DERIVATION OF MONOGENIC FUNCTIONS AND APPLICATIONS

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Abstract. The paper studies two types of results on inducing monogenic functions in $\mathbf{R}_1^n$. One is based on McIntosh's formula and the other is along the line of Fueter's Theorem. Applications are summarized and a new application on monogenic sinc function interpolation is introduced.

1. Background

Denote by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ the *basic elements* that satisfy

$$\mathbf{e}_i^2 = -1, \mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i, \; i, j = 1, 2, \ldots, n, \; i < j.$$

We will work on the following spaces:

$$\mathbf{R}^n = \{\underline{x} = x_1\mathbf{e}_1 + \cdots x_n\mathbf{e}_n : x_i \in \mathbf{R}, i = 1, \ldots, n\},$$

$$\mathbf{R}_1^n = \{x = x_0 + \underline{x} : x_0 \in \mathbf{R}, \underline{x} \in \mathbf{R}^n\},$$

$\mathbf{R}^{(n)}$ is the Clifford algebra generated by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ over the real number field $\mathbf{R}$;

$\mathbf{C}^{(n)}$ is the Clifford algebra generated by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ over the complex number field $\mathbf{C}$.

We adopt the notation $x \in \mathbf{R}^{(n)}$(or $\mathbf{C}^{(n)}$) implies $x = \sum_s x_s\mathbf{e}_s$, where $x_s \in \mathbf{R}$ (or $\mathbf{C}$), and s runs over all the possible ordered sets

$$s = \{0 \le j_1 < \cdots < j_k \le n\}, \text{ or } s = \emptyset, \qquad \text{and}$$

$$\mathbf{e}_s = \mathbf{e}_{j1} \cdots \mathbf{e}_{jk}, \quad \mathbf{e}_0 = \mathbf{e}_\emptyset = 1.$$

The functions we will study will be defined in subsets of $\mathbf{R}_1^n$, and take their values in $\mathbf{R}^{(n)}$ or $\mathbf{C}^{(n)}$.

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The Dirac operator D for functions in $\mathbf{R}_1^n$ is defined by

$$D = D_0 + \underline{D},\ D_0 = \frac{\partial}{\partial x_0},\ \underline{D} = \frac{\partial}{\partial x_1}\mathbf{e}_1 + \cdots + \frac{\partial}{\partial x_n}\mathbf{e}_n.$$

It applies from the left- and right- hand sides to the function, in the manners

$$Df = \sum_{i=0}^{n}\sum_{s}\frac{\partial f_s}{\partial x_i}\mathbf{e}_i\mathbf{e}_s \text{ and } fD = \sum_{i=0}^{n}\sum_{s}\frac{\partial f_s}{\partial x_i}\mathbf{e}_s\mathbf{e}_i,$$

respectively. If $Df = 0$, then f is said to be left-monogenic; and, if $fD = 0$, then right-monogenic. If f is both left- and right- monogenic, then it is said to be monogenic.

Examples:

(1) The case $n = 1$ corresponds to the complex number field: $\mathbf{e}_1 = \mathbf{i}$, $D = \frac{\partial}{\partial x} + \mathbf{i}\frac{\partial}{\partial y}$, $f(z) = u(x,y) + \mathbf{i}v(x,y)$ and $Df = 0$ if and only if the Cauchy-Riemann equations hold:

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}. \end{cases}$$

(2) The case $n = 2$ corresponds to the space of Hamilton quaternions:

$$\vec{\mathbf{i}} = \mathbf{e}_1, \vec{\mathbf{j}} = \mathbf{e}_2, \vec{\mathbf{k}} = \mathbf{e}_1\mathbf{e}_2,$$

$$q = q_0 + q_1\vec{\mathbf{i}} + q_2\vec{\mathbf{j}} + q_3\vec{\mathbf{k}}, q_k \in \mathbf{R}.$$

Profound studies on Clifford analysis have been conducted since Fueter's school in the 1930's till the present time (see, for instance, [Ma], [CS] and [Q1] and their references).

(3) Let $u_j(x), j = 0, 1, \ldots, n$, be defined in $\mathbf{R}_1^n$ with values in $\mathbf{C}$. Set

$$U = -u_0 + u_1\mathbf{e}_1 + \cdots + u_n\mathbf{e}_n$$

Then

$$DU = 0$$

if and only if these functions form a conjugate harmonic system (or satisfy the generalized Cauchy-Riemann equations, see [St] and [KQ1]):

$$\begin{cases} \displaystyle\sum_{j=0}^{n}\frac{\partial u_j}{\partial x_j} = 0 \\ \dfrac{\partial u_k}{\partial x_j} = \dfrac{\partial u_j}{\partial x_k}, 0 \le k < j \le n. \end{cases}$$

For monogenic functions there hold Cauchy's Theorem and Cauchy's formula. The Cauchy kernel in the context is $\mathbf{E}(x) = \frac{\overline{x}}{|x|^{n+1}}$, where for $x = x_0 + \underline{x}$, we denote $\overline{x} = x_0 - \underline{x}$ .. It is observed that the Clifford structure of $\mathbf{R}_1^n$ is the "true" analogue of the one complex variable structure of $\mathbf{R}_1^1$.

2. C-K Extension and McIntosh's Formula

It can be proved that if we have a real analytic function defined in an open set O of $\mathbf{R}^n$, then we can always monogenically extend it to an open set Q of $\mathbf{R}_1^n$ where $O = \mathbf{R}^n \cap Q$ (*C-K extension*, see, for instance [BDS]). The extension can be realized by the operation $\mathbf{e}^{-x_0\underline{D}} f(\underline{x})$, understood in the symbolic way. In fact, formally we have

$$\begin{aligned} D(e^{-x_0\underline{D}} f(\underline{x})) &= (D_0 + \underline{D})(e^{-x_0\underline{D}} f(\underline{x})) \\ &= (-\underline{D}) e^{-x_0\underline{D}} f(\underline{x}) + \underline{D} e^{-x_0\underline{D}} f(\underline{x}) = 0. \end{aligned}$$

Examples:

(1) If $f(x) = x_j$, then

$$e^{-x_0\underline{D}}(x_j) = (1 + (-x_0\underline{D}) + \frac{1}{2!}(-x_0\underline{D})_2 + \cdots)x_j = x_j\mathbf{e}_0 - x_0\mathbf{e}_j \stackrel{\triangle}{=} z_j.$$

(2) The extension of $x_i x_j, i \neq j$, is

$$\frac{1}{2}(z_i z_j + z_j z_i),$$

etc.

In practice the C-K extension and the related forms are, in general, complicated and not easy to use. On the contrary, McIntosh's formula, somehow plays the role of Fourier-Laplace transform in $\mathbf{R}_1^n$, has been playing a crucial role in a number of questions in function theory ([LMcQ], [PQ], [Q4], [KQ1], [KQ2]). The formula first appeared in late 1980's ([Mc1]) and formally published in [Mc2] and [LMcQ] in 1994. The formula involves a set of notations: If f is defined in $\mathbf{R}^n$ with Fourier transform, then the possible monogenic extension of f is given by

$$f(x) = \frac{1}{(2\pi)^2} \int_{\mathbf{R}^n} e(x, \underline{\xi}) \hat{f}(\underline{\xi}) d\underline{\xi}, \qquad \text{(McIntosh's formula)},$$

provided that the integral on the right-hand-side is properly defined, where

$$e(x, \underline{\xi}) = e^{\mathbf{i}\underline{x}\cdot\underline{\xi}} \{ e^{-x_0|\underline{\xi}|} \chi_+(\underline{\xi}) + e^{x_0|\underline{\xi}|} \chi_-(\underline{\xi}) \},$$

$$\chi_\pm(\underline{\xi}) = \frac{1}{2}(1 + \mathbf{i}\frac{\underline{\xi}}{|\underline{\xi}|}), \quad x = x_0 + \underline{x}, \quad \underline{x}, \underline{\xi} \in \mathbf{R}^n.$$

In [BDS] a wide range of similar notions are introduced. It is exactly M^cIntosh's form, however, that has been effectively used, especially in problems related to Fourier transformation.

In the formulas for the projections $\chi_{\pm}$, if we take $n = 1$ and $\mathbf{e}_1 = -\mathbf{i}$, then we have

$$\chi_{\pm}(\underline{\xi}) = \pm\mathrm{sgn}\xi,$$

where $\xi = \mathbf{i}\underline{\xi}$.

This indicates that the formula provides a decomposition of a function into functions similar to those in the Hardy spaces. Indeed, we have,

$$f(x) = f^+(x) + f^-(x),$$

where

$$f^{\pm}(x) \stackrel{\triangle}{=} \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} \mathbf{e}^{\pm}(x,\underline{\xi}) \stackrel{\wedge}{f}(\underline{\xi})d\underline{\xi},$$

$$\mathbf{e}^{\pm}(x,\underline{\xi}) = \mathbf{e}^{i\underline{x}\cdot\underline{\xi}}\mathbf{e}^{\mp x_0|\underline{\xi}|}\chi_{\pm}(\underline{\xi}).$$

We can further show that for $x_0 > 0$,

$$f^+(x) = \frac{1}{w_n}\int_{\mathbf{R}^n} E(x-\underline{y})\cdot f(\underline{y})d\underline{y};$$

while $f^-(x), x_0 > 0$, is the monogenic extension of $f^-(x)$ for $x_0 < 0$, where the latter is also of the Cauchy's integral form of f. For $x_0 < 0$ we have the analogous notation.

Under the context of the classical Paley-Wiener Theorem in the case $n = 1$, viz. $f \in L^2(\mathbf{R})$ and

$$\mathrm{supp} \stackrel{\wedge}{f}(\xi) \subset [-\delta,\delta], \quad \delta > 0,$$

there follows

$$f(z) = f^+(z) + f^-(z).$$

For $z = x + iy,\ y > 0$, we have

$$e^{-y|\underline{\xi}|}\chi_{[0,\delta]}(\underline{\xi}) = e^{-y\xi}\chi_{[0,\delta]}(\underline{\xi}), \qquad e^{y|\underline{\xi}|}\chi_{[-\delta,0]}(\underline{\xi}) = e^{-y\xi}\chi_{[-\delta,0]}(\underline{\xi}),$$

and so

$$f^+(z) = \frac{1}{2\pi}\int_0^{\delta} \mathbf{e}^{ix\xi}\mathbf{e}^{-y\xi} \stackrel{\wedge}{f}(\xi)d\xi = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(t)}{t-z}dt,$$

and

$$f^-(z) = \frac{1}{2\pi}\int_{-\delta}^{0} \mathbf{e}^{ix\xi}\mathbf{e}^{-y\xi} \stackrel{\wedge}{f}(\xi)d\xi,$$

where $f^-(z)$ is well defined, but not be expressible by a Cauchy integral. In fact, since $y = \mathrm{Im}\ z > 0$, $f^-(z)$ is the holomorphic extension to the upper-half complex plane of the Cauchy integral of f in the lower-half complex plane. By virtue of M^{c}Intosh's formula we have exactly the same notion in $\mathbf{R}_1^n$.

We will mention two applications of M^{c}Intosh's formula.

(1) Paley-Wiener Theorem in $\mathbf{R}_1^n$

In a recent paper we proved the following theorem ([KQ1]).

Theorem. *Let $f \in L^2(\mathbf{R}^n)$. Then f can be monogenically extended to $\mathbf{R}_1^n$ with the estimate*

$$|f(x)| \leq c\mathbf{e}^{\mathbf{R}|x|}$$

if and only if

$$\text{supp}\, \hat{f} \subset \overline{B(0,R)},$$

where

$$B(0,R) = \{x \in \mathbf{R}^n : |\underline{x}| < R\}.$$

In the case we have

$$f(x) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e(x,\underline{\xi})\hat{f}(\underline{\xi})d\underline{\xi}, \quad x \in \mathbf{R}_1^n.$$

In the literature higher dimensional versions of the Paley-Wiener theorem have been sought (see the references of [KQ1]). We wish to make the point that the version with the Clifford algebra setting provides the precise analogue. The commonly adopted proofs of the classical Paley-Wiener Theorem are not readily applicable to the Clifford setting owing to the defect that products of monogenic functions are no longer again monogenic in general. However, a particular proof for the one complex variable case can be closely followed through a non-trivial computation based on M^{c}Intosh's formula([KQ1]).

(2) Monogenic sinc function with Shannon sampling for functions in the Paley-Wiener classes

Define the class of functions

$$PW(R) = \Big\{ f : \mathbf{R}_1^n \to \mathbf{C}^{(n)} : \quad f \text{ is monogenic in the whole } \mathbf{R}_1^n \text{ and satisfies } |f(x)| \leq C\mathbf{e}^{R|x|} \Big\}.$$

The monogenic sinc function is defined to be

$$\text{sinc}(x) = \frac{1}{(2\pi)^n}\int_{[-\pi,\pi]^n} \mathbf{e}(x,\underline{\xi})d\underline{\xi}.$$

The following exact interpolation of functions in the $PW(R)$ classes is proved in ([KQ2]).

Theorem. *If $f \in PW(\frac{\pi}{h})$, then*

$$f(x) = \sum_{\underline{k}\in\mathbf{Z}^n} f(h\underline{k})\,\text{sinc}\left(\frac{x-h\underline{k}}{h}\right),$$

where the convergence is in the pointwise sense independent of the order of summation.

The proof is based on estimates of the monogenic sinc function derived from M^{c}Intosh's formula.

3. Fueter's Theorem and Generalizations

This addresses the problem of deriving monogenic and harmonic functions from those of the same kind but in lower dimensional spaces.

Let $f^0(z)$ be a function of one complex variable analytic in an open set O of the upper-half complex plane $\mathbf{C}^+$. If $f(z) = u(x,y) + \mathbf{i}v(x,y)$, $z = x + iy$, we introduce

$$\vec{f^0}(x) = u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|}v(x_0, |\underline{x}|).$$

and set

$$\tau(f^0)(x) = \Delta^{\frac{n-1}{2}}\vec{f^0}(x), x \in \mathbf{R}_1^n.$$

Theorem of Fueter (1935). *When $n = 3$, interpreted as the quaternionic space, the mapping τ maps an analytic function $f^0(z)$ in O to a quaternionic monogenic function in*

$$\vec{O} = \{q = q_0 + \underline{q} : q_0 + \mathbf{i}|\underline{q}| \in O\}.$$

Theorem of Sce (1957). *For n being an odd integer the mapping τ maps $f^0(z)$ to a monogenic function in*

$$\vec{O} = \{x = x_0 + \underline{x} : x_0 + \mathbf{i}|\underline{x}| \in O\}.$$

These results were extended in [Q2] in 1997 to the cases n being an integer and the operator $\Delta^{\frac{n-1}{2}}$ interpreted as the Fourier multiplier operator with symbol $|\xi|_{n-1}$. We note that

$$\tau(\frac{1}{z})(x) = E(x) = \frac{1}{|x|^{n+1}}.$$

In [Q2-3] for any integer $n \geq 2$ a corresponding relationship between the functions $f^0(z) = z^k$ and certain monogenic functions $P^{(k)}(x)$ of homogeneity of degree k is established:

$$\tau(\frac{1}{z^k})(x) = P^{(-k)}(x), k = 1, 2, \ldots,$$

and

$$P^{(k-1)} = I(P^{(-k)}), k = 1, 2, \ldots,$$

where I is the Kelvin inversion defined by

$$If(x) = E(x)f(x^{-1}).$$

It is noted that if n is an odd integer, then

$$P^{(k-1)} = \tau(z^{n+k-2}).$$

The sequence $P^{(k)}, k \in \mathbf{Z}$, is used to establish the bounded holomorphic functional calculus of the Dirac operator on Lipschitz perturbations,denoted by D_Σ, of the unit sphere in $\mathbf{R}_1^n$ (and similarly on $\mathbf{R}^n$).

We now describe the result. Set

$$\mathbf{S}_w = \{0 \neq z \in \mathbf{C} : z = x + iy, \frac{|y|}{|x|} < \tan w\},\ 0 < w < \frac{\pi}{2},$$

$$\tan w > \text{the Lipschitz constant of } \Sigma,$$

$$\mathbf{H}^\infty(\mathbf{S}_w) = \{b : \mathbf{S}_w \to \mathbf{C} : f^o \text{ is bounded and analytic in } \mathbf{S}_w\}.$$

Given $b \in \mathbf{H}^\infty(\mathbf{S}_w)$, and set, formally,

$$b(D_\Sigma)f = \frac{1}{2\pi i}\int_\gamma b(\xi)(I - D_\Sigma)^{-1} d\xi f,$$

where γ is a certain curve in $\mathbf{S}_w$ surrounding the spectrum of D_Σ. The operators $b(D_\Sigma)$ are proved to be equal to the Fourier multiplier operators

$$M_b f(x) = \sum_{k=1}^\infty b(k)P_k f(x) + \sum_{k=1}^\infty b(-k)Q_k f(x),$$

where $P_k f$ and $Q_k f$ are projections of f onto the spaces of monogenic functions of homogeneity degree k and $-k$, respectively. They are also equal to the singular integral operators

$$S_\Phi f(x) = \lim_{n\to\infty}\left\{\frac{1}{w_n}\int_{|x-y|>\epsilon}\Phi(y^{-1}x)E(y)n(y)f(y)d\sigma(y) + \Phi^1(\epsilon, x)f(x)\right\},$$

where in a certain sense $\Phi = \overset{\vee}{b}$ (the inverse Fourier transform of b) and $\Phi^1(\epsilon, x) =$ the average of Φ on the sphere centered at x of radius ϵ.

That is

$$b(D_{\sum}) = M_b = S_\phi.$$

The boundedness of the operators $b(D_\Sigma)$, $b \in \mathbf{H}^\infty(S_w)$, is proved through their singular integral expressions S_Φ based on the estimates of the kernels Φ and Φ^1. The derivation of the estimates are reduced to the similar estimates in the one complex variable case via the correspondence between the functions z^k and $P^{(k)}$ ([Q5]).

On Lipschitz perturbations of higher dimensional spheres the theory cannot be done through the Poisson Summation method based on the graph case as in the unit circle case ([Q5]). It encountered some difficulties and hence was first achieved in the quaternionic space ([Q1]), and then in general Euclidean spaces ([Q3]).

Further generalizations of Fueter's Theorem include the following.

(i) In a recent paper F.Sommen proved that if n is an odd positive integer and $x \in \mathbf{R}_1^n$, then for $f^0(z) = u(s,t) + \mathbf{i}v(s,t), z = s + it$, analytic in an open set $O \subset \mathbf{C}^+$, then for $x \in \vec{O}$

$$D\Delta^{k+\frac{n-1}{2}}((u(x_0, |x|) + \frac{x}{|\underline{x}|}v(x_0, |\underline{x}|))P_k(\underline{x})) = 0,$$

where P_k is any polynomial in $\underline{x}$ of homogeneity k, left-monogenic with respect to the Dirac operator $\underline{D}$ ([So]).

(ii) K.I.Kou and T.Qian extended Sommen's result to the cases when n is an even positive integer and Sommen extended his result to the cases $k + \frac{n-1}{2}$ being non-negative integers, no matter whether k is an integer ([KQS]).

(iii) The derivation of monogenic functions can be reduced to that of harmonic functions, based on the following observations.

A. If h is harmonic in $x_0, x_1, \ldots, x_n$, then $\overline{D}h$ is monogenic, where $\overline{D} = D_0 - \underline{D}$.

B. If f is monogenic, then there exists a harmonic function h such that $f = \overline{D}h$.

The following result for harmonic functions is obtained in a recent paper of T.Qian and F.Sommen ([QS]).

Denote

$$\underline{x}^{(r)} = x_1^{(r)}\mathbf{e}_1^{(r)} + \cdots + x_{p_r}^{(r)}\mathbf{e}_{p_r}^{(r)} \in \mathbf{R}^{p_r},$$

where $r = 1, \ldots, d, \sum_{r=1}^d p_r = m$, and

$$\mathbf{e}_i^{(r)}\mathbf{e}_{i'}^{(r')} = -\mathbf{e}_{i'}^{(r')}\mathbf{e}_i^{(r)}, \text{wherever } (r,i) \neq (r', i').$$

Let $h(s_1, \ldots, s_d)$ be a harmonic function in the d variables $s_1, \ldots, s_d$. Them, if $p_r, r = 1, \ldots, d$, are odd and $m = \sum_{r=1}^d p_r$ is even, then

$$\Delta^{\frac{m}{2}} h(|\underline{x}^{(1)}|, \ldots, |\underline{x}^{(d)}|) = 0,$$

where Δ is the Laplacian for all the m variables $x_i^r, r = 1, \ldots, d$, $i = 1, \ldots, p_r$.

(iv) The latest result along this line is by K.I.Kou and T.Qian ([KQ3]), as follows.

In the above notation we have

$$\Delta^{(k_1+\ldots+k_d)+\frac{m}{2}}[h(|\underline{x}^{(1)}|, \ldots, |\underline{x}^{(d)}|)P_{k_1}^{(1)} \cdots P_{k_d}^{(d)}(\underline{x}^{(d)})] = 0,$$

where for any $r = 1, 2, \ldots, d$, $P_{k_r}^{(r)}(\underline{x}^{(r)})$ is a left-monogenic functions with respect to $\underline{D}^{(r)}$, homogeneous of degree k_r, where k_r is any non-negative integer.

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