WEAK TYPE (1,1) ESTIMATES OF MAXIMAL TRUNCATED SINGULAR OPERATORS

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ABSTRACT. Let $\mathcal{X}$ be a space of homogeneous type and T a singular integral operator which is bounded on $L^2(\mathcal{X})$. We give a sufficient condition on the kernel of T so that the maximal truncated operator T_*, which is defined by $T_*f(x) = \sup_{\epsilon>0} |T_\epsilon f(x)|$, to be of weak type $(1,1)$. Our condition is weaker than the usual Hörmander type condition. Applications include the dominated convergence theorem of holomorphic functional calculi of linear elliptic operators on irregular domains.

1. INTRODUCTION AND MAIN THEOREM

Let us consider a space of homogeneous type $(\mathcal{X}, d, \mu)$ which is a set $\mathcal{X}$ endowed with a distance d and a non-negative Borel measure μ on $\mathcal{X}$ such that the doubling condition

$$\mu(B(x, 2r)) \leq c\mu(B(x, r)) < \infty$$

holds for all $x \in \mathcal{X}$ and $r > 0$, where $B(x, r) = \{y \in \mathcal{X} : d(x, y) < r\}$. A more general defintion can be found in [CW, Chapter 3].

The doubling property implies the following strong homogeneity property,

$$\mu(B(x; \lambda r)) \leq c\lambda^n \mu(B(x; r)) \tag{1.1}$$

for some $c, n > 0$ uniformly for all $\lambda \geq 1$. The parameter n is a measure of the dimension of the space. There also exist c and $N, 0 \leq N \leq n$ so that

$$\mu(B(y; r)) \leq c\left(1 + \frac{d(x, y)}{r}\right)^N \mu(B(x; r)) \tag{1.2}$$

uniformly for all $x, y \in \mathcal{X}$ and $r > 0$. Indeed, the property (1.2) with $N = n$ is a direct consequence of triangle inequality of the metric d and the strong homogeneity property. In the case of Euclidean spaces $\mathbb{R}^n$ and Lie groups of polynomial growth, N can be chosen to be 0.

[1]2000 *Mathematics Subject Classification*: 42B20, 47F, 47G10

[2]Both authors are partially supported by a grant from Australia Research Council, and the second author is also partially supported by NSF of China.

We consider the following "generalised approximations to the identity" which was introduced in [DM].

DEFINITION 1.1. *A family of operators $\{A_t, t > 0\}$ is said to be a "generalised approximation to the identity" if, for every $t > 0$, A_t is represented by kernels $a_t(x,y)$ in the following sense: for every function $f \in L^p(\mathcal{X}), p \geq 1$,*

$$A_t f(x) = \int_{\mathcal{X}} a_t(x,y) f(y) d\mu(y);$$

and the following condition holds:

$$|a_t(x,y)| \leq h_t(x,y) = \frac{1}{\mu(B(x;t^{1/m}))} s(d(x,y)^m t^{-1}), \tag{1.3}$$

where m is a positive fixed constant and s is a positive, bounded, decreasing function satisfying

$$\lim_{r\to\infty} r^{n+N+\epsilon} s(r^m) = 0 \tag{1.4}$$

for some $\epsilon > 0$, where n and N are two constants in (1.1) and (1.2).

The operators we are going to consider henceforth were introduced in [DM]. They are defined in the following way.

(1.5) T is a bounded operator on $L^2(\mathcal{X})$ with an associated kernel $k(x,y)$ such that for $f \in L_0^\infty(\mathcal{X})$,

$$T(f)(x) = \int_{\mathcal{X}} k(x,y) f(y) d\mu(y), \quad \text{for } \mu-\text{almost every } x \notin \text{supp} f.$$

(1.6) There exists a "generalised approximation of the identity" $\{A_t, t > 0\}$ such that TA_t have associated kernels $k_t(x,y)$ and there exist constants $c_1, c_2 > 0$ so that

$$\int_{d(x,y)\geq c_1 t^{1/m}} |k(x,y) - k_t(x,y)| d\mu(x) \leq c_2, \quad \text{for all } y \in \mathcal{X}.$$

(1.7) There exists a "generalised approximation of the identity" $\{B_t, t > 0\}$ such that $B_t T$ have kernels $K_t(x,y)$ which satisfy

$$|K_t(x,y)| \leq c_4 \frac{1}{\mu(B(x;t^{1/m}))}, \quad \text{when } \ d(x,y) \leq c_3 t^{1/m}$$

and

$$|K_t(x,y) - k(x,y)| \leq c_4 \frac{1}{\mu(B(x;d(x,y)))} \frac{t^{\alpha/m}}{d(x,y)^\alpha}, \text{ when } d(x,y) \geq c_3 t^{1/m},$$

for some constants $c_3, c_4, \alpha > 0$.

We assume that T is an operator satisfying (1.5), (1.6) and (1.7). The maximal operator T_* is the supremum of the truncated integrals, namely,

$$T_*f(x) = \sup_{\epsilon>0} |T_\epsilon f(x)| = \sup_{\epsilon>0} \left| \int_{d(x,y)\geq\epsilon} k(x,y)f(y)d\mu(y) \right|.$$

It is proved in [DM] that if T verifies (1.5) and (1.6), then it is of weak type (1,1) and of strong type (p,p) for $1 < p \leq 2$. In addition, if (1.7) is also satisfied, the operator T is bounded on $L^p(\mathcal{X})$ for all $1 < p < \infty$. Furthermore, Theorem 3 [DM] shows that T_* is bounded on $L^p(\mathcal{X}), 1 < p < \infty$. Implicitly in the proof we can find the following Cotlar type inequality:

$$T_*f(x) \leq CM(Tf)(x) + CMf(x),$$

where M is the Hardy-Littlewood maximal function. Hence, boundedness of T_* follows from boundedness of T and M.

The following is the main result of this paper.

THEOREM 1.2. *Let T be an operator satisfying the assumptions (1.5) and (1.7). Also assume the following condition (1.8): there exists a "generalised approximation of the identity" $\{A_t, t > 0\}$ so that the kernels $(\mathcal{K}_{\epsilon,t}(x,y) - K_\epsilon(x,y))$ of the operators $(B_\epsilon TA_t - B_\epsilon T)$ satisfy*

$$\sup_\epsilon \int_{d(x,y)\geq\beta t^{1/m}} |\mathcal{K}_{\epsilon,t}(x,y) - K_\epsilon(x,y)| d\mu(x) \leq C \tag{1.8}$$

for some constants C and β, and for all $y \in \mathcal{X}$. Then,

(i) the maximal truncated operator T_* is bounded on $L^p(\mathcal{X})$ for $1 < p < \infty$.

(ii) When $p = 1$, T_* is of weak-type $(1,1)$, that is,

$$\mu(\{x : |T_*f(x) > \alpha\}) \leq \frac{C}{\alpha}\|f\|_1, \quad \text{for all } \alpha > 0.$$

NOTE: (i) Comparing the above Theorem 1.2 with Theorem 3 of [DM], the assumption (1.8) is stronger than (1.6), but we obtain new endpoint estimates, i.e. the weak type (1,1) estimates in (ii).

(ii) Theorem 1.2 improves the classical results of Calderón-Zygmund operators. See [St, Chapter 1, Corollary 2] for Euclidean spaces $\mathcal{X} = \mathbb{R}^n$, and [CW] for spaces of homogeneous type. Let us note that there is no regularity assumptions in the space variables. In comparison with the classical Calderón-Zygmund operators, the Hörmander type inequalities are replaced by (1.7) and (1.8) which involved the "generalised approximations of the identity". In fact, for suitable "generalised approximations of the identity", it is proved in [DM] that conditions (1.6) and (1.7) are weaker than the usual assumptions of Calderón-Zygmund operators (Proposition 2, [DM]). We also show that our condition (1.8) is actually a consequence of the condition (1.6) (Proposition 2.1). As applications, we get the dominated convergence theorem of holomorphic functional calculi of linear elliptic operators on irregular domains.

2. Proof of Theorem 1.2

We first prove (ii). For a fixed $\epsilon > 0$, one writes $T_\epsilon u(x) = B_{\epsilon^m} Tu(x) - (B_{\epsilon^m} T - T_\epsilon)(u)(x)$. Since the class of operators B_t satisfies the conditions (1.3) and (1.4), we have

$$|B_{\epsilon^m} Tu(x)| \leq cM(|Tu(x)|) \tag{2.1}$$

where c is a constant independent of ϵ. Similarly to the proof of Theorem 3 in [DM], using the condition (1.7) we also have

$$\sup_{\epsilon>0} |(B_{\epsilon^m} T - T_\epsilon)u(x)| \leq cM(|u|(x)). \tag{2.2}$$

Theorem 1.2 then follows from (2.2) if we can prove that the operator

$$T_*^B u(x) = \sup_\epsilon |B_{\epsilon^m} Tu(x)|$$

is of weak-type $(1,1)$. Following the idea of Theorem 1 of [DM], we first use the Calderón-Zygmund decomposition to decompose an integrable function into "good" and "bad" parts (see, for example, [CW]).

Given $f \in L^1(\mathcal{X}) \cap L^2(\mathcal{X})$ and $\alpha > \|f\|_1 (\mu(\mathcal{X}))^{-1}$, then there exist a constant c independent of f and α, and a decomposition

$$f = g + b = g + \sum_i b_i,$$

so that

(a) $|g(x)| \leq c\alpha$ for all almost $x \in \mathcal{X}$;

(b) there exists a sequence of balls Q_i so that the support of each b_i is contained in Q_i and

$$\int_{\mathcal{X}} |b_i(x)| d\mu(x) \leq c\alpha\mu(Q_i);$$

(c) $\sum_i \mu(Q_i) \leq c\alpha^{-1} \int_{\mathcal{X}} |f| d\mu(x)$;

(d) each point of $\mathcal{X}$ is contained in at most a finite number N of the balls Q_i.

Note that conditions (b) and (c) imply that $\|b\|_1 \leq c\|f\|_1$ and hence that $\|g\|_1 \leq (1 + c)\|f\|_1$.

We have

$$\mu(\{x: |T_*^B f(x)| > \alpha\}) \le \mu(\{x: |T_*^B g(x)| > \alpha/2\}) + \mu(\{x: |T_*^B b(x)| > \alpha/2\}).$$

It follows from (2.1), (1.5) and boundedness of the Hardy-Littlewood maximal function that T_*^B is bounded on $L^2(\mathcal{X})$. Since $|g(x)| \le c\alpha$, we obtain

(2.3)
$$\mu(\{x: |T_*^B g(x)| > \alpha/2\}) \le 4\alpha^{-2}\|T_*^B g\|_2^2 \le c\alpha^{-2}\|g\|_2^2 \le c\alpha^{-1}\|f\|_1.$$

Concerning the "bad" part $b(x)$, we temporarily fix a b_i whose support is contained in Q_i, then choose $t_i = r_i^m$ where m is the constant appearing in (1.3), and r_i is the radius of the ball Q_i. We then decompose

$$\sup_\epsilon \Big|B_\epsilon T \sum_i b_i(x)\Big| \le \sup_\epsilon \bigg(\Big|B_\epsilon T \sum_i A_{t_i} b_i(x)\Big| + \Big|B_\epsilon T \sum_i (I - A_{t_i}) b_i(x)\Big|\bigg).$$

It follows from the decay assumption (1.3) that

$$\Big\|\sum_i A_{t_i} b_i\Big\|_2 \le c\alpha\Big(\sum_i \mu(Q_i)\Big)^{1/2} \le c\alpha^{1/2}\|f\|_1^{1/2}.$$

See details in the proof of estimate (10) in [DM]. Combining this with L^2-boundedness of T_*^B, we have

$$\begin{aligned}(2.4)\quad \mu(\{x: \sup_\epsilon |B_\epsilon T \textstyle\sum_i A_{t_i} b_i(x)| > \alpha/4\}) &\le 16\alpha^{-2}\Big\|\sum_i T_*^B A_{t_i} b_i\Big\|_2^2 \\ &\le c\alpha^{-2}\Big\|\sum_i A_{t_i} b_i\Big\|_2^2 \\ &\le \frac{c}{\alpha}\|f\|_1.\end{aligned}$$

On the other hand

$$\mu(\{x: \sup_\epsilon |B_\epsilon T \textstyle\sum_i (I - A_{t_i}) b_i(x)| > \alpha/4\}) \le \displaystyle\sum_i \mu(B_i) + \sum_i \frac{4}{\alpha}\int_{(B_i)^c} \sup_\epsilon |B_\epsilon T(I - A_{t_i}) b_i(x)| d\mu(x),$$

where $(B_i)^c$ denotes the complement of B_i which is the ball with the same centre y_i as that of the ball Q_i in the Calderón-Zygmund decomposition but with radius increased by a factor of $(1+c_1)$, where c_1 is the constant in (1.6). Because of property (c) of the Calderón-Zygmund decomposition and doubling volume property of $\mathcal{X}$, we have

$$\sum_i \mu(B_i) \leq c \sum_i \mu(Q_i) \leq c\alpha^{-1}\|f\|_1. \tag{2.5}$$

Using assumption (1.8), we have

$$\int_{(B_i)^c} \sup_\epsilon |B_\epsilon T(I - A_{t_i})b_i(x)|d\mu(x)$$
$$\leq \int_{(B_i)^c} \sup_\epsilon \left| \int_{\mathcal{X}} (K_\epsilon(x,y) - \mathcal{K}_{\epsilon,t_i}(x,y))b_i(y)d\mu(y) \right| d\mu(x)$$
$$\leq \int_{\mathcal{X}} \|b_i(y)\| \left\{ \sup_y \sup_\epsilon \int_{d(x,y)\geq c{t_i}^{1/m}} |K_\epsilon(x,y) - \mathcal{K}_{\epsilon,t_i}(x,y)|d\mu(x) \right\} d\mu(y)$$
$$\leq C\|b_i\|_1, \qquad \text{because } B(y; ct_i^{1/m}) \subset B_i.$$

Therefore

$$\sum_i \frac{1}{\alpha} \int_{(B_i)^c} \sup_\epsilon |B_\epsilon T(I - A_{t_i})b_i(x)|d\mu(x) \leq C\alpha^{-1} \sum_i \|b_i\|_1 \leq \frac{C}{\alpha}\|f\|_1. \tag{2.6}$$

Combining the above estimates (2.2), (2.3), (2.4), (2.5) and (2.6), we have for any $\alpha > \|f\|_1(\mu(\mathcal{X}))^{-1}$,

$$\mu(\{x : \ |T_* f(x)| > \alpha\}) \leq \frac{C}{\alpha}\|f\|_1.$$

If $\mathcal{X}$ is unbounded, the proof is done because the former inequality holds for every $\alpha > 0$. Otherwise, we have to consider what happens for $0 < \alpha \leq \|f\|_1(\mu(\mathcal{X}))^{-1}$. Since $\mathcal{X}$ is bounded we can write $\mathcal{X} = B(x_0, r)$ for some $r > 0$. We conclude

$$\mu(\{x : |T_* f(x)| > \alpha\}) \leq \mu(\mathcal{X}) \leq \frac{C}{\alpha}\|f\|_1$$

for any $\alpha > 0$.

We now prove (i). For any $1 < p \leq 2$, L^p-boundedness of T_* follows from the Marcinkiewicz interpolation theorem. Using a standard duality argument, T_* is proved to be a bounded operator on $L^p(\mathcal{X})$ for all $2 < p < \infty$.

The proof of Theorem 1.2 is complete.

In the next proposition, we show that, for suitable chosen B_t, our condition (1.8) is actually a consequence of condition (1.6).

PROPOSITION 2.1. *Let T be a bounded linear operator on $L^2(\mathcal{X})$ with kernel $k(x,y)$. Assume there exists a "generalised approximation of* *the identity" $\{A_t, t>0\}$ so that the kernels $k_t(x,y)$ of TA_t satisfy the condition (1.6), i.e. there exist constant c and $\delta>1$ so that*

$$\int_{d(x,y)\geq \delta t^{1/m}} |k(x,y)-k_t(x,y)|d\mu(x)\leq c \tag{2.7}$$

for all $y\in\mathcal{X}$.

Then, there exists a "generalised approximation of the identity" $\{B_t, t>0\}$ which is represented by kernels $b_t(x,y)$ in the following sense: for any $f\in L^p(\mathcal{X}), p\geq 1$,

$$B_tf(x)=\int_{\mathcal{X}} b_t(x,y)f(y)d\mu(y),$$

so that the kernels $(\mathcal{K}_{\epsilon,t}(x,y)-K_\epsilon(x,y))$ of the operators $(B_\epsilon TA_t-B_\epsilon T)$ satisfy

$$\sup_\epsilon \int_{d(x,y)\geq \beta t^{1/m}} |\mathcal{K}_{\epsilon,t}(x,y)-K_\epsilon(x,y)|d\mu(x)\leq C$$

for some constants C and β, and for all $y\in\mathcal{X}$.

Proof. Choose $\delta>1$ and let $\beta=3\delta/2$. For any $\epsilon>0$, we choose $b_\epsilon(x,z)=0$ when $d(x,z)\geq(\delta/2)t^{1/m}$. Then, for $x,y\in\mathcal{X}$ so that

$$\mathcal{K}_{\epsilon,t}(x,y)=\int_{\mathcal{X}} b_\epsilon(x,z)k_t(z,y)d\mu(z),$$

$$\text{and}\qquad K_\epsilon(x,y)=\int_{\mathcal{X}} b_\epsilon(x,z)k(z,y)d\mu(z).$$

For all $y\in\mathcal{X}$,

$$\begin{aligned}
&\int_{d(x,y)\geq \beta t^{1/m}} |\mathcal{K}_{\epsilon,t}(x,y)-K_\epsilon(x,y)|d\mu(x)\\
&\quad\leq \int_{d(x,y)\geq \beta t^{1/m}}\int_{\mathcal{X}} |b_\epsilon(x,z)||k_t(z,y)-k(z,y)|d\mu(x)d\mu(z)\\
&\quad\leq \left(\sup_{z\in\mathcal{X}}\int_{\mathcal{X}}|b_\epsilon(x,z)|d\mu(x)\right)\times\int_{d(z,y)\geq\delta t^{1/m}}|k_t(z,y)-k(z,y)|d\mu(z)\\
&\quad\leq c_1\int_{d(z,y)\geq\delta t^{1/m}}|k_t(z,y)-k(z,y)|d\mu(z)\\
&\quad\leq C,
\end{aligned}$$

where the last inequality follows from (2.7) and the third inequality is using the estimate

$$\int_{\mathcal{X}}|b_\epsilon(x,z)|d\mu(x)\leq\int_{\mathcal{X}}h_\epsilon(x,z)d\mu(z)\leq c_1.$$

As a consequence of the boundedness of the maximal truncated operator T_*, we obtain pointwise almost everywhere convergence of $\lim_{\epsilon\to 0} T_\epsilon f(x)$. More precisely, we have the following corollary.

COROLLARY 2.2. *Assume that the operator T satisfies the conditions of Theorem 1.2. Assume that the kernel $k(x,y)$ of T satisfies the estimate*

$$|k(x,y)| \leq c(\mu(B(x;d(x,y)))^{-1}.$$

Then there exist a sequence of positive functions $\epsilon_j(x)$ such that $\lim\limits_{j\to\infty} \epsilon_j(x) = 0$, and a function $m \in L^\infty(\mathcal{X})$ such that for $f \in L^p(\mathcal{X})$, $1 \leq p < \infty$,

$$Tf(x) = m(x)f(x) + \lim_{j\to\infty} \int_{|x-y|\geq \epsilon_j(x)} k(x,y)f(y)d\mu(y)$$

for almost every $x \in \mathcal{X}$.

Proof. Corollary 2.2 follows from a standard argument of proving the existence of almost everywhere pointwise limits as a consequence of the corresponding maximal inequality. See, for example, [CM, Chapter 7, Theorem 6] for Euclidean spaces $\mathcal{X} = \mathbb{R}^n$, and [CW] for spaces of homogeneous type.

REMARK 2.3.

As in Section 3 of [DM], Theorem 1.2 and Corollary 2.2 can be modified so that they are still true when the space of homogeneous type $\mathcal{X}$ is replaced by one of its measurable subsets Ω. In this sense, it is sufficient that condition (1.3) on the upper bound $h_t(x,y)$ of the kernel $a_t(x,y)$ is replaced by

$$h_t(x,y) = (\mu(B^{\mathcal{X}}(x;t^{1/m})))^{-1}s(d(x,y)^m t^{-1}),$$

where $B^{\mathcal{X}}(x;t^{1/m})$ is the ball of centre x, radius $t^{1/m}$ in the space $\mathcal{X}$. For the details, see Section 3, [DM].

3. Applications: Holomorphic functional calculi of linear elliptic operators

We first review some definitions regarding the holomorphic functional calculus as introduced by McIntosh [Mc]. Let $0 \leq \omega < \pi$ be given. Then

$$S_\omega = \{z \in \mathbb{C} : |\arg z| \leq \omega\} \cup \{0\}$$

denotes the closed sector of angle ω and S_ω^0 denotes its interior, while $\dot{S}_\omega = S_\omega \backslash \{0\}$. An operator L on some Banach space E is said to be of type ω if L is closed and densely defined, $\sigma(L) \subset S_\omega$, and for each $\theta \in (\omega, \pi]$ there exists a constant C_θ such that

$$|\eta| \, \|(\eta I - L)^{-1}\|_{\mathcal{L}(E)} \leq C_\theta, \qquad \eta \in -\dot{S}_{\pi-\theta}.$$

If $\mu \in (0, \pi]$, then

$$H_\infty(S^0_\mu) = \{f : S^0_\mu \to \mathbb{C}; f \text{ is holomorphic and } ||f||_\infty < \infty\},$$

where $||f||_{H_\infty} = \sup\{|f(z)| : z \in S^0_\mu\}$. In addition, we define

$$\Psi(S^0_\mu) = \left\{ g \in H_\infty(S^0_\mu) : \exists s > 0, \exists c \geq 0 : |g(z)| \leq c\frac{|z|^s}{1 + |z|^{2s}} \right\}.$$

If L is of type ω and $g \in \Psi(S^0_\mu)$, we define $g(L) \in L(E)$ by

$$g(L) = -\frac{1}{2\pi i} \int_\Gamma (\eta I - L)^{-1} g(\eta) d\eta, \tag{3.1}$$

where Γ is the contour $\{\xi = re^{\pm i\theta} : r \geq 0\}$ parametrised clockwise around S_ω, and $\omega < \theta < \mu$. If, in addition, L is one-one and has dense range and if $g \in H_\infty(S^0_\mu)$, then

$$f(L) = [h(L)]^{-1}(fh)(L), \tag{3.2}$$

where $h(z) = z(1 + z)^{-2}$. It can be shown that $g(L)$ is a well-defined linear operator in E and that this definition is consistent with definition (3.2) for $g \in \Psi(S^0_\mu)$. The definition of $g(L)$ can even extended to encompass unbounded holomorphic functions; see [Mc] for details. L is said to have a bounded holomorphic functional calculus on the sector S_μ if

$$||g(L)||_{\mathcal{L}(E)} \leq N||g||_\infty$$

for some $N > 0$, and for all $g \in H_\infty(S^0_\mu)$.

Assume that Ω is a measurable subset of a space of homogeneous type $(\mathcal{X}, d, \mu)$. Let L be a linear operator on $L^2(\Omega)$ with $\omega < \pi/2$ so that $(-L)$ generates a holomorphic semigroup $e^{-zL}, 0 \leq |\text{Arg}(z)| < \pi/2 - \omega$ which possesses the following two properties:

(3.3) The holomorphic semigroup $e^{-zL}, |\text{Arg}(z)| < \pi/2 - \omega$ is represented by kernels $a_z(x, y)$ which satisfy, for all $\theta > \omega$, an upper bound

$$|a_z(x, y)| \leq c_\theta h_{|z|}(x, y)$$

for $x, y \in \Omega$, and $|\text{Arg}(z)| < \pi/2 - \theta$, where $h_{|z|}$ is defined on $\mathcal{X} \times \mathcal{X}$ by (1.3).

(3.4) The operator L has a bounded holomorphic functional calculus in $L^2(\Omega)$. That is, for any $\nu > \omega$ and $g \in H_\infty(S^0_\nu)$, the operator $g(L)$ satisfies

$$||g(L)f||_2 \leq c_\nu ||g||_\infty \|f\|_2.$$

Applying Theorem 1.2, Corollary 2.2 and Remark 2.3, we have

THEOREM 3.1. *Let L be an operator verifying the assumptions (3.3) and (3.4). Assume that for $g \in H_\infty(S^0_\nu)$, the kernel $k(x,y)$ of $g(L)$ satisfies the estimate*

$$|k(x,y)| \le c(\mu(B(x;d(x,y)))^{-1}, \quad \text{for all } x,y \in \Omega. \tag{3.5}$$

If we denote $T = g(L)$, then

(i) If $1 < p < \infty$, then $\|T_* f\|_p \le C\|g\|_\infty \|f\|_p$.

(ii) If $f \in L^1(\mathcal{X})$, then the map $f \to T_* f$ is weak type $(1,1)$.

(iii) There exist a sequence of positive functions $\epsilon_j(x)$ such that $\lim_{j\to\infty} \epsilon_j(x) = 0$, and a function $m(x) \in L^\infty(\Omega)$ such that for $f(x) \in L^p(\Omega), 1 \le p < \infty$,

$$Tf(x) = m(x)f(x) + \lim_{j\to\infty} \int_{|x-y|\ge \epsilon_j(x)} K(x,y)f(y)d\mu(y)$$

for almost every $x \in \Omega$.

Proof. We follow an idea of Theorem 6 in [DM]. Choose operators $B_t = A_t = e^{-tL}$. As in Theorem 6 of [DM], there exist some constants $c, c_1, \alpha > 0$ such that the kernels $K_t(x,y)$ of $B_t T$ satisfy

$$|K_t(x,y)| \le c\frac{1}{\mu(B^{\mathcal{X}}(x;t^{1/m}))},$$

for all $x, y \in \Omega$ such that when $d(x,y) \le c_1 t^{1/m}$;

$$|K_t(x,y) - k(x,y)| \le c\frac{1}{\mu(B^{\mathcal{X}}(x;d(x,y)))}\frac{t^{\alpha/m}}{d(x,y)^\alpha}$$

for all $x, y \in \Omega$ such that when $d(x,y) \ge c_1 t^{1/m}$.

Using the methods of Theorems 5 and 6 [DM], it is not difficult to check that the kernels $(\mathcal{K}_{\epsilon,t}(x,y) - K_\epsilon(x,y))$ of the operators $(B_\epsilon T A_t - B_\epsilon T)$ satisfy

$$\sup_\epsilon \int_{d(x,y)\ge \beta t^{1/m}} |\mathcal{K}_{\epsilon,t}(x,y) - K_\epsilon(x,y)| d\mu(x) \le C \tag{3.6}$$

for some constants C and β, and for all $y \in \Omega$.

For the proof of (3.6), we leave it to readers. Then, Theorem 3.1 follows from Remark 2.3.

REMARK 3.2.

The condition (3.5) is satisfied by large classes of linear operators on $\mathbb{R}^n$ or a domain Ω of $\mathbb{R}^n$ without any condition on smoothness of the boundary of Ω. For example, if the function $h_t(x,y)$ of (1.3) is bounded above by the Gaussian bounds

$$ct^{-n/2}\exp\{-\alpha|x-y|^2/t\}$$

for some $\alpha > 0$, or by the Poisson bounds

$$\frac{ct}{(t^2+|x-y|^2)^{(n+1)/2}},$$

then (3.5) is a direct result from straightforward integration.

One example of an operator L which possesses Gaussian bounds on its heat kernel is the Schrödinger operator with potential V, defined by

$$L = -\triangle + V(x),$$

where V is a nonnegative function on $\mathbb{R}^n$. See, Lecture 7 in [ADM]. Another example of an operator L on such a domain, which possesses Gaussian bounds on its heat kernel, is the Laplacian on an open subset of $\mathbb{R}^n$ subject to Dirichlet boundary conditions. More general operators on open domain of $\mathbb{R}^n$ which possess Gaussian bounds can be found in [AE], [DM].

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