of the transversal part of Laplacian on the cross-sections of the wave-guides Ω_s . Then the matrix $\mathbf{S}$ on the first spectral band $(\frac{\pi}{\delta})^2 < \lambda < 4(\frac{\pi}{\delta})^2$ is a matrix of an operator in the channel space $H = \oplus\sum_{s=1}^4 L_2(\delta_s)$ defined as

$$\mathbf{S}(\lambda) = \sum_{s,t=1}^{4} S_{st}^1 e_s^1\rangle\langle e_t^1 + \sum_{s,t=1}^{4}\sum_{l=2}^{\infty} s_{st}^l e_s^l\rangle\langle e_t^1 := \mathcal{S}^1(\lambda) + \mathbf{s}^1(\lambda). \tag{3}$$

The part $\mathcal{S}^1(\lambda)$ of the parent S-matrix $\mathbf{S}(\lambda)$ in the *open* first channel (below the second threshold) is exactly the Scattering Matrix on the first spectral band : for any *incident* linear combination of incoming modes from the first spectral band $\sum_s \psi_s e_s^1 e^{-i\sqrt{\lambda-(\frac{\pi}{|\delta|})^2}\,x} := \Psi_{in} e^{-i\sqrt{\lambda-(\frac{\pi}{|\delta|})^2}\,x}$ the corresponding scattered wave is constructed in form

$$\Psi = \Psi_{in} e^{-i\sqrt{\lambda-(\frac{\pi}{|\delta|})^2}\,x} + \mathcal{S}^1(\lambda)\Psi_{in} e^{i\sqrt{\lambda-(\frac{\pi}{|\delta|})^2}\,x} + \mathbf{s}^1(\lambda)\Psi_{in}$$

where the last term is exponentially decreasing at infinity if λ is below the second threshold: $\mathbf{s}^1(\lambda) = \sum_{l=2}^{\infty} e^{-\sqrt{(\frac{l\pi}{|\delta|})^2-\lambda}\,x} s_{st}^l e_s^l >< e_t^l$. Both operators $\mathcal{S}^1(\lambda),\ \mathbf{s}^1(\lambda)$ act in $L_2(\delta) = \oplus\sum_{s=1}^4 L_2(\delta_s)$ as matrices with elements $S_{st},\ s_{st}$ which are uniquely defined from the condition of matching of components of the scattered wave Ψ in the wires Ω_s with the solution of the Schrödinger equation on the basic domain $\hat{\Omega}_0$.

To describe the matching conditions in convenient form for general case when several channels are open $(\frac{\pi}{|\delta|})^2 < \lambda < (m+1)^2(\frac{\pi}{|\delta|})^2$ [5],

[4]Convenience of this infinite analytic matrix-function of the spectral parameter was explained to the author by Professor D.P. Kouzov,[58].

[5]We shall deliberately use the term *channel* both for the domains Ω_s, $s = 1,2,3,4$, and for the invariant subspaces of the Laplacian on them. Still we hope that any confusion is avoided, since the term is used for subspaces with adjectives "open" or "closed".

we consider the decomposition of the space $L_2(\delta) = H$ of the square-integrable functions on the cross-sections into an orthogonal sum of the channel-spaces $H = H_+ \oplus H_-$ which correspond to the cross-sections of *open* ($l \leq m$) and *closed* ($l \geq m+1$) channels :

$$H_{+,s} = \bigvee_{l=1}^{l=m} e_s^l, \ \ \sum_{s=1}^{4} H_{+,s} := H_+,$$

$$H_{-,s} = \bigvee_{l=m+1}^{l=\infty} e_s^l, \ \sum_{s=1}^{4} H_{-,s} := H_-$$

There are two types of oscillating exponential modes $e^{\pm\, i\sqrt{\lambda - (\frac{l\pi}{|\delta|})^2}\, x} e_s^l$ in the open channels and only one type of a bounded exponential mode $e^{-\sqrt{(\frac{l\pi}{|\delta|})^2 - \lambda}\, x} e_s^l$ in the closed channels. These modes fulfill inside the wires the Helmholtz equation with the spectral parameter λ and the Dirichlet boundary conditions on the walls $y = 0, \delta$. We introduce the diagonal operators $\mathcal{K}_s = \oplus \sum_{l=1}^{\infty} i\sqrt{\lambda - \frac{l^2\pi^2}{|\delta|^2}} e_s^l\rangle\langle e_s^l$ and $\mathcal{K} = \sum_{s=1}^{4} \mathcal{K}_s$. The operator $\mathcal{K}$ for complex values of the spectral parameter, $\Im\sqrt{\lambda - \frac{\pi^2 l^2}{|\delta|^2}} > 0$, plays the role of the Dirichlet-to-Neumann map on the infinite domain $\cup_{i=1}^{4}\Omega_i$. It acts on the channel space H and fulfills the condition $\Im K > 0$ on the spectral sheet of the spectral parameter λ with a cut along the spectrum. The limit values of it on the real axis of the spectral parameter exist in each channel and may be decomposed into an orthogonal sum of two parts $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$, $\mathcal{K}_s = \mathcal{K}_{+,s} \oplus \mathcal{K}_{-,s}$ which correspond to the open and closed channels respectively :

$$\mathcal{K}_{+,s} = \oplus \sum_{l=1}^{m} i\sqrt{\lambda - \frac{l^2\pi^2}{|\delta|^2}}\ \ e_s^l\rangle\langle e_s^l\ ,\ \ \mathcal{K}_+ = \sum_{s=1}^{m} \mathcal{K}_{+,s},$$

$$\mathcal{K}_{-,s} = \oplus \sum_{l=m+1}^{\infty} i\sqrt{\lambda - \frac{l^2\pi^2}{|\delta|^2}}\ \ e_s^l\rangle\langle e_s^l\ ,\ \ \mathcal{K}_- = \sum_{s=m+1}^{\infty} \mathcal{K}_{-,s}.$$

Both operators $\mathcal{K}_\pm$ have bounded inverse in H, if λ does not coincide with thresholds. The operator $\mathcal{K}_-^{-1}(\lambda')$ is small for given $\lambda\ \frac{\pi^2}{|\delta|^2} < \lambda' << \frac{\pi^2(m+1)^2}{|\delta|^2}$ if $m >> 1$ and acts as an operator from $W_2^{\beta-1/2}(\delta)$ into $W_2^{\beta+1/2}(\delta)$. It is self-adjoint and negative on real axis of the spectral parameter λ below the threshold $\frac{\pi^2(m+1)^2}{|\delta|^2}$. The operator $\mathcal{K}_+$ is anti-hermitian (the operator $i\mathcal{K}_+$ is self-adjoint).

One may consider a similar decomposition for the DN-map $\hat{\Lambda}$ of the modified domain $\hat{\Omega}_0$, restricted onto the cross-sections of the channels $\hat{\Lambda}|_H = \hat{\Lambda}|_{H_+} \oplus \hat{\Lambda}|_{H_-} := \hat{\Lambda}_+ \oplus \hat{\Lambda}_-$. The restrictions of the eigenfunctions e_s^l on the bottom cross-sections δ_s of the channels Ω_s may be continued by zero values from $\delta = \cup_{s=1}^4 \delta_s$ onto the complement $\partial\hat{\Omega}_0 \setminus \delta$. The resulting functions lie in $W_2^{\beta+1/2}(\partial\hat{\Omega}_0)$, with any $\beta < 1$. We denote the continued functions by e_s^l as well. They belong to the domain of the properly defined in H operators $\hat{\Lambda},\ \mathcal{K}$. This fact permits to consider restriction of the DN-map $\hat{\Lambda}$ onto the channel space H as densely-defined unbounded operator in H. In particular for real values of the spectral variable λ one may consider the corresponding symmetric operator enclosed by the orthogonal projections [6] P_H acting from $L_2(\partial\hat{\Omega}_0)$ onto H:

$$P_H \hat{\Lambda}|_H.$$

It is defined in H on the linear variety of all elements from $W_2^{\beta+1/2}(\delta)$ and takes values in $W_2^{\beta-1/2}(\delta)$ on the bottom section. We may consider the representation of it by a Hermitian matrix in the above orthogonal decomposition $H = H_+ \oplus H_-,\ P_H = P_+ \oplus P_-$:

$$P_H \hat{\Lambda}|_H = P_+ \hat{\Lambda} P_+ + P_+ \hat{\Lambda} P_- + P_- \hat{\Lambda} P_+ + P_- \hat{\Lambda} P_- :=$$
$$\hat{\Lambda}_{++} + \hat{\Lambda}_{+-} + \hat{\Lambda}_{-+} + \hat{\Lambda}_{--} =$$
$$\begin{pmatrix} \hat{\Lambda}_{++} & \hat{\Lambda}_{+-} \\ \hat{\Lambda}_{-+} & \hat{\Lambda}_{--} \end{pmatrix}.$$

The following statement is actually slightly modified version of a similar statement announced in [21].

Theorem 2.1. *The Scattering Matrix $\mathcal{S}$ is a contracting matrix-function on the spectral sheet of the variable λ and it may be represented via components of the DN-map $P_H \hat{\Lambda} P_H$ of the modified domain as*

$$\mathcal{S} = -\left\{\hat{\Lambda}_{++} - \hat{\Lambda}_{+-}\left[\hat{\Lambda}_{--} - \mathcal{K}_-\right]^{-1}\hat{\Lambda}_{-+} - \mathcal{K}_+\right\}^{-1} \times \left\{\hat{\Lambda}_{++} - \hat{\Lambda}_{+-}\left[\hat{\Lambda}_{--} - \mathcal{K}_-\right]^{-1}\hat{\Lambda}_{-+} + \mathcal{K}_+\right\}. \tag{4}$$

Here the operator $\hat{\Lambda}_{++} - \hat{\Lambda}_{+-}\left[\hat{\Lambda}_{--} - \mathcal{K}_-\right]^{-1}\hat{\Lambda}_{-+} := \mathcal{D}(\lambda)$ is an operator-function with a positive imaginary part in the lower half-plane $\Im\lambda < 0$ and negative imaginary part in the upper half-plane $\Im\lambda > 0$. It is a

[6] The orthogonal projections act in $L_2(\partial\hat{\Omega}_0)$ as multiplications by the indicators ξ_s of $\delta_s \subset \partial\hat{\Omega}_0$.

finite-dimensional hermitian matrix - function in H_+ on the interval $\left(\frac{\pi^2}{\delta^2},(m+1)^2\frac{\pi^2}{\delta^2}\right)$ of the real axis of the spectral parameter λ, outside of the spectrum of $\hat{L}$. The operator-functions $\mathcal{K}_\pm$ are analytic with positive imaginary parts in the upper half-plane $\Im\lambda > 0$, and negative imaginary part in the lower half-plane, $\mathcal{K}_+$ is anti-hermitian and $\mathcal{K}_-$ is real negative on the interval $\left(\frac{\pi^2}{\delta^2},(m+1)^2\frac{\pi^2}{\delta^2}\right)$ of the real axis λ below the $m+1$ threshold and the whole matrix $\mathcal{S}$ is unitary in H_+ on the spectrum of the operator L below the $m+1$ threshold.

Proof Consider the *background* scattering problem for the Schrödinger operator on the channels Ω_t with zero boundary conditions on the boundary $\partial\Omega_t$, $t=1,2,3,4$. Components of the corresponding scattered waves on the open channel H_l may be presented as

$$\Psi_t^0 = \psi_t^{0l} e_t^l \left[e^{-i\sqrt{\lambda - \frac{\pi^2 l^2}{|\delta|^2}}x} - e^{i\sqrt{\lambda - \frac{\pi^2 l^2}{|\delta|^2}}x} \right],\ l < m+1,$$

and the whole background scattered wave is

$$\Psi^0(x) = \left[e^{-\mathcal{K}x} - e^{\mathcal{K}x}\right]\vec{\psi}^0,$$

where $\vec{\psi}_0 = \sum_{t,l}\psi_t^{0l}e_t^l$ defines the amplitude of the incident wave in each channel Ω_t, $t=1,2,3,4$. The above functions Ψ^0 may play a role of eigenfunctions of the absolutely-continuous spectrum of the Laplacian in $\oplus\sum_{s=1}^4 L_2(\Omega_s)$. The components on the wires of the *perturbed* eigenfunctions Ψ incident by the same amplitude may be presented as linear combinations of the non-perturbed wave Ψ_t^0 and *outgoing*[7] exponential solution combined of both oscillating $l < m+1$ and exponentially decreasing components, in *all* channels :

$$\Psi = \left[e^{-\mathcal{K}x} - e^{\mathcal{K}x}\right]\vec{\psi}_0 + \sum_{l,l'<m+1;t,t'} e_t^l e^{i\sqrt{\lambda - \frac{\pi^2 l^2}{|\delta|^2}}\,x}\mathcal{T}_{ll'}^{tt'}\psi_{t'}^{0l'} +$$

$$\sum_{l\geq m+1;t,t'} e_t^l e^{i\sqrt{\lambda - \frac{\pi^2 l^2}{|\delta|^2}}\,x} s_{ll'}^{tt'}\psi_{t'}^{0l'} :=$$

$$\left[e^{-\mathcal{K}x} - e^{\mathcal{K}x}\right]\vec{\psi}_0 + e^{\mathcal{K}x}\mathcal{T}\vec{\psi}_0. \tag{5}$$

[7]Following [46] we call the solution of the homogeneous equation *outgoing* if it may be analytically continued as a bounded function onto the spectral plane λ.

The operator $\mathcal{T}$ in H — so-called T-matrix [8] - introduced above by (5) is connected to the parent scattering matrix $\mathbf{S}$ by the formula:

$$\mathbf{S} = \mathcal{S} + \mathbf{s} = -I + \mathcal{T}$$

and to S-matrix by the formula $\mathcal{S} = -I_+ + \mathcal{T}_+$, where the index $+$ indicates to the corresponding space H_+. The matching condition of the scattered wave on the bottom cross-sections δ_t of the wires $\Omega_t,\ t = 1, 2, 3, 4$ with the solution Ψ_0 of the Schrödinger equation inside the modified domain $\hat{\Omega}_0$ gives the following equation :

$$\hat{\Lambda}\mathcal{T}|_\delta \vec{\psi}_0 = \frac{\partial}{\partial n_\delta}\Psi_0 =$$

$$\mathcal{K}\left[-2 + \mathcal{T}\right]\vec{\psi}_0, \tag{6}$$

which may be transformed into operator equation by cancelling the arbitrary incident vector $\vec{\psi}_0 \in H_+$. One can rewrite this equation in form :

$$\mathcal{T} = -\left[\hat{\Lambda} - \mathcal{K}\right]^{-1} 2\mathcal{K}. \tag{7}$$

After framing the result by the projections P_+ onto the open channels we obtain

$$\mathcal{T}_+ = -P_+ \frac{2}{\hat{\Lambda} - \mathcal{K}} P_+ \mathcal{K}_+$$

or

$$\mathbf{S} = -\left[\hat{\Lambda} - \mathcal{K}\right]^{-1}\left[\hat{\Lambda} + \mathcal{K}\right].$$

Using the negativity of the imaginary part of the denominator in the upper half-plane $\Im\lambda > 0$ one may conclude that the parent S-matrix exists and is a contraction in the upper half-plane. Similarly one may check that it is an analytic contracting function in the lower half-plane of the spectral plane λ. Now the Scattering Matrix may be obtained as $\mathcal{S} = P_+\mathbf{S}P_+$. Hence is a contraction in the upper half-plane too. One may derive from [47] that it has contracting boundary values almost everywhere on the real axis of the spectral parameter. We obtain an explicit expression for the Scattering matrix using the above orthogonal decomposition of H and the techniques for operator matrices, developed in [40], [41]. Really,we may construct an explicit expression for the parent scattering matrix via solving the equation

$$\left[\hat{\Lambda} - \mathcal{K}\right]\mathbf{S}e = -\left[\hat{\Lambda} + \mathcal{K}\right]e.$$

[8] In fact the defined object should be called *parent T-matrix*. The conventional T-matrix may be obtained from it via restriction onto the open channels

which may be presented for $e \in H_+$ as a system of two equations for $(\mathbf{S}e)_+ = \mathcal{S}e$, $(\mathbf{S}e)_- = \mathbf{s}e$ in the above orthogonal decomposition $H = H_+ \oplus H_-$:

$$(\hat{\Lambda} - \mathcal{K})\mathcal{S}e + (\hat{\Lambda} - \mathcal{K})\mathbf{s}e = -(\hat{\Lambda} + \mathcal{K})\mathcal{S}e - (\hat{\Lambda} + \mathcal{K})\mathbf{s}e. \tag{8}$$

An explicit expression for the Scattering Matrix in the complex plane may be obtained as a restriction of the parent S-matrix $\mathbf{S}$ onto the sum of open channels H_+. The equation (8) may be reduced to the pair of equations via multiplication of it from the left side by $P_\pm$ respectively:

$$\left(\hat{\Lambda}_{++} - \mathcal{K}_+\right)\mathcal{S}e + \hat{\Lambda}_{+-}\mathbf{s}e = -\left(\hat{\Lambda}_{++} + \mathcal{K}_-\right)e,$$

$$\hat{\Lambda}_{+-}\mathcal{S}e + \hat{\Lambda}_{--}\mathbf{s}e - \mathcal{K}_-\mathbf{s}e = -\hat{\Lambda}_{-+}e.$$

The amplitudes of the exponential modes may be eliminated with use of the second equation :

$$\mathbf{s}e = \left(\hat{\Lambda}_{--} - \mathcal{K}_-\right)^{-1}\left[-\hat{\Lambda}_{-+}\right](e + \mathcal{S}e).$$

Inserting them into the first equation we obtain the announced expression (4) for the Scattering Matrix.

According to the above remark the the Scattering Matrix has contracting boundary values on the real on the spectrum of the operator L below the $m+1$ threshold. The imaginary part of the denominator of the Scattering Matrix on the upper shore of the spectrum of L below the $m+1$-th threshold is negative due to presence of the anti-hermitian term $\mathcal{K}_+$, $\Im\mathcal{K}_+ > 0$.The limit values of the matrix-function $\mathcal{D} = \hat{\Lambda}_{++} - \hat{\Lambda}_{+-}\left[\hat{\Lambda}_{--} - \mathcal{K}_-\right]^{-1}\hat{\Lambda}_{-+}$ give a self-adjoint matrix-function in the channel space H_+, and hence the S-matrix

$$\mathcal{S} = -\frac{\mathcal{D} + \mathcal{K}_+}{\mathcal{D} - \mathcal{K}_+}$$

is unitary in H^+ on the interval of real axis below the $m+1$-th threshold. □

The straight formula for the Scattering Matrix announced in the above theorem may be used not only for complex, but also for real values of the spectral parameter λ if the operator $\hat{\Lambda}_{--}(\lambda) - \mathcal{K}_-(\lambda)$ is invertible at those values, that is, if and only if neither of its eigenvalues coincides with zero. Vice versa, if it has a zero eigenvalue at the real spectral point λ_0 with the eigenvector $\vec{\psi}_0 \in W_2^{\beta+1/2}(\delta) \in H_-$, then the operator $\hat{L}$ has an embedded eigenvalue λ_0 with the eigenvector

$$\Psi_0 = \begin{cases} e^{\mathcal{K}_-\, x}\,\vec{\psi}_0, & \text{on the channels } \Omega_s,\ s = 1,2,3,4 \\ \hat{P}_0\vec{\psi}_0 & \text{inside the domain } \hat{\Omega}. \end{cases}$$

Note that the investigation of properties of the scattering matrix on a network combined of a quantum well and several channels may be accomplished in course of the straightforward construction of the Green function and scattered waves based on [22, 23], see for instance [25, 26, 27, 28]. In particular, in [26] an importance of a special finite matrix was noticed, which defines the scattering on the corresponding resonator. This matrix is an analog of the re-normalized reduced DN-map $D(\lambda) = \hat{\Lambda}_{++} - \hat{\Lambda}_{+-}\left[\hat{\Lambda}_{--} - \mathcal{K}_{-}\right]^{-1}\hat{\Lambda}_{-+}$ constructed above. Different approach to similar problem was developed in [24] and in [29].In the next section we follow the ideas of [30, 32, 15], taking a special care of the case when the channels *are not thin* comparing with the typical wavelength of the scattering process we explore.

The above analysis shows that the spectral characteristics of the modified domain are important, being directly connected with the Scattering Matrix. In the next section we will reduce calculation of the resolvent of the operator $\hat{L}$ on the modified domain to an integral equation with a finite-dimensional matrix-function. Similar integral equation may be derived for the scattered waves.

3. Matching Domains

For standard domains like a unit disc or rectangle the Dirichlet-to-Neumann map may be practically obtained via separation of variables or by the perturbation techniques, for equations with non-zero potential. In the previous section we derived an expression for the scattering matrix on a simplest network based on DN-map for a modified domain $\hat{\Omega}_0$ with flat pieces on the boundary obtained by removing from the domain Ω_0 some segments ω_s bordered by the arcs $\gamma_s \subset \partial\Omega_0$ and the corresponding chords δ_s which may serve as the bottom sections of the attached rectangular channels Ω_s. In actual section we fill the gap between the spectral analysis on the basic domain and on the corresponding modified domain, suggesting the procedure of obtaining spectral characteristics of the modified domain with only one segment removed,

$$\hat{\Omega}_0 = \Omega_0 \backslash \omega.$$

For other segments the procedure may be iterated. The procedure will result in obtaining a finite-dimensional equation which contains some finite-dimensional matrix function of the spectral parameter λ. We begin with derivation of the corresponding equation containing a contracting operator.

Let us denote common ends of the chord δ and the arc γ, following in anti-clockwise order on the boundary of the basic domain Ω_0, by symbols a, b. Consider the reflection $\hat{\Omega}_0'$ of the domain $\hat{\Omega}_0$ with respect to the direct line passing through the points a, b. Denote by $\mathbf{\Omega}_0$ the joining of $\hat{\Omega}_0' \cup \hat{\Omega}_0$. Assuming that the original domain Ω_0 is convex [9] we may construct a disc $D \supset \hat{\Omega}_0$ bordered by the circle ∂D containing only two common points a, b with $\partial\hat{\Omega}_0$. We may choose the disc such that the smallest open segment of it bordered by the arc $\partial D \cap \omega$ and the chord δ is contained strictly inside ω. The boundary of the disc may be decomposed as $\partial D = \Gamma_\delta \cup \Gamma^\delta$, $\Gamma_\delta \in \omega$, Γ^δ belongs to the complement of $\hat{\Omega}_0$. We assume that a similar construction is done also for the domain Ω_0' obtained from Ω_0 via reflection in (a, b). This way we obtain the reflected arc $\gamma' \subset \partial\Omega_0'$, D', the segment ω' and other similar details labeled with primes. We consider also a circle $\Sigma \cup \Sigma'$, $\Sigma \subset \Omega_0$, $\Sigma' \subset \Omega_0'$, constructed on (a, b) as a diameter and the corresponding disc B. We assume, that the chord is not too large, so that we have $B \in \mathbf{\Omega}$, otherwise we have to construct several non-intersecting open discs whose diameters cover $[a, b]$.

The following simple observation was used in [32, 15] when constructing integral equations for Green functions of composite domains combined of two simple parts joined by a small opening [32] or a thin channel [15] :

OBS 1 *Consider a classical solution u to the Dirichlet problem for the Laplace equation in D with the boundary data*

$$u|_{\Gamma_\delta} = 1, \ u|_{\Gamma^\delta} = 0.$$

Then $0 < u(x) < \frac{\theta_\delta}{\pi}$, where θ_δ is the angle at the point x subtended by the chord δ.

This statement may be verified with use of maximum principle and the corresponding fact in the half-plane, which is obviously true since the harmonic measure of the boundary interval δ at the point x in a half-plane coincides with the above ratio $\frac{\theta_\delta}{\pi}$. This estimate may be used to prove that the integral operator in the equation (10) is contracting in the space M_γ of bounded measurable functions on the arc γ or in the corresponding space of continuous functions on the arc, for negative λ or for small δ, similarly to [32, 15].

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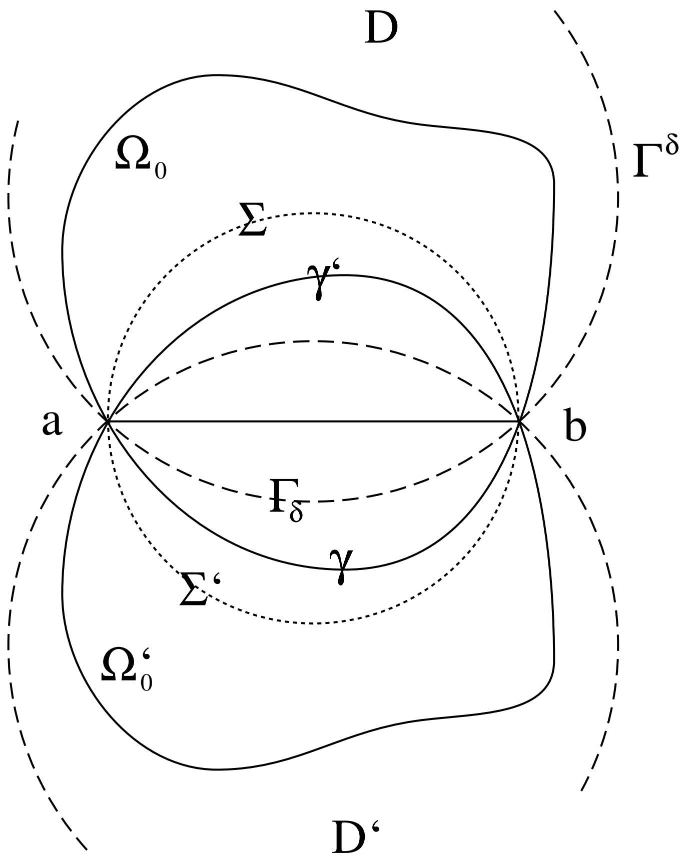
[9]One may see from the Fig.1 that this construction is possible for some non-convex domains too.



We aim now on derivation of an integral equation for the Green function of a composite domain when the contact opening *is not small*. In such a case the equation the corresponding operator is not a contraction but may be represented as a sum of a contraction plus a finite-dimensional rational function of the spectral parameter. The following auxiliary statement permits to derive the integral equation in the space of continuous functions for the values of the resolvent kernel of the Schrödinger operator on the arc γ.

Lemma 3.1. *Consider the resolvent R_λ of the Schrödinger operator L with zero boundary condition on the basic domain Ω_+ and the corresponding Poisson kernel P_λ. Then the product $R_\lambda P_{-M}$, $M > 0$ is a compact operator from the space $L_2(\gamma)$, $\kappa > 0$ into the space $W_2^{1+\beta}(F)$ on any compact sub-domain $G \in \Omega_+$ and into the space of continuous functions C_F on the sub-domain, and the following estimation of the* *remainder of the spectral series*

$$R_\lambda P_{-M} h = \sum_{l=1}^{\infty} \frac{\varphi_l}{\lambda_l - \lambda} \langle P_{-M} h,\ \varphi_l \rangle$$

holds :

$$|\sum_{l=N}^{\infty} \frac{\varphi_l}{\lambda_l - \lambda} \langle P_{-M} h,\ \varphi_l \rangle|_{C(F)} \leq$$

$$|\sum_{l=N}^{\infty} \frac{\varphi_l}{\lambda_l - \lambda} \langle P_{-M} h,\ \varphi_l \rangle|_{W_2^{1+\beta}(F)} \leq$$

$$\left[\sum_{l=N}^{\infty} \frac{1}{|\lambda_l|^{1+\alpha-\beta}}\right]^{1/2} \frac{C}{\kappa M^{1/4-\alpha/2}} |h|_{L_2(\gamma)}, \ \ 0 < \beta < \alpha < 1/2 \tag{9}$$

for the values of the spectral parameter λ from any compact domain on the spectral plane Λ which does not contain any eigenvalues of the Dirichlet problem for the operator.

Proof: If $0 < \alpha < 1/2$ then the operator $[-\triangle +1]^{\alpha/2}$ does not require any boundary conditions and

$$|P_{-M} h|^2_{W_2^{\alpha}(\Omega)} \leq \frac{C}{M^{1/2-\alpha}} |h|_{L_2(\gamma)}.$$

The announced estimate for the spectral series follows from the embedding theorem

$$|\varphi_l|_{C(F)} \leq C_0 |\varphi_l|_{W_2^{1+\beta}(\Omega_+)} = C_0 |(-\triangle +1)^{1/2+\beta/2} \varphi_l|_{L_2(\Omega_+)} = C_0 \lambda^{1/2+\beta/2},$$

for the normalized eigenfunctions. This implies both compactness and the estimation of the C-norm of the remainder of the spectral series for $\alpha > \beta > 0$. Really, due to the above estimate of the C-norm of the eigenfunctions we may estimate the remaining of the spectral series as

$$\sum_{N}^{\infty} \frac{|\langle \varphi_l, (-\triangle +1)^{\alpha/2} P_{_M} h \rangle|}{|\lambda_l - \lambda| \lambda_l^{-1/2+\alpha/2-\beta/2}} := \mathbf{\Sigma}.$$

The last sum may be estimated with use of the Parseval inequality for the orthogonal and normalized system of eigenfunctions of the Schrödinger operator and the spectral asymptotic for the corresponding eigenvalues. On any compact sub-domain of the complement of spectrum of the operator L we have:

$$\mathbf{\Sigma} \leq \left[\sum_{N} \frac{1}{\lambda_l^{1+\alpha-\beta}}\right]^{1/2} \frac{C_0}{M^{1/4-\alpha/2}} |h|_{L_2(\gamma)}$$

□

Note that the statement of the above lemma may be derived from the estimates of the Poisson map obtained in [14].

Now we may derive the integral equation for the Green- function on the composite domain $\Omega \cup \Omega'$ following ideas used in [21]. Assume that the the Green function $G(\lambda, x, y)$ and the corresponding Poisson maps $P_\lambda,\ P'_\lambda$ for Ω and Ω' are known. If the pole y sits in upper domain Ω, then the Green-function $\mathbf{G}$ in the joined domain $\mathbf{\Omega} = \Omega \cup \Omega'$ may be presented via the values u_γ of it on γ:

$$\mathbf{G}(\lambda,\, x,\, y) = G(\lambda, x,\, y) + P(\lambda)\, u_\gamma\,(x),\ \ x \in \Omega,$$
$$\mathbf{G}(\lambda,\, x,\, y) = P'(\lambda)\mathbf{G}(\lambda, x,\, y)|_{\gamma'}(x) =$$
$$P'(\lambda)\,(G(\lambda,\, x,\, y) + P(\lambda)u_\gamma(x))\,|_{x\in\gamma'},\ \ x \in \Omega'.$$

Then using the Poisson map P^D_λ in the disc bordered by the semi-circles $\Sigma,\ \Sigma'$ we may obtain the integral equation for u_γ restricting $\mathbf{G}(\lambda, x, y)$ onto γ:

$$u_\gamma(x) = P^D_\lambda\,\{\mathbf{G}(\lambda,\, x,\, y)|_{x\in\Sigma} + \mathbf{G}(\lambda, x,\, y)|_{x\in\Sigma'}\}\,|_\gamma =$$

(10)
$$P^D_\lambda\,\{[G(\lambda, x,\, y) + P(\lambda)u_\gamma(x)]|_{x\in\Sigma} + [P'(\lambda)\,(G(\lambda, x,\, y) + P(\lambda)u_\gamma)]|_{x\in\Sigma'}\}|_\gamma.$$

Theorem 3.1. *The integral operator $\mathbf{K}_\lambda$ on $C(\gamma)$ in the equation (10)*

$$\mathbf{K}_\lambda : v \longrightarrow P^D_\lambda\,\{[P(\lambda)v]|_\Sigma + [P'(\lambda)\,(P(\lambda)v)]|_{\Sigma'}\}\,|_\gamma$$

is represented as a sum of a finite-dimensional operator function $\mathbf{k}_N(\lambda)$ and a contracting operator-function $\mathbf{k}_\epsilon(\lambda)$ which admits an uniform estimation $|\mathbf{k}_\epsilon(\lambda)|_{C(\gamma)} \leq [1/2 + \epsilon]$ on any compact sub-domain $\mathcal{B}$ of the intersection of complements of spectra of the Schrödinger operators in Ω, Ω', D,

$$\mathbf{k} = \mathbf{k}_N + \mathbf{k}_\epsilon$$

in the space of $C(\gamma)$ of continuous functions on γ.

Proof is based on the above observation and the lemma (3.1). Really,due to the Hilbert identity the Poisson map may be presented as

$$P_\lambda = P_{-M} + (\lambda + M)R_\lambda P_{-M}.$$

The first summand in the right side generates an operator with the norm $\leq 1/2$ acting from $C(\gamma)$ into $C(\Sigma)$. The second summand, according to the above lemma (3.1), is a compact operator-function from $C(\gamma)$ into $C(\Sigma)$. Moreover, the corresponding spectral series admits the uniform estimate of the remainder. Using the spectral decomposition of the resolvent we may represent it as a finite sum over eigenvalues localized in $\mathcal{B}$ and an infinite complementary sum, which is convergent uniformly, according to (3.1), and may be uniformly estimated inside $\mathcal{B}$. The finite sum admits a uniform estimate on the intersection of $\mathcal{B}$

with the regularity field of the Schrödinger operator. Hence denoting by $\mathcal{B}'$ the complement of $\mathcal{B}$ we obtain

$$P_\lambda(x,y) = P_{-M}(x,y) - (\lambda + M)\sum_{\lambda_l\in\mathcal{B}} \frac{\varphi_l(x)\frac{\partial\varphi_l}{\partial n_y}(y)}{(\lambda_l+M)(\lambda_l-\lambda)} -$$

$$(\lambda + M)\sum_{\lambda_l\in\mathcal{B}'} \frac{\varphi_l(x)\frac{\partial\varphi_l}{\partial n_y}(y)}{(\lambda_l+M)(\lambda_l-\lambda)}.$$

Similar decomposition may be used for all other Poisson maps constituting the integral operator $\mathbf{k}_\lambda$, but instead of the contracting property the maximum principle should be used. Now we may join all addenda containing the finite-dimensional operators in a special sum, and form the contracting term of the P_M and "small" terms appearing from the remaining of the spectral sums. This accomplishes the proof of the theorem. □

It follows from the above theorem that calculation of the Green function $\mathbf{G}$ in the composite domain $\Omega \cup \Omega'$ may be reduced to an equation of a form

$$[-I + \mathbf{k}_N + \mathbf{k}_\epsilon]v = f.$$

One may choose ϵ and a family of small discs D_s centered at the eigenvalues of the operator L in Ω and in D such that on the complementary domain $\mathcal{B}_\epsilon = \mathcal{B}\backslash \cup_s D_s$ the operator-function $[I - \mathbf{k}_\epsilon]^{-1}$ exists and is uniformly bounded.Then the above equation may be rewritten as

$$v - [I - \mathbf{k}_\epsilon]^{-1}\mathbf{k}_N v = -[I - \mathbf{K}_M - \mathbf{k}_\epsilon]^{-1} f$$

thus becoming a finite-dimensional one. In particular the eigenvalues of the operator in a composite domain $\Omega \cup \Omega'$ may be found from the corresponding determinant condition.

Now the Green-function $\hat{G}(\lambda, x, y)$ of the modified domain $\Omega\backslash\omega$ may be obtained from the Green-function $\mathbf{G}(\lambda, x, y)$ of the composite domain $\Omega \cup \Omega'$ via reflection $y \to y' \in \Omega'$ with respect to the chord (a, b)

$$\hat{G}(\lambda, x, y) = \mathbf{G}(\lambda, x, y) - \mathbf{G}(\lambda, x, y').$$

The Green function of the modified domain with several segments removed may be obtained by iteration of above procedure. Now the DN-map of the modified domain may be obtained via general formula , see Appendix 1 , and the Scattering Matrix is obtained as described in the previous section, see also [21], where the resonances are discussed.

4. Appendix 1. Dirichlet-to-Neumann map: basic facts

We describe here general features of the DN-map, see also [31, 10, 56], for Schrödinger Operator defined in the space $L_2(\Omega)$ by the differential expression

$$L_{_D} v = - \triangle v + q(x)v$$

on W_2^2- smooth functions vanishing on the piecewise smooth boundary $\Gamma = \partial\Omega$ of the compact domain Ω with no inner angles. Together with the operator $L := L_{_D}$ we may consider the operator L_N defined by the same differential expression L with homogeneous Neumann conditions on the boundary

$$\frac{\partial v}{\partial n}|_{\partial\Omega} \stackrel{W_2^{1/2}(\partial\Omega)}{=} 0,$$

Both $L := L_{_D}$ and $L_{_N}$ are self-adjoint operators in $L_2(\Omega)$. Corresponding resolvent kernels $G_{N,D}(x,\, y, \lambda)$ and the Poisson kernel

$$\mathcal{P}_\lambda(x,\, y) = -\frac{\partial G_D(x,\, y, \lambda)}{\partial n_y},\, y \in \partial\Gamma,$$

for regular values of the spectral parameter λ are locally W_2^2-smooth if $x \neq y$ and square integrable in Ω with boundary values

$$G_{N,D}(x,\, y, \lambda) \in \, W_2^{3/2}(\partial\Omega),\, x \in \partial\Omega,\, y \in \Omega$$

and

$$\mathcal{P}(x,\, y,\, \lambda) \in \, W_2^{1/2}(\partial\Omega),\, y \in \partial\Omega, x \in \Omega.$$

The behaviour of $G_N(x,\, y,\, \lambda)$ when both $x,\, y$ are near to the boundary $\Gamma = \partial\Omega$ is described by the following asymptotic which may be derived from the integral equations of potential theory:

$$G_N(x, x_\Gamma, \lambda) =$$

$$\frac{1}{\pi} \log \frac{1}{|x - x_\Gamma|} + \mathbf{Q}_\lambda + o(1), \tag{11}$$

where the term $\mathbf{Q}_\lambda$ contains a spectral information, see [9, 8]. The spectra $\sigma_{N,D}$ of operators $L_{N,D}$ are discrete and real. Solutions of classical boundary problems for operators $L_{N,D}$ may be represented for regular λ by the "re-normalized" simple layer potentials - for the Neumann problem

$$Lu = \lambda u,\, u \in W_2^2(\Omega),$$

$$\frac{\partial u}{\partial n}|_{\partial\Omega} = \rho \in W_2^{1/2}\partial\Omega,,$$

$$u(x) = \int_{\partial\Omega} G_\lambda(x,\, y)\rho(y)d\Gamma, \tag{12}$$

and by the re-normalized double-layer potentials - for Dirichet problem:

$$Lu = \lambda u,\, u \in W_2^2(\Omega),$$

$$u|_{\partial\Omega} = \hat{u} \in W_2^{3/2}\partial\Omega,$$

$$u(x) = \int_{\partial\Omega} \mathcal{P}_D(x,\, y,\, \lambda)\hat{u}(y)d\Gamma. \tag{13}$$

Generally the DN-map is represented for regular points λ of the operator L_D as

$$(\Lambda(\lambda)\hat{u}\,)\,(x_\Gamma) =$$

$$\frac{\partial}{\partial n}|_{x=x_\Gamma} \int_{\partial\Omega} \mathcal{P}_D(x,\, y,\, \lambda)\hat{u}(y)d\Gamma. \tag{14}$$

The inverse map of $W_2^{1/2}(\Gamma)$ onto $W_2^{3/2}(\Gamma)$ may be presented at the regular points of the operators $L_N^{in,out}$ as

$$\left(Q^{in,out}(\lambda)\rho^{in,\,out}\,\right)(x_\Gamma) =$$

$$\pm \int_\Gamma G_N^{in,\,out}(x,\, y,\, \lambda)\rho^{in,\,out}(y)d\Gamma. \tag{15}$$

The high-energy asymptotic behaviour of the symbol of the DN-map was studied in [7]. It is essentially defined by the local properties of the boundary. In case of non-regular (Lipshitz-class) boundaries, see for instance [57, 59] analogs of DN-maps are not still known, despite the straight connection between the DN-map and the spectral function of the corresponding Operator with Dirichlet boundary conditions similar to thoroughly studied connection between the Weyl-function and spectral function for ordinary differential equations and systems.

In scattering problems we need to evaluate DN - map on real axis of the spectral parameter λ. One can see from the straightforward integration by parts with W_2^2-solutions of the boundary problem that the DN-map is an analytic function of the spectral parameter λ with a positive imaginary part for exterior boundary problem and with a negative imaginary part for interior one :

$$\Im\big\langle\, \Lambda^S_{out} u_\Gamma, u_\Gamma\,\big\rangle|_\Gamma = \Im\big\langle\, \frac{\partial u}{\partial n}|_\Gamma,\, u_\Gamma\,\big\rangle|_\Gamma =$$

$$\Im \int_\Gamma \frac{\partial G^{out}(x_\Gamma, y_\Gamma, \lambda)}{\partial n(x_{_\Gamma})\partial n(y_{_\Gamma})} u(x_{_\Gamma})\bar{u}(y_{_\Gamma})dx_{_\Gamma}dy_{_\Gamma} > 0,$$

$$\Im\big\langle\, \Lambda^S_{in} u_\Gamma, u_\Gamma\,\big\rangle|_\Gamma = \Im\big\langle\, \frac{\partial u}{\partial n}|_\Gamma,\, u_\Gamma\,\big\rangle|_\Gamma =$$

$$-\Im \int_\Gamma \frac{\partial G^{in}(x_\Gamma, y_\Gamma, \lambda)}{\partial n(x_\Gamma)\partial n(y_\Gamma)} u(x_\Gamma)\bar{u}(y_\Gamma) dx_\Gamma dy_\Gamma < 0,$$

for $u_\Gamma \in W_2^{3/2}$, $\Im\lambda \neq 0$. It follows from [47], that there exist weak non-tangential boundary values of the operator-function on real axis from upper and lower half-planes, which we denote by $\Lambda_{out}(\lambda \pm i0)$. Similarly the weak limits of $\Lambda_{in}(\lambda)$ exist and are bounded operators from $W_2^{3/2}(\Gamma)$ onto $W_2^{1/2}(\Gamma)$ on the complement of the spectrum σ_D of the inner Dirichlet problem in Ω. The following simple statement, see [56], shows, that the singularities of the DN-map $\Lambda_{in}(\lambda)$ as an unbounded operator in $L_2(\Gamma)$ and the poles at the eigenvalues of the inner Dirichlet problems may be separated :

Theorem 4.1. *Let us consider the Schrödinger operator $L = -\triangle + q(x)$ in $L_2(\Omega)$ with real measurable essentially bounded potential q and homogeneous Dirichlet boundary condition at the C_2-smooth boundary Γ of Ω. Then the DN-map Λ_{in}^C of L has the following representation on the complement of the corresponding spectrum Σ_L in complex plane λ, $M > 0$:*

$$\Lambda_{in}(\lambda) = \Lambda_{in}(-M) - (\lambda + M)\mathcal{P}_{-M}^{+}\mathcal{P}_{-M} - (\lambda + M)^2\mathcal{P}_{-M}^{+} R_\lambda \mathcal{P}_{-M}, \tag{16}$$

where R_λ is the resolvent of L, and P_M is the Poisson kernel of it. The operators

$$\Lambda_{in}(-M),\ \left(\mathcal{P}_M^{+}\mathcal{P}_M\right)(x_\Gamma, y_{\Gamma)}$$

are bounded respectively from $W_2^{3/2}(\Gamma)$ onto $W_2^{1/2}(\Gamma)$ and in $W_2^{3/2}(\Gamma)$, and the operator

$$\left(\mathcal{P}_M^{+} R_\lambda \mathcal{P}_M\right)(x_\Gamma, y_\Gamma) = \sum_{\lambda_s \in \Sigma_L} \frac{\frac{\partial\varphi_s}{\partial n}(x_\Gamma)\frac{\partial\varphi_s}{\partial n}(y_\Gamma)}{(\lambda_s + M)^2(\lambda_s - \lambda)}$$

is compact in $W_2^{3/2}(\Gamma)$.

Similar statement is true for DN-map in exterior domain,

$$\Lambda_{out}(\lambda) = \Lambda_{out}(-M) + (\lambda + M)\mathcal{P}_{-M}^{+}\mathcal{P}_{-M} + (\lambda + M)^2\mathcal{P}_{-M}^{+} R_\lambda \mathcal{P}_{-M}, \tag{17}$$

with only difference that first terms of the decomposition contain the DN-map and Poisson kernel for the exterior domain and the last term is represented via the integral over the absolutely continuous spectrum

$\sigma_L^a = [0, \infty)$ of L, and the integrand combined of the normal derivatives of the corresponding scattered waves $\psi(x, |k|, \nu)$, $k = |k|\nu$, $|\nu| = 1$:

$$\mathcal{P}^+_{-M} R_\lambda \mathcal{P}_{-M} = \frac{1}{(2\pi)^3} \int_{|k|^2 \in \Sigma_L^a} \frac{\frac{\partial \psi}{\partial n}(x_\Gamma, |k|, \nu) \frac{\partial \bar{\psi}_s}{\partial n}(y_\Gamma |k|, \nu)}{(|k|^2 + M)^2 (|k|^2 - \lambda)} d^3 k.$$

There exists an important connection between DN-map and the celebrated Krein formula [4, 3] from the Operator Extension theory, see for instance [56]. Using Krein's approach one may connect the resolvent and scattering matrix of the operator L with the resolvent and the scattering matrix of the orthogonal sum of operators $L_N^{in} \oplus L_N^{out}$ defined in $L_2(\Omega_{in}) \oplus L_2(\Omega_{out})$ with homogeneous Neumann boundary conditions at the boundary Γ. In this case the deficiency indices of operators restricted to the domain of all elements from domains $L_{in,\, out}$ with non-jumping normal derivatives, are infinite and deficiency elements of the restricted operators $L_{in,\, out}$ have *non jumping* boundary values from $W_2^{3/2}(\Gamma)$ and satisfy the adjoint homogeneous equation

$$L_0^+ u_\rho := - \triangle\, u_\rho + q u_\rho = \lambda u_\rho.$$

They may be represented in form of a *re-normalized simple-layers* formed of re-normalized unit-charge potentials $G^{in,\, out}(x,\, s_\Gamma,\, \lambda)$:

$$u^{in,\, out}_{\rho, \bar{\lambda}}(x) = \int_\Gamma G^{in,\, out}_\lambda (x,\, s) \rho(s) d\Gamma,\ \Im \lambda \neq 0. \tag{18}$$

with densities $\rho \in W_2^{1/2}(\Gamma)$. They have the normal boundary values $\hat{u}^{in,out}_{\rho,\, \lambda} \in W_2^{3/2}(\Gamma)$, $\frac{\partial \hat{u}^{in,out}_{\rho,\, \lambda}}{\partial n} \in W_2^{1/2}(\Gamma)$, which may be evaluated via integration by parts:

$$\hat{u}^{in,out}_{\rho,\, \lambda}(x) = \int_\Gamma G^{in,out}_\lambda (x, s) \rho^{in,out}(s) d\Gamma_s :=$$

$$\left(\mathcal{Q}^{in,\, out}_\lambda * \rho^{in,out} \right)(x),\ x \in \Gamma,$$

$$\frac{\partial \hat{u}^{in,out}_{\rho,\, \lambda}}{\partial n}(x) = \pm \rho^{in,out}(x),\ x \in \Gamma. \tag{19}$$

From the last formula (19) we see, that the boundary values of non-jumping deficiency elements $u : [u]|_\Gamma = 0$ of the operators $l_0^{in,\, out}$ are connected by the integral analog of Krein's Q-function:

$$\hat{u}^{in,out} = \pm \mathcal{Q}^{in,\, out}_\lambda * \frac{\partial \hat{u}^{in,out}_{\rho,\, \lambda}}{\partial n}, \tag{20}$$

where the integral operators $\mathcal{Q}_\lambda^{in,\,out}*$ transform $W_2^{1/2}(\Gamma)$ into $W_2^{3/2}(\Gamma)$. We see, that these operators, if exist for given λ, are inverse of the corresponding Dirichlet-to-Neumann maps $\Lambda_{in,\,out}$. Taking into account the continuity of the piece-wise defined solution $\hat{u}$ and non-jumping condition for its normal derivatives

$$\hat{u}^{in}(x_\Gamma) = \hat{u}^{out}(x_\Gamma),$$

$$\rho^{in}(x_\Gamma) = \frac{\partial \hat{u}_\lambda^{in}}{\partial n}(x_\Gamma) =$$

$$\frac{\partial \hat{u}_\lambda^{out}}{\partial n}(x_\Gamma) = -\rho^{out}(x_\Gamma)$$

one obtain the

Theorem 4.2. *The resolvent kernel $G_\lambda(x, y)$ of the operator L for x, y in Ω_{out} may be represented by the Krein formula*

$$G(x, y, \lambda) = G^{out}(x, y, \lambda) -$$

$$G^{out}(x, *\lambda)\left[\mathcal{Q}_\lambda^{in} + \mathcal{Q}_\lambda^{out}\right]^{-1} G^{out}(*, y\lambda), \tag{21}$$

where the star stands for variables on Γ. The corresponding formula for the scattered waves ψ_ν in outer domain has the form

$$\psi_\nu(x,\, \lambda) = \psi_\nu^{out}(x,\, \lambda) -$$

$$G^{out}(x, *\lambda)\left[\mathcal{Q}_\lambda^{in} + \mathcal{Q}_\lambda^{out}\right]^{-1} \psi_\nu^{out}(*,\, \lambda), \tag{22}$$

where all operator-functions are calculated as limits from the upper half-plane. The expression for the scattering amplitude of the operator L is given by the formula:

$$a(\omega, \nu, \lambda) = a^{out}(\omega, \nu, \lambda) +$$

$$\psi_\omega^{out}(*,\, \lambda)\left[\mathcal{Q}_\lambda^{in} + \mathcal{Q}_\lambda^{out}\right]^{-1} \psi_\nu^{out}(*,\, \lambda). \tag{23}$$

where all operator-functions are calculated as limits from the upper half-plane.

The previous formula (23) is actually a generalization of the formula (4) from the second section of our paper. It may be applied also in a case when infinite number of incoming-outgoing channels are open. Actually this is exactly the case of real devices which are implemented now as patterns on the surface of semiconductors, see [51]: the roles of wires are played by conic domains. Dirichlet-to Neumann maps for 2-d cones, $\arg x \in (\alpha, \beta)$, $|x| > R$ may be easily obtained by separation of variables, and the techniques developed in the previous section may be used to match the cones with the well. But the resulting Scattering Matrix is this case not finite-dimensional.

5. Appendix 2. Estimation of the life-time of resonances

The expression for the Scattering Matrix obtained in Theorem 2.1 may be used to calculate important characteristics of the device such that a life-time of the resonance or an electron's current through the device. It may be done numerically based on techniques developed in [21] and the above Theorem 3.1. In this appendix we suggest an approximate estimation of the position (and the life time) of the resonance arising from the resonance eigenvalue λ_0 in the basic domain. The estimation will be done in non-dimensional coordinates.

The resonance may be found as a vector - zero of the numerator of the scattering matrix, that is a pair of a point λ in the upper half-plane and a corresponding normalized resonance vector $e_\lambda \in H_+$ such that

$$\left\{\hat{\Lambda}_{++}(\lambda) - \hat{\Lambda}_{+-}(\lambda)\left[\hat{\Lambda}_{--}(\lambda) - K_-\right]^{-1}\hat{\Lambda}_{-+}(\lambda) + K_+\right\} e_\lambda = 0. \tag{24}$$

We assume that the resonance eigenvalue λ_0 is situated on the first spectral band of the system of wave-guides $\Omega_1, \Omega_2, \dots, \frac{\pi^2}{\delta^2} < \lambda_0 < 4\frac{\pi^2}{\delta^2}$. The parts K_+, K_- of the DN-map K of the system of wave-guides are operator-functions with a positive imaginary part in the upper half-plane

$$K_+ = i\sqrt{\lambda - \frac{\pi^2}{\delta^2}}P_+, \quad K_- = i\sum_{l\geq 2}\sqrt{\lambda - \frac{l^2\pi^2}{\delta^2}}P_-^l.$$

Here P_+ and $\sum_{l\geq 2} P_-^l$ are, respectively, projections on the open and closed channels.One can see that the operator-function K_- is real and monotone on the first spectral band, Introducing the notations $\varphi_0 = P_+\frac{\partial\hat{\Psi}_0}{\partial n_\delta}$, $\varphi_1 = P_-\frac{\partial\hat{\Psi}_0}{\partial n_\delta}$ we may rewrite expressions for components of the previous equation (24) as follows[10]:

$$\hat{\Lambda}_{++}(\lambda) = -\frac{\varphi_0\rangle\langle\varphi_0}{\lambda_0 - \lambda} + Q_0(\lambda),$$

with some finite-dimensional analytic matrix-function $Q_0 : H_+ \to H_+$ of the spectral parameter and possible poles at the other eigenvalues of $\hat{L}_0$,

$$\hat{\Lambda}_{--}(\lambda) = -\frac{\varphi_1\rangle\langle\varphi_1}{\lambda_0 - \lambda} + Q_1(\lambda),$$

with some infinite-dimensional analytic matrix-function $Q_1 : H_- \to H_-$ of the spectral parameter and possible poles at the other eigenvalues

[10] We denote by $\hat{\Psi}_0$ the resonance eigenfunction of the operator $\hat{L_0}$ in the modified domain

of $\hat{L}_0$, and

$$\hat{\Lambda}_{+-}(\lambda) = -\frac{\varphi_0\rangle\langle\varphi_1}{\lambda_0 - \lambda} + Q_{01}(\lambda),$$

$$\hat{\Lambda}_{-+}(\lambda) = -\frac{\varphi_1\rangle\langle\varphi_0}{\lambda_0 - \lambda} + Q_{10}(\lambda),$$

with two bounded analytic operator-functions Q_{01}, Q_{10} respectively from H_- to H_+ and from H_+ to H_-, with possible poles at the other eigenvalues of $\hat{L}_0$. Using the introduced notations one may rewrite the above equation (24) as

(25)

$$0 = K_+ e - \frac{\varphi_0\rangle\langle\varphi_0, e\rangle}{\lambda_0 - \lambda} + Q_0(\lambda)e -$$

$$\left[\frac{\varphi_0\rangle\langle\varphi_1}{\lambda - \lambda_0} + Q_{01}(\lambda)\right]\left[\frac{\varphi_1\rangle\langle\varphi_1}{\lambda - \lambda_0} + Q_1(\lambda) - K_-\right]^{-1}\left[\frac{\varphi_1\rangle\langle\varphi_0, e\rangle}{\lambda - \lambda_0} + Q_{10}(\lambda)e\right].$$

We may find an approximate solution of the above equation assuming that the component K_+ of the DN-map of the channels dominates each of the components of the DN-map of the modified basic domain in a neighborhood of the resonance eigenvalue $\lambda : ||K_-^{-1}Q_1|| << 1$, $||K_-^{-1}Q_{10}|| << 1$, $||Q_{01}K_-^{-1}|| << 1$. This assumption corresponds to the relatively narrow channels and central position of the Fermi-level inside the first spectral band : $||Q_{10}|| < \sqrt{3}\frac{\pi}{\delta}$. Then neglecting above terms compared with $K_\pm$ we may simplify the equation (25). Really the inverse of the mid-term may be found as a solution of the equation

$$\left[-\frac{\varphi_1\rangle\langle\varphi_1}{\lambda_0 - \lambda} + Q_1(\lambda) - K_-\right] u = f_1, \quad f_1 \in H_-. \tag{26}$$

Introducing the notation $\langle u, \varphi_1\rangle = \beta$, we obtain the solution of the above equation (26) in form:

$$u = -\left(I - K_-^{-1}Q_1(\lambda)\right)^{-1}\left[\beta\frac{K_-\varphi_1}{\lambda_0 - \lambda} + K_-^{-1}f_1,\right]$$

which gives the value of the constant β

$$\beta = -\frac{(\lambda_0 - \lambda)\left\langle \left(I - K_-^{-1}Q_1\right)^{-1} K_-^{-1}f_1, \varphi_1 \right\rangle}{(\lambda_0 - \lambda) + \left\langle \left(I - K_-^{-1}Q_1\right)^{-1} K_-^{-1}\varphi_1, \varphi_1 \right\rangle},$$

and, after neglecting $K_-^{-1}Q_1$ compared with the unit operator, an approximate expression for the solution of the equation (26) :

$$u \approx -K_-^{-1}f_1 + \frac{K_-^{-1}\varphi_1\rangle\langle K_-^{-1}\varphi_1,\ f_1\rangle}{(\lambda_0 - \lambda) - \langle K_-^{-1}f_1, \varphi_1\rangle}.$$

Now we may substitute here $f_1 = -\frac{\varphi_1\rangle\langle\varphi_0, e\rangle}{\lambda_0-\lambda} - Q_{10}e$, and cancel the terms $K_-^{-1}Q_{10}$, compared with unity. This gives:

$$u = \frac{K_-^{-1}\varphi_1\rangle\langle\varphi_0, e\rangle}{(\lambda_0 - \lambda) + \langle K_-^{-1}\varphi_1, \varphi_1\rangle}.$$

It remains to insert the result as an argument of the left factor in the triple product of the above equation (25)

$$\left[-\frac{\varphi_0\rangle\langle\varphi_1}{\lambda_0 - \lambda}\right] u$$

and neglect the term $Q_{01}K_-^{-1}$ compared with unity. We obtain finally the approximate expression for the whole triple product in form

$$-\frac{\langle K_-^{-1}\varphi_1, \varphi_1\rangle}{(\lambda_0 - \lambda) + \langle K_-^{-1}\varphi_1, \varphi_1\rangle}\ \varphi_0\rangle\langle\varphi_0, e\rangle.$$

Inserting the approximate expression for the triple term into the equation (25) and neglecting Q_0 compared with K_+ we obtain a simplified expression for it

(27)

$$-\frac{1}{\lambda_0 - \lambda}\left[1 - \frac{\langle K_-^{-1}\varphi_1, \varphi_1\rangle}{(\lambda_0 - \lambda) + \langle K_-^{-1}\varphi_1, \varphi_1\rangle}\right]\varphi_0\rangle\langle\varphi_0, e\rangle + i\sqrt{\lambda - \frac{\pi^2}{\delta^2}}e = 0.$$

From the last equation (27) we obtain immediately an approximate expression for the resonance vector $e = \varphi_0$, which is in agreement with our previous guess, see [18, 21]. Note, that the approximate calculation of the resonance vector (zero-vector of the Scattering Matrix) presented in [21] is not accurate enough, and hence it could not give the resonance vector anticipated in our previous papers with proper precision. The resonance vector obtained now corresponds exactly to the expectations formulated in [18].

On the other hand we may obtain from above calculation a scalar equation for the resonance itself - the zero of the Scattering matrix :

$$\lambda = \lambda_0 + \langle K_-^{-1}(\lambda)\varphi_1, \varphi_1\rangle + \frac{i|\varphi_0|^2}{\sqrt{\lambda - \frac{\pi^2}{\delta^2}}}.$$

An approximate value for the resonance arising from the resonance eigenvalue λ_0 of the quantum well Ω_0 embedded into the first spectral band is equal to

$$\lambda_{res} = \lambda_0 + \langle K_-^{-1}(\lambda_0)\varphi_1, \varphi_1\rangle + \frac{i}{\sqrt{\lambda_0 - \frac{\pi^2}{\delta^2}}}.$$

The inverse life time is found as an imaginary part of it.

One may also use the above approximate expression (27) for the numerator of the Scattering Matrix to obtain the approximate values of transmission coefficients from the input- wire to the terminals. Really, the approximate expression for the Scattering matrix may be presented as

$$\mathcal{S}(\lambda) = -I - \frac{2i\sqrt{\lambda_0 - \frac{\pi^2}{\delta^2}}}{\frac{|\varphi_0|^2}{\lambda_0 - \lambda + \langle K_-^{-1}\varphi_1, \varphi_1\rangle} + i\sqrt{\lambda_0 - \frac{\pi^2}{\delta^2}}} P_0, \tag{28}$$

where P_0 is an orthogonal projection onto the subspace spanned by the resonance vector φ_0. It follows from (28) that the transmission amplitude from the input wire Ω_1 to the output wire Ω_s is equal to

$$\mathcal{S}_{1s} = -\frac{2i\sqrt{\lambda_0 - \frac{\pi^2}{\delta^2}}}{\frac{|\varphi_0|^2}{\lambda_0 - \lambda + \langle K_-^{-1}\varphi_1, \varphi_1\rangle} + i\sqrt{\lambda_0 - \frac{\pi^2}{\delta^2}}} \; \frac{\langle \frac{\partial \Psi_0}{\partial n_{\delta_1}}, e_1^1\rangle \langle \frac{\partial \Psi_0}{\partial n_{\delta_s}}, e_s^1\rangle}{\sum_{t=1}^4 |\langle \frac{\partial \Psi_0}{\partial n_{\delta_t}}, e_t^1\rangle|^2},$$

hence is proportional to the average values of the normal derivative of the resonance eigenfunction on the Quantum Well, as it was anticipated in our paper [18].

The approximate calculation presented in this section may be easily transformed into accurate estimations for the resonance parameters - just via using the Neumann series for the operator $\left(1 - K_-^{-1}Q_1\right)$ in the mid-term of the triple product. It will be done in the subsequent publication by A.Mikhailova.

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