DIVERGENT SUMS OF SPHERICAL HARMONICS

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Abstract. We combine the Cantor-Lebesgue Theorem and Uniform Boundedness Principle to prove a divergence result for Cesàro and Bochner-Riesz means of spherical harmonic expansions.

1. Background

Fix an integer \( d > 1 \) and consider the unit sphere \( S^d \) in \( \mathbb{R}^{d+1} \), equipped with normalized rotation-invariant measure. For each \( n \geq 0 \) let \( \mathcal{H}_n \) denote the space of spherical harmonics of degree \( n \) restricted to \( S^d \), so that \( L^2(S^d) = \oplus_{n=0}^{\infty} \mathcal{H}_n \). See [22, Section 4.2] for details. Every distribution \( \psi \) on \( S^d \) has a spherical harmonic expansion

\[
\sum_{n=0}^{\infty} Y_n(\psi)(x), \quad \forall x \in S^d, \text{ where } Y_n(\psi) \in \mathcal{H}_n, \quad \forall n \geq 0.
\]

This is the expansion of \( \psi \) in eigenfunctions of the Laplace-Beltrami operator on \( S^d \). It is known [14] that if \( 1 \leq p < 2 \) then there is an \( \psi \in L^p(S^d) \) for which (1) diverges almost everywhere. That leaves open the general behaviour of spherical harmonic expansions for elements of \( L^2(S^d) \). A partial step in this direction follows from the localization principle [18].

Theorem 1.1 (Localization). Suppose \( \psi \) is a distribution on \( S^d \) and \( U \subset S^d \) is an open set disjoint from the support of \( \psi \). For each \( x \in U \), the expansion \( \sum_{n=0}^{\infty} Y_n(\psi)(x) \) converges if and only if \( Y_n(\psi)(x) \to 0 \) as \( n \to \infty \).

Corollary 1.2. If \( \psi \in L^2(S^d) \) and \( U \subset S^d \) is an open set on which \( \psi \) is zero almost everywhere, then the expansion \( \sum_{n=0}^{\infty} Y_n(\psi)(x) \) converges to zero almost everywhere on \( U \).

There are special cases where a function \( \psi \in L^2(S^d) \) can be guaranteed to have an almost everywhere convergent spherical harmonic expansion, if \( \psi \) is in an \( L^2 \)- Sobolev space \( W^{2,s} \) of positive index \( s \) [16] or if it is zonal [1]. (Recall that a function \( f \) on \( S^d \) is said to be zonal about a point \( y \in S^d \) when \( f(x) \) depends only on \( x \cdot y \) for all \( x \in S^d \).)

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Carleson’s theorem [3] has been extended to zonal functions [11]. Let $p_c$ be the critical index

$$p_c = \frac{2d}{d+1}.$$

**Theorem 1.3.** If $p_c < p \leq 2$ and $f \in L^p(S^d)$ is zonal about a point $y \in S^d$, then its spherical harmonic expansion is convergent almost everywhere.

**Corollary 1.4.** Suppose $\psi \in L^2(S^d)$, $U \subset S^d$ is an open set, $f_1 \in \bigcup_{s>0} W^{2,s}(S^d)$, $f_2 \in L^2(S^d)$ is a finite sum of zonal functions, and $\psi = f_1 + f_2$ almost everywhere on $U$. Then $\sum_{n=0}^{\infty} Y_n(\psi)(x)$ converges almost everywhere on $U$.

The two corollaries 1.2 and 1.4 would be rendered trivial if there were a higher dimensional version of Carleson’s theorem. They do suggest that when considering convergence of expansions, we should examine the term-wise behaviour away from the support of a distribution.

In the early 1980’s we showed [17] that Theorem 1.3 is sharp and that localization fails at the critical index.

**Theorem 1.5.** For each $y \in S^d$ and $1 \leq p \leq p_c$ there is a $\psi \in L^p(S^d)$, supported in the hemisphere $\{x : x \cdot y \geq 0\}$ whose spherical harmonic expansion diverges almost everywhere.

This was proved by a combination of the Cantor-Lebesgue theorem, knowledge of the $L^p$-norms of the zonal spherical functions, and the uniform boundedness principle. Kanjin [13] showed that these methods could be combined with a result of Hardy and Riesz [12] to deal with Riesz means for radial functions on Euclidean space. This approach was also used in [20] for Riesz means of radial functions on non-compact rank one symmetric spaces.

Here we prove a similar result for Cesàro and Riesz means of spherical harmonic expansions of zonal functions. This shows the sharpness of the results in [4]. See [2, 7] for earlier work on Cesàro means of spherical harmonic expansions. See [21, 5] for results in a more general setting.

2. **Cesàro & Riesz means**

2.1. **Cesàro means.** The Cesàro means [24, pages 76–77] of order $\delta$ of the expansion (1) are defined by

$$\sigma_N^\delta \psi(x) = \sum_{n=0}^{N} \frac{A_{N-n}^\delta}{A_N^\delta} Y_n(\psi)(x), \quad \forall N \geq 0, x \in S^d,$$

where $A_n^\delta = \binom{n + \delta}{n}$. Theorem 3.1.22 in [24] says that if the Cesàro means converge, then the terms of the series have controlled growth.
Lemma 2.1. Suppose that \( \lim_{N \to \infty} \sigma_N^\delta \psi(x) \) exists for some \( x \in X \) and \( \delta > -1 \). Then

\[
|Y_N(\psi)(x)| \leq C_\delta N^\delta \max_{0 \leq n \leq N} |\sigma_n^\delta \psi(x)|, \quad \forall n \geq 0.
\]

2.2. Riesz means. Hardy and Riesz [12] had proved a similar result for Riesz means. Recall that the Riesz means of order \( \delta \geq 0 \) are defined for each \( r > 0 \) by

\[
S_r^\delta \psi(x) = \sum_{0 \leq n < r} \left(1 - \frac{k}{r}\right)^\delta Y_n(\psi)(x).
\]

Theorem 21 of [12] tells us how the convergence of \( S_r^\delta \psi(x) \) controls the size of the partial sums \( S_0^\delta \psi(x) \).

Lemma 2.2. Suppose that \( \psi \) is a distribution on the sphere for which there is some \( \delta > 0 \) and \( x \in X \) at which its Riesz means \( S_r^\delta \psi(x) \) converges to \( c \) as \( r \to \infty \) then

\[
|S_0^\delta \psi(x) - c| \leq A_\delta r^\delta \sup_{0 \leq t \leq r+1} |S_t^\delta \psi(x)|.
\]

Note that this implies

\[
Y_n(\psi)(x) = O(n^\delta)
\]

and we have the same growth estimates as in Lemma 2.1.

Gergen[9] wrote formulae relating the Riesz and Cesàro means of order \( \delta \geq 0 \), from which it follows that the two methods of summation are equivalent.

3. Zonal Functions and Jacobi Polynomials

3.1. Notation. Suppose that \( f \) is a function on \( S^d \) with \( f(x) \) depending only on \( x \cdot y \), for a fixed \( y \in S^d \), so that \( f(x) = f_0(x \cdot y) \). The spherical harmonic expansion of \( f \) is

\[
\sum_{n=0}^{\infty} c_n(f_0) h_n^{-1} P_n^{(\alpha,\alpha)}(x \cdot y)
\]

where \( \alpha = (d - 2)/2 \), \( P_n^{(\alpha,\alpha)} \) is the Jacobi polynomial of degree \( n \) and index \( (\alpha, \alpha) \),

\[
h_n = \int_{-1}^{1} |P_n^{(\alpha,\alpha)}(t)|^2 (1 - t^2)^\alpha \, dt,
\]

and the coefficients are

\[
c_n(f_0) = \int_{-1}^{1} f_0(t) P_n^{(\alpha,\alpha)}(t) (1 - t^2)^\alpha \, dt, \quad \forall n \geq 0.
\]

See section 4.7 of Szegö’s book [23] for details about these special functions. Let \( m_\alpha \) be the measure on \([-1, 1]\) given by

\[
dm_\alpha(t) = (1 - t^2)^\alpha \, dt,
\]
so that \( \{ P_n^{(\alpha,\alpha)} : n \geq 0 \} \) is an orthogonal basis of \( L^2(m_\alpha) \). From (4.3.3) in [23] we know that the normalization constants \( h_n \) satisfy
\[
(5) \quad h_n \sim A n^{-1} \text{ as } n \to \infty
\]

### 3.4. Asymptotics.
This shows that \( |G| \leq n^{1/2} \) for an interval in the real line with the asymptotic property
\[
\int_{E} |F_n(\theta)|^2 d\theta = |c_n|^2 \left( \int_{E} |\nabla |^2 \right) = |c_n|^2 \left( \frac{|E|}{2} + \frac{e^{2n\gamma_n}}{4} \chi_E(2M_n) + \frac{e^{-2n\gamma_n}}{4} \chi_E(-2M_n) + o(1) \right).
\]
The Riemann-Lebesgue Theorem [24, Thm. II.4.4] says that the Fourier transforms \( \hat{G}(\pm 2M_n) \to 0 \) as \( M_n \to \infty \). If we know that there is some function \( G \) for which \( |F_n(\theta)| \leq G(n) \) uniformly on \( E \) for all \( n \) then there is an \( n_0 > 0 \) for which
\[
\int_E |F_n(\theta)|^2 d\theta \leq G(n)^2 |E|, \quad \forall n \geq n_0.
\]
This shows that \( |c_n| \leq 2G(n) \) for all \( n \geq n_0 \).

### 3.5. Uniform Boundedness.
Suppose there is a number \( 1 < q \leq \infty \) and some positive number \( A \) with
\[
\| P_n^{(\alpha,\alpha)} \|_{L^q(m_\alpha)} \geq c n^A, \quad \forall n \geq 1.
\]

The formation of the coefficient
\[
F \mapsto c_n(F) = \int_{-1}^{1} F(t) P_n^{(\alpha,\alpha)}(t) dm_\alpha(t)
\]
is then a bounded linear functional on the dual of \( L^q(m_\alpha) \) with norm bounded below by a constant multiple of \( n^A \). The uniform boundedness principle implies that for \( p \) conjugate to \( q \) and each \( 0 < \varepsilon < A \) there is an \( F \in L^p(m_\alpha) \) so that
\[
(6) \quad c_n(F)/n^\varepsilon \to \infty \text{ as } n \to \infty.
\]

### 3.3. Cantor-Lebesgue Theorem.
This idea is explained in [19] and is based on [24, Section IX.1]. Suppose we have a sequence of functions \( F_n \) on an interval in the real line with the asymptotic property
\[
F_n(\theta) = c_n (\cos(M_n \theta + \gamma_n) + o(1)), \quad \forall n \geq 0
\]
uniformly on a set \( E \) of finite positive measure, and with \( M_n \to \infty \) as \( n \to \infty \). Integrating \( |F_n|^2 \) over \( E \) gives
\[
\int_E |F_n(\theta)|^2 d\theta = |c_n|^2 \left( \int_E \cos^2(M_n \theta + \gamma_n) d\theta + o(1) \right)
\]
\[
= |c_n|^2 \left( \frac{|E|}{2} + \frac{e^{2n\gamma_n}}{4} \chi_E(2M_n) + \frac{e^{-2n\gamma_n}}{4} \chi_E(-2M_n) + o(1) \right).
\]

The Riemann-Lebesgue Theorem [24, Thm. II.4.4] says that the Fourier transforms \( \hat{G}(\pm 2M_n) \to 0 \) as \( M_n \to \infty \). If we know that there is some function \( G \) for which \( |F_n(\theta)| \leq G(n) \) uniformly on \( E \) for all \( n \) then there is an \( n_0 > 0 \) for which
\[
\int_E |F_n(\theta)|^2 d\theta \leq G(n)^2 |E|, \quad \forall n \geq n_0.
\]

This shows that \( |c_n| \leq 2G(n) \) for all \( n \geq n_0 \).

### 3.4. Asymptotics.
Theorem 8.21.8 in Szegö’s book[23] gives the following asymptotic behaviour for the Jacobi polynomials \( P_n^{(\alpha,\alpha)} \). For \( \alpha \geq -1/2 \) and \( \varepsilon > 0 \) the following estimate holds uniformly for all \( \varepsilon \leq \theta \leq \pi - \varepsilon \) and \( n \geq 1 \).
\[
(7) \quad P_n^{(\alpha,\alpha)}(\cos \theta) = n^{-1/2} k(\theta) \cos (M_n \theta + \gamma) + O \left( n^{-3/2} \right).
\]
Here \( k(\theta) = \pi^{-1/2} (\sin \theta/2)^{-\alpha-1/2} \), \( M_n = n + (2\alpha + 1)/2 \), and \( \gamma = -(\alpha + 1/2)\pi/2 \).
From Egoroff’s theorem and Lemma 2.1 we can say that if the series (4) is Cesàro summable of order \( \delta \) on a set of positive measure in \( S^d \) then there is a set of positive measure \( E \subset [0, \pi] \) on which
\[
\left| c_n(f_0) h_n^{-1} P_n^{(\alpha,\alpha)}(\cos \theta) \right| \leq A n^\delta
\]
and hence
\[
\left| c_n(f_0)n^{(1/2) - \delta}(\cos (M_n \theta + \gamma) + O(n^{-1})) \right| \leq A
\]
uniformly for \( \theta \in E \). The argument of subsection 3.3 shows that
\[
\left| c_n(f_0)n^{(1/2) - \delta} \right| \leq A, \quad \forall n \geq 1.
\]

Lemma 3.1. If \( f \) is a zonal function on the unit sphere whose spherical harmonic expansion is Cesàro summable of order \( \delta \) on a set of positive measure, then there is a constant \( A > 0 \) for which
\[
\left| c_n(f_0) \right| \leq A n^{\delta - (1/2)}, \quad \forall n \geq 1.
\]

\[
q_c = \frac{4(\alpha + 1)}{2\alpha + 1} = \frac{2d}{d - 1}.
\]
Equation (2.2) in [15] gives the following lower bounds on these norms.

Lemma 3.2. For real number \( \alpha > -1/2, 1 \leq q < \infty, \) and \( r > -1/q, \)
\[
\left( \int_0^1 \left| P_n^{(\alpha,\alpha)}(x) \right|^q (1 - x)^\alpha dx \right)^{1/q} \sim \begin{cases} n^{-1/2} & \text{if } q < q_c, \\ n^{-1/2} (\log n)^{1/q} & \text{if } q = q_c, \\ n^{\alpha - (2\alpha + 2)/q} & \text{if } q > q_c. \end{cases}
\]
Notice that these integrals are taken over \([0, 1]\) rather than all of \([-1, 1]\).

4. Main Result

Theorem 4.1. For each \( 1 \leq p < p_c = 2d/(d + 1), \)
\[
0 \leq \delta < \frac{d}{p} - \frac{d + 1}{2},
\]
and \( y \in S^d \), there is a function in \( L^p(S^d) \) which is zonal about \( y \), supported in the hemisphere \( \{ x : x \cdot y \geq 0 \} \), and whose spherical harmonic expansion has Cesàro and Riesz means which diverge almost everywhere.

Proof. Suppose that a series (4) has Cesàro means of order \( \delta \) which converge on a set of positive measure. Then Lemma 3.1 implies that
\[
c_n(f_0) = O \left( n^{\delta - (1/2)} \right), \quad \text{as } n \to \infty.
\]
Compare this inequality with the last line of Lemma 3.2 and section 3.2. If \( q > q_c \), \((1/p) + (1/q) = 1\) and
\[
\alpha - \frac{(2\alpha + 2)}{q} > \delta - \frac{1}{2}
\]
then there must be a zonal function \( f \in L^p(S^d) \) with \( f_0 \) supported on \([0, 1]\) for which the estimate (10) fails. Remembering the definition of \( \alpha \) in terms of the dimension \( d \), we are considering
\[
\delta - \frac{1}{2} < \frac{d - 1}{2} - d \left(1 - \frac{1}{p}\right)
\]
which means
\[
\delta < \frac{d}{p} - \frac{(d + 1)}{2}.
\]

**Remark 4.1.** In [19] we applied this technique to produce an analogous theorem for Laguerre expansions.

### 5. Central Function on SU(2)

We conclude with a simple three dimensional example. Suppose that \( G = SU(2) \) is equipped with the normalized translation invariant measure \( \mu \) and that \( T \) is the maximal torus of diagonal elements of \( G \). For each \( \ell \in \hat{G} = \{k/2 : k \in \mathbb{Z}, k \geq 0\} \) there is an irreducible unitary representation of \( G \) with dimension \( 2\ell + 1 \) and character
\[
\chi_\ell \left( \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right) = \frac{\sin \left( (2\ell + 1)\theta \right)}{\sin (\theta)}.
\]
Every central function on \( G \) is determined by its restriction to \( T \). The Fourier series of central functions are expansions in the characters. If \( f \in L^1(G, \mu) \) is central then
\[
f \sim \sum_{\ell=0}^{\infty} c_\ell \chi_\ell
\]
with
\[
c_\ell = \int_G f(x) \overline{\chi_\ell(x)} \, d\mu(x), \quad \forall \ell \in \hat{G}.
\]
In [8] and [10], Dooley, Giulini, Soardi, and Travaglini estimated the Lebesgue norms of characters of compact Lie groups. The group \( SU(2) \) provides the simplest case of these estimates. For each \( q > 3 \)
\[
\|\chi_\ell\|_q \geq c (2\ell + 1)^{1-3/q}, \quad \forall \ell \in \hat{G}.
\]
If \( 1/p + 1/q = 1 \) then
\[
1 - \frac{3}{q} = 1 - 3 \left(1 - \frac{1}{p}\right) = \frac{3}{p} - 2.
\]
Uniform boundedness then says that if $1 \leq p < 3/2$ and $a < (3/p) - 2$ then there is a central function $f \in L^p(G)$ for which the coefficients in (11) have

$$c_\ell/(2\ell + 1)^a \text{ unbounded as } \ell \to \infty.$$ 

Suppose that (11) is Cesàro summable of order $\delta$ on a set of positive measure. Then Lemma 2.1 says that

$$c_\ell \sin ((2\ell + 1)\theta) = O(\ell^\delta) \text{ as } \ell \to \infty,$$

on a set of positive measure. The Cantor-Lebesgue Theorem then says that

$$c_\ell = O(\ell^\delta) \text{ as } \ell \to \infty.$$ 

**Theorem 5.1.** For $1 \leq p < 3/2$ and $0 \leq \delta < (3/p) - 2$ there is a central function $f \in L^p(SU(2))$ for which the Cesàro and Riesz means of order $\delta$ are divergent almost everywhere.

This shows the sharpness of results in Clerc’s paper [6].

**References**


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