ALGEBRAIC OPERATORS, DIVIDED DIFFERENCES, FUNCTIONAL CALCULUS, HERMITE INTERPOLATION AND SPLINE DISTRIBUTIONS.

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Abstract. This article combines three components corresponding to real analysis, algebra and operator theory. A simple and universal way of finding a Hermite interpolation (quasi)polynomial and Gel'fond's formula covering both real and complex cases and based on the study of the sufficient conditions for the continuity of divided differences with respect to two types of the simultaneous convergence of nodes is presented. Relying on it, we establish both algebraic and topological Jordan decomposition for algebraic operators and algebraic and topological properties of their calculus, such as a constructive algebraic characterization of the solvability and an explicit estimate of the related a priori constant and a new characterization of bounded algebraic operators in terms of orbits. Further applications include explicit relations between the barycentric or uniform distributions on convex polyhedra and, correspondingly, B-splines or Steklov splines and short analytic proofs of some classical results from the polynomial arithmetics, wavelet theory and discrete Fourier transform. We also provide correct definitions, typical properties and representations for the holomorphic calculus of the closed operators with mixed spectra including both (double)sectorial and bounded components.

1. Introduction.

The topic of this work is quite classical and situated on the junction of real analysis, operator theory and algebra. The article is devoted to the development and application of explicit algebraic relations based on our approach to the Hermite interpolation problem and divided differences (§3) in the theories of algebraic operators (§4), holomorphic functional calculus (§5), geometric probability (§3.5), polynomial rings (Remark 3.1, §3.4 and §4) and wavelets (§3.4). Since the topic has very deep classical roots, historical remarks accompany most of the results.

Section 3 contains the description of our approach to the Hermite interpolation problem and its generalizations. To deduce Gel'fond's formulas in both real and complex settings simultaneously, paying attention to the smoothness requirements in the real setting, we study the continuity of divided differences with respect to the merging and non-merging convergences showing that the former requires less smoothness and using only elementary methods of the classical calculus. Some ring structures on the classes of quasi-polynomials related to our generalizations are discussed. In §3.4, we demonstrate applications to the polynomial arithmetics, showing close ties with the discrete Fourier transform and an algebraic result used in wavelet theory, while the Steklov and B-splines appear as the density functions for the projections of the uniform and barycentric distributions on convex polyhedra in

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linear spaces in §3.5, along with various representations for the divided and classical differences.

The entire §4 deals with algebraic operators containing both purely algebraic results for abstract linear spaces and for Banach (or complete linear metric) spaces. Jordan decomposition and the criteria for the solvability and openness of the operators $f(A)$ with an algebraic operator A are established there in constructive manner in terms of explicit relations. We also provide a criterion for a closed operator to be bounded and algebraic.

Starting with A. Taylor's setting, we discuss approaches to introducing a holomorphic calculus of a closed operator with non-empty resolvent set paying attention to the mixed setting of an operator with both sectorial and bounded spectral components in §5. We check the correctness, uniqueness, continuity and other standard properties of a functional calculus and provide a representation based on results from previous sections.

The formulas are numbered independently in every logical unit of the text, such as a definition, remark, theorem, lemma, corollary and their proofs.

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2. Definitions.

Let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ be extended complex plane with the base of the vicinities of ∞ consisting of the exteriors of the circles in $\mathbb{C}$. Assume also that $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a sequence of sets $\{A_k\}_{k\in\mathbb{N}}$, let us recall that

$$\liminf_{k\to\infty} A_k = \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} A_k.$$

Definition 2.1 (Cauchy domain, contour)**.** An open $G \subset \mathbb{C}_\infty$ is a *Cauchy domain* if G has a finite number of components (maximal connected subsets) $\{G_i\}_{i=1}^m$ with disjoint closures $\bar{G}_i \cap \bar{G}_j = \emptyset$ for $i \neq j$, such that the boundary $\partial G = \cup_{i=1}^m \partial G_i$ is composed of a finite number of closed rectifiable and pairwise-disjoint Jordan curves. A *contour* is an oriented boundary of a Cauchy domain. The positive orientation of ∂G designated by $+\partial G$ is defined by the positive (counterclockwise) orientation $+\partial G_i$ of ∂G_i for every i.

If $F \subset \Omega \subset \mathbb{C}_\infty$ for a closed F (in $\mathbb{C}_\infty$) and open Ω, a contour $\gamma = \partial G$ envelopes F in Ω if $F \subset G$ and $\bar{G} \subset \Omega$, and G is bounded if, and only if, F is bounded. Here we also assume that $\infty \in \Omega$ (or $\Omega \subset \mathbb{C}$) if, and only if, $\infty \in F$ (or $F \subset \mathbb{C}$).

Remark 2.1. It is shown in [29] (Theorem 3.1 in [29]) that there always exists a contour γ enveloping F in ω if $\partial\Omega$ is bounded.

For a linear spaces X and Y, let B be linear operator from $D(B) \subset X$ into Y, and let E be linear subset (manifold) of $D(B)$. Then $B_{|E}$ is the restriction of B to E. The relation $A \subset B$ between two operators means that $D(A) \subset D(B)$ and $A = B_{D(A)}$. One also has $D(A+B) = D(A) \cap D(B)$.

For an injective A, the inverse operator A^{-1} is the operator with the domain $D(A^{-1}) = \operatorname{Im}(A)$ satisfying $AA^{-1} = I_{|\operatorname{Im}(A)}$ (right inverse) and $A^{-1}A = I_{|D(A)}$ (left inverse).

For (complex) Banach spaces X and Y, let $\mathcal{L}(X,Y)$ and $\mathcal{C}(X,Y)$ be, correspondingly, the spaces of bounded and closed operators A from X ($D(A) \subset X$) into Y. We say that $\{A_k\}_{k\in\mathbb{N}} \subset \mathcal{C}(X,Y)$ converges to A in $\mathcal{C}(X,Y)$ if

$$D(A) = \liminf_{k\to\infty} D(A_k) \text{ and } \lim_{i\to\infty} A_k x = Ax \text{ for every } x \in D(A).$$

Assume also that $\mathcal{L}(X) = \mathcal{L}(X, X)$ and $\mathcal{C}(X) = \mathcal{C}(X, X)$.

For $n \in \mathbb{N}$, a Banach space X and a closed operator $A \in \mathcal{C}(X)$, we assume that $D_0(A) = X$,

$$D_n(A) = D(A^n) = \{x \in D(A): \ A^i x \in D(A) \text{ for } 1 \le i < n\}$$

and

$$D_\infty(A) = \bigcap_{n\in\mathbb{N}} D_n(A).$$

Definition 2.2. For a Banach space X and a closed $A \in \mathcal{C}(X)$, the *resolvent set* $\rho(A)$ is the open set of $\lambda \in \mathbb{C}$, such that there exists $(A - \lambda I)^{-1} \in \mathcal{L}(X)$. The *spectrum* $\sigma(A)$ is the complement $\mathbb{C} \setminus \rho(A)$. We shall always assume that $\infty \in \sigma(A)$ if, and only if, $A \notin \mathcal{L}(X)$, and $\rho(A) \neq \emptyset$.

If λ is an isolated point of $\sigma(A)$, then the operator-valued function $(z - A)^{-1}$ (called the *resolvent*) is analytic in $H\setminus\{\lambda\}$, where an open $H \subset \mathbb{C}$ is a neighborhood of λ disjoint with $\sigma(A) \setminus \{\lambda\}$, and can be expanded into the Laurent series

$$(z - A)^{-1} = \sum_{i\in\mathbb{Z}} A_i (z - \lambda)^i.$$

For $n \in \mathbb{N}$, the point λ is a *pole* of the order n if $A_{-n} \neq 0$ but $A_i = 0$ for $i < -n$.

Definition 2.3 (Classes of holomorphic functions)**.** For an open $\Omega \subset \mathbb{C}_\infty$, let $H(\Omega)$ be the class of all *bounded holomorphic functions* on Ω endowed with the topology of the uniform convergence on the compact subsets of $\Omega \cap \mathbb{C}$. Let $H_\infty(\Omega)$ be the class of all bounded holomorphic functions on Ω endowed with the norm inherited from $L_\infty(\Omega)$.

For $\Omega \supset \mathbb{C} \setminus C$, where C is a disc in $\mathbb{C}$, and $d \in \mathbb{N}$, let $H^d(\Omega)$ be the subspace of $H(\Omega)$ consisting of the functions with the pole at ∞ of an order not greater than d, and let $H^{\mathbb{N}}(\Omega) = \bigcup_{n\in\mathbb{N}} H^n(\Omega)$. Assume also that $H^0(\Omega) = H(\Omega \cup \{\infty\})$, and $H^{-1}(\Omega \cup \{\infty\})$ is the subspace of $H^0(\Omega)$ consisting of the functions with zeros of order, at least, 1 at ∞, i.e. satisfying

$$f(\infty) = \lim_{|z|\to\infty} f(z) = 0.$$

Definition 2.4 (Holomorphic functional calculus)**.** For a subspace $Y \subset H^{\mathbb{N}}(\Omega)$ and a closed operator $B : X \supset D(B) \to X$ with $\sigma(B) \subset \Omega$, we say that B possess the Y*-functional calculus* if there exists a mapping $\mathcal{F}: \ Y \to \mathcal{C}(X), \ f \mapsto f(B)$ satisfying

1) $\mathcal{F}(0) = 0$;
2) $\mathcal{F}(1) = I$ if $1 \in Y$;
3) $\mathcal{F}(z) = B$ if $z \in Y$;
4) $\mathcal{F}(\alpha f + \beta g) = \alpha\mathcal{F}(f) + \beta\mathcal{F}(g)$ if $\alpha, \beta \in \mathbb{C}$ and $f, g \in Y$;
5) $\mathcal{F}(fg)x = \mathcal{F}(f)\mathcal{F}(g)x$ if $f, g, fg \in Y$ and $x \in D(\mathcal{F}(fg)) \bigcap D(\mathcal{F}(f)\mathcal{F}(g))$;
6) $\lim_{k\to\infty} f_k(B) \overset{\mathcal{C}(X)}{=} f(B)$ if $\{f_k\}_{k=1}^\infty \subset Y$ and $\lim_{k\to\infty} f_k \overset{H(\Omega)}{=} f$ in the topology of the uniform convergence on compact subsets inherited from $H(\Omega)$;
7) $g(B) = f(B)$ if $g = f_{|G}$ for an open G with the bounded ∂G satisfying $\sigma(B) \subset G$ and $\bar{G} \subset \Omega$.

The operator B is also said to possess the bounded $H_\infty(\Omega)$-functional calculus if, in addition, one has $\mathrm{Im}(\mathcal{F}) \subset \mathcal{L}(X)$ and $\mathcal{F} \in \mathcal{L}(H_\infty(G), \mathcal{L}(X))$:

8) $\|\mathcal{F}(f)\|_{\mathcal{L}(X)} \le C\|f\|_{H_\infty(G)}$ for every $f \in H_\infty(G)$.

The value of the best constant C in 6) is the norm $\|\mathcal{F}|\,\mathcal{L}(H_\infty(G), \mathcal{L}(X))\|$ of the functional calculus operator $\mathcal{F}$.

3. Gel'fond's formula: real and complex cases

This section is devoted to Gel'fond's formula and its real line counterpart that is the outcome of the work started by Ch. Hermite [15] in 1877 and going back to A.-L. Cauchy, J.L. Lagrange, G. Leibniz, C. Maclaurin, I. Newton and B. Taylor.

3.1. **Complex case and background.** Assume that $F = \{z_j\}_{j=1}^n \subset G \subset \mathbb{C}$, where $z_i \neq z_j$ for $i \neq j$, $G \subset \mathbb{C}$ is an open subset of a complex plane, and that a contour γ envelopes $\{z_j\}_{j=1}^n$ in G.

In 1871, G. Frobenius [12] found explicit formulas for the coefficients of the Newton series and for the remainder in terms of Cauchy integrals. Naturally, the remainder term appeared coinciding with that in the formula (cFH) below (see [3] for more details).

In 1877, Ch. Hermite [15] has described how to find the polynomial p_f that solves the complex Hermite interpolation problem of finding a polynomial of the degree less than m satisfying

$$p_f^{(i)}(z_j) = f^{(i)}(z_j) \text{ for } 1 \leq j \leq n \text{ and } 0 \leq i < m_j \in \mathbb{N}_0, \tag{cH}$$

where $m = \sum_{j=1}^n m_j$ and f is a given function from $H(G)$. He was primarily concerned with establishing the following integral representation for the residual term:

$$f(z) - p_f(z) = \frac{\omega(z)}{2\pi i} \oint_\gamma \frac{f(\zeta)}{(\zeta - z)\omega(\zeta)} d\zeta. \tag{cFH}$$

where

$$\omega(z) = \prod_{j=1}^n (z - z_j)^{m_j}.$$

The case $m_j = 1$ for $1 \leq j \leq n$ corresponds to the Lagrange approximation problem considered much earlier, while the case $n = 1$ is the Taylor interpolation problem.

In 1883, Sylvester used Lagrange's solution to the Lagrange approximation problem to define a function $f(A)$ of a matrix A with the minimal polynomial $\omega(A) = 0$ as the polynomial $p_f(A)$, where p_f is the Lagrange polynomial

$$p_f(z) = \sum_{j=1}^n f(z_j) \frac{\omega_j(z)}{\omega_j(z_j)}, \text{ where } \omega_j(z) = \frac{\omega(z)}{z - z_j}.$$

In 1886, Buchheim [4] has lifted the restriction $m_j = 1$ defining $f(A)$ as $p_f(A)$, where p_f is the solution to the Hermite interpolation problem:

$$p_f(z) = \sum_{j=1}^n \omega_j(z) \sum_{i=0}^{m_j - 1} (i!)^{-1} \left(\frac{f}{\omega_j}\right)^{(i)} (z_j)(z - z_j)^i, \text{ where}$$

$\omega_j(z) = \omega(z)/(z - z_j)^{m_j}$.

Eventually, with the aid of his contribution to the theory of divided differences of analytic functions, Gel'fond [13] has established the following remarkable explicit formula: for $f \in H(G)$ and $z \in G$, one has

$$f(z) = \sum_{j=1} \omega_j(z) \sum_{i=0}^{m_j - 1} (i!)^{-1} \left(\frac{f}{\omega_j}\right)^{(i)} (z_j)(z - z_j)^i + \omega(z) \triangle_{\{(z,1)\} \cup \{(z_j, m_j)\}_{j=1}^n} f, \tag{G}$$

where $\omega_j(z) = \omega(z)/(z - z_j)^{m_j}$ and $\triangle\{z\} \cup F f$ is the divided difference of f with the nodes $\{z\} \cup \{z_j\}_{j=1}^n$ with the multiplicities 1 and $\{m_j\}_{j=1}^n$ correspondingly (see [13] for various equivalent definitions in complex case). We should note that in complex case the divided difference with the multiple nodes $F^* = \{(z_j, m_j)\}_{j=1}^n$ can be defined by

$$\triangle_{F^*} f = \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{\omega(\zeta)} d\zeta$$

meaning that the residual in Gel'fond's formula is, in fact, in Hermite's form, while p_f is the outcome of Hermite's method. Gel'fond's proof is rather simple and uses only the Cauchy representation formula and its traditional corollaries.

3.2. **Abstract Hermite decomposition and divided differences.** In this subsection we first define and investigate the properties of the divided differences in the case of functions on open subsets of the real line with minimal differentiability requirements, and, then, utilise Gel'fond's method to deduce a counterpart of his formula in the real case. Note that the same approach works in the complex case as well thanks to the remarkable differentiability properties of holomorphic functions. In particular, we provide very simple proof of the continuous dependence of the divided difference on its nodes and the derivation of the Hermite interpolation polynomial.

We shall deal with the specific classes (linear algebras) of differentiable functions defined as follows.

Definition 3.1. For $n \in \mathbb{N}$ and $\{m_j\}_{j=1}^n$, let $F = \{x_j\}_{j=1}^n \subset \mathbb{R}$ (or $\mathbb{C}$) be distinct points, let $F^* = \{(x_j, m_j)\}_{j=1}^n$ be F with multiplicities $\{m_j\}_{j=1}^n$, and $m = \sum_{j=1}^n m_j$. Assume also that open $G = \cup_{j=1}^n I_j \subset \mathbb{R}$ (or $\mathbb{C}$), where the finite or infinite interval I_j satisfies $I_j \cap F = \{x_j\}$ for every j.

Let $C^{F^*}(G)$ be the linear algebra of the functions f possessing the derivatives $\{f^{(i)}(x)\}_{i=0}^{m_j-1}$ in the interval I_j and the m_jth derivative $f^{(m_j)}(x_j)$ at x_j for every $1 \le j \le n$.

Let also $\omega(x) = \omega_{F^*}(x) = \prod_{j=1}^n (x - x_j)^{m_j}$ and $\omega_j(x) = \omega_{F^* - (x_j, m_j)}(x) = \prod_{i \ne j} (x - x_i)^{m_i}$.

Definition 3.2 (Set operations)**.** For a finite subset $F \subset \mathbb{R}$, its *multiplicity relation* $F^* \subset \mathbb{R} \times \mathbb{N}$ (or $\mathbb{C} \times \mathbb{N}$) is defined by the *multiplicity function* $\mu_{F^*} : \mathbb{R} \to \mathbb{N}_0$:

$$\mu_{F^*}(x) = \begin{cases} m_x & \text{if } (x, m_x) \in F^*; \\ 0 & \text{if } x \notin F. \end{cases} ; \text{ and } F = \operatorname{supp} \mu_{F^*}.$$

Let also $m(F^*) = \sum_{x \in F} \mu_{F^*}(x)$ and $\psi(F^*) = \sum_{x \in F} x \mu_{F^*}(x)$.

Let $F^*, H^* \subset \mathbb{R} \times \mathbb{N}_0$ be two multiplicity relations. Let the *partial relation* $F \le H$ be defined by $\mu_{F^*} \le \mu_{H^*}$. In this case, we also define their *difference* $G^* = F^* - G^*$ by $\mu_{G^*} = \mu_{H^*} - \mu_{F^*}$. Let also its *sum* $D^* = F^* + H^*$ be defined by $\mu_{D^*} = \mu_{H^*} + \mu_{F^*}$. We also assume that $\emptyset^* = \{(\emptyset, 0)\}$ and $\mu_{\emptyset^*} = 0$.

For a system $\{F_j^*\}_{j=1}^n$ of multiplicity relations, let $F_{\max}^* = \max_{j=1}^n F_j^*$ and $F_{\min}^* = \min_{j=1}^n F_j^*$ be defined, correspondingly, by the relations

$$\mu_{F_{\max}^*} = \max_j \mu_{F_j^*} \text{ and } \mu_{F_{\min}^*} = \min_j \mu_{F_j^*}.$$

The next simple lemma is a direct corollary of the Leibniz formula.

Lemma 3.1. *For $n \in \mathbb{N}$, a point $x_0 \in I \subset \mathbb{R}$ (or $\mathbb{C}$) and open I, let $\{f, g, h\} \cup \{f_k\}_{k \in \mathbb{N}}$ possess the derivatives up to order $m - 1$ (inclusive) in I and up to order m at x_0. Then, one has*
a) $(fg)^{(i)}(x_0) = (fh)^{(i)}(x_0)$ for $0 \le i \le m$ if $g^{(i)}(x_0) = h^{(i)}(x_0)$ for $1 \le i \le m$;
b) $\lim_{k \to \infty} (f_k g)^{(i)}(x_0) = (fg)^{(i)}(x_0)$ for $0 \le i \le m$ if $\lim_{k \to \infty} f_k^{(i)}(x_0) = f^{(i)}(x_0)$ for $1 \le i \le m$.

In the next theorem we establish an abstract Hermite (AH) decomposition providing a linear projection onto a general space of quasi-polynomials. This generalization has a simpler proof and will be used in §5 in relation with a functional calculus.

Theorem 3.1 (AH-decomposition). *For $n \in \mathbb{N}$ and a multiplicity relation $F^* = \{(z_j, m_j)\}_{j=1}^n$, let $\{f\} \cup \{g_j\}_{j=1}^n \subset C^{F^*-F\times\{1\}}(G)$ and f/g_j have the derivatives up to the order $m_j - 2$ in I_j (see Def. 3.1) and up to the order $m_j - 1$ at x_j, while $g^{(l)}(z_k) = 0$ for $1 \le j, k \le n$, $k \ne j$ and $0 \le l < m_k$. Then the following $\vec{g}$-quasi-polynomial solves the Hermite problem for the quasi-polynomials $\sum_{j=1}^n g_j \mathcal{P}_{m_j-1}$:*

$$p_{\vec{g},F^*} f(x) = \sum_{j=1} g_j(x) \sum_{i=0}^{m_j-1} \frac{1}{i!} \left(\frac{f}{g_j}\right)^{(i)} (x_j)(x - x_j)^i, \text{ and} \tag{1}$$

we have the decomposition

a) $f(x) = p_{\vec{g},F^*} f(x) + \omega_{F^*-F\times\{1\}}(x) r(x)$ for $f \in C^{F^*}(G)$, where $r \in C(G)$, $r(z_j) = 0$ and there exists $r^{(m_j-2)}(x)$ for $x \in I_j \setminus \{z_j\}$ (see Def. 3.1) for every $1 \le j \le n$;

b) $f(x) = p_{\vec{g},F^*} f(x) + \omega_{F^*}(x) r_{\vec{g}}(x)$ for $f \in H(G)$, $F \subset G \subset \mathbb{C}$, where $r_{\vec{g}} \in H(G)$, if $\{g_j, 1/g_j\}_{j=1}^n \subset H(G)$.

The proof of Theorem 3.1. According to Part *a*) of Lemma 3.1 the function $\phi_j = T_{m_j-1}(f/g_j, z_j) g_j$, where $T_{m_j-1}(f/g_j, z_j)$ is the Taylor polynomial of the order $m_j - 1$ at z_j possesses the derivatives

$$\phi_j^{(l)}(z_k) = \delta_{kj} f^{(l)}(z_k) \text{ for } k \ne j,\ 0 \le l < m_k$$

because $T_{m_j-1}(f/g_j, z_j)$ has the same derivatives up to the order $m_k - 1$ at z_k for every $1 \le k \le n$ as f. Hence, $\sum_{j=1}^n \phi_j$ solves the Hermite interpolation problem in the class of the $\vec{q}$-quasi-polynomials $\sum_{j=1}^n g_j \mathcal{P}_{m_j-1}$. The factorisation of the remainder term is established by induction with the aid of Part *a*) of Theorem 3.4 and the Bezout theorem (Theorem 3.2) for analytic functions in the cases of Parts *a*) and *b*) correspondingly. Indeed, $q_0 = p - p_f$ possesses 0 derivatives up to order $m_j - 1$ at z_j for $1 \le j \le n$. Therefore, $q_0(x) = q_1(x)(x - z_1)^{m_1-1}$ thanks to Part *a*) of Theorem 3.4 and q_0 is continuous in G and possesses 0 derivatives up to order $m_j - 1$ at z_j for $2 \le j \le n$. Eventually, we set $r = q_n$. If $f \in H(G)$, we use the stronger factorisation steps $q_{j-1}(z) = (z - z_j)^{m_j} a_j(z)$. □

Remark 3.1. Let us note that the multiplication operation $(f, g) \mapsto p_{\vec{g},F^*}(fg)$ for $f, g \in \sum_{j=1}^n g_j \mathcal{P}_{m_j-1}$ provides (together with the natural linear structure) a non-trivial ring structure for $\sum_{j=1}^n g_j \mathcal{P}_{m_j-1}$. In classical purely polynomial choice of $\vec{g}$ considered in the next theorem, the ring $\sum_{j=1}^n g_j \mathcal{P}_{m_j-1} = \mathcal{P}_{m-1}$ endowed with this structure appears to be isomorphic to the quotient $\mathcal{P}/\omega_{F^*}\mathcal{P}$, where $\mathcal{P} = \bigcup_{l\in\mathbb{N}} \mathcal{P}_l$ (see Theorem 3.3 and Corollaries 3.1 and 3.4 and pages 52–53 in [3]).

We shall also need the Bezout theorem that easily follows from the Taylor representation of a polynomial.

Theorem 3.2. *For $l \in \mathbb{N}_0$, let p be a polynomial (or rational, or analytic function at x_0). Then we have $p^{(i)}(x_0) = 0$ for $0 \le i \le l - 1$ if, and only if, $p(x) = (x - x_0)^l r(x)$, where $r(x)$ is a polynomial (or rational, or analytic function at x_0).*

The next theorem is well-known (for example, see [2] p. 147) but we provide a short proof based on the previous lemma avoiding the traditional usage of either the L'Hôpital rule or the Taylor expansions of rational functions. It should be noted that the interpolation polynomial does not depend on G as far as G contains the nodes F. This theorem is a particular case $g_j = \omega_j$ of the previous theorem. We provide the same proof to compare its simplicity with the existing proofs.

Theorem 3.3 (Hermite interpolation polynomial)**.** *Let $f \in C^{F^*-F\times\{1\}}(G)$. Then the real Hermite interpolation problem of finding a polynomial of a degree less than m satisfying*

$$p^{(i)}(x_j) = f^{(i)}(x_j) \text{ for } 1 \le j \le n \text{ and } 0 \le i < m_j \in \mathbb{N}, \tag{rH}$$

where $m = \sum_{j=1}^n m_j$, has the unique solution

$$p_{F^*}f(x) = \sum_{j=1} \omega_j(x) \sum_{i=0}^{m_j-1} \frac{1}{i!} \left(\frac{f}{\omega_j}\right)^{(i)}(x_j)(x-x_j)^i. \tag{1}$$

Moreover, if p is a polynomial of a degree $l \ge m$ satisfying the conditions (rH), then

$$p(x) = p_{F^*}f(x) + \omega_{F^*}(x)q(x),$$

where $q(x)$ is a polynomial of the degree $l-m$.

The proof of Theorem 3.3. Applying the Bezout theorem to $q_0 = p - p_f$ at $x_0 = x_1$, then to q_1 at $x_0 = x_2$ and so on, we obtain the factorizations $r(x) = (x - x_1)^{m_1} r_1(x)$ ... $q_0 = \omega q$. It also gives the uniqueness.

Let us now find the polynomial p_j of the minimal degree, satisfying, for some $1 \le j \le n$,

$$p_j^{(i)}(x_k) = \delta_{kj} f^{(i)}(x_j) \text{ for } 1 \le k \le n \text{ and } 0 \le i < m_j \in \mathbb{N}, \tag{$rH(j)$}$$

According to the Bezout theorem, $p_j = \omega_j r_j$ for a polynomial r_j of degree less then m_j. Eventually, thanks to part a) of Lemma 3.1, we can take as r_j the Taylor polynomial of the degree $m_j - 1$ for $\frac{f}{\omega_j}$ at the point x_j because it has the same derivatives at x_j as $\frac{f}{\omega_j}$:

$$p_j(x) = \omega_j(x) \sum_{i=0}^{m_j-1} \frac{1}{i!} \left(\frac{f}{\omega_j}\right)^{(i)}(x_j)(x-x_j)^i.$$

Thus, one has $p_j^{(i)}(x_j) = f^{(i)}(x_j)$ since $f = \omega_j \frac{f}{\omega}$ in a neighborhood of x_j. We finish the proof by taking $p_f = \sum_{j=1}^n p_j$. □

Corollary 3.1. *For $f, g \in C^{F^*}(G)$ and $\vec{g}$ as in Theorem 3.1, one has*

$$p_{\vec{g},F^*}(fg) = p_{\vec{g},F^*}\left(f p_{\vec{g},F^*} g\right) = p_{\vec{g},F^*}\left(p_{\vec{g},F^*} f p_{\vec{g},F^*} g\right).$$

The proof of Corollary 3.1 According to Theorem 3.1, p_{F^*} depends on the derivatives $\{f^{(k)}(x)\}_{\{(x,k)\} \le F^*}$, and these derivatives are the same for the products fg, $fp_{F^*}g$ and $p_{F^*}fp_{F^*}g$ thanks to Lemma 3.1. □

Now we are in a position to define a divided difference with the nodes of arbitrary multiplicities.

Definition 3.3 (Divided difference)**.** For $F^* = \{(z_j, m_j)\}_{j=1}^n$ as in Definition 3.1 and $m = m(F^*)$, let $f \in C^{F^*}(G)$. Then the *divided difference* $\triangle_{F^*}f$ is the coefficient near the senior power x^{m-1} of the Hermite interpolation polynomial p_f, i.e.

$$\triangle_{F^*}f = \frac{(p_{F^*}f)^{(m-1)}}{(m-1)!} = \sum_{j=1}^n \frac{1}{(m_j-1)!}\left(\frac{f}{\omega_j}\right)^{(m_j-1)}(z_j). \tag{$\triangle_{F^*}$}$$

This definition (without the explicit expression for the divided difference) was given by G. Kowalewski [20] in 1932. In 1938, Chakalov [5] gave an explicit formula in the form

$$\triangle_{F^*} f = \sum_{j=1}^{n} \frac{1}{(m_j-1)!} \sum_{k=0}^{m_j-1} a_{j,k} f^{(k)}(x_j), \text{ where}$$

$$\frac{1}{\omega_{F^*}(z)} = \sum_{j=1}^{n} \frac{1}{(m_j-1)!} \sum_{k=0}^{m_j-1} a_{j,k} \frac{k!}{(z-z_j)^{k+1}},$$

under the condition that f is smooth enough to be approximated (together with some of its derivatives) by the linear combinations of the functions $(\cdot - z)^{-1}$ for z from $\mathbb{R}$ or $\mathbb{C}$ (if f is analytic).

The next lemma is a the well-known generalization of the *Leibnitz rule*. According to de Boor [3], the earliest proofs are provided by T. Popoviciu (1933) and Steffensen (1939) We provide a short direct proof in terms of our notation. The term "Leibniz rule" is due to C. de Boor (see Corollaries 28 and 30 in [3] for a different proof and a generalisation).

Lemma 3.2 (Leibnitz rule)**.** *For a multiplicity relation F^* with $m(F^*) = m$, let $\{F_k^*\}_{k=0}^m$ be a maximal monotone sequence ("ladder") of multiplicity relations satisfying*

$$F_0^* = \emptyset^*, \ F_m^* = F^*, \ F_k^* - F_{k-1}^* = (x, 1) \text{ for some } x \in F \text{ and } 0 < k \leq m.$$

Assume also that $f, g \in C^{F^*}(G)$ and $\bar{F}_k^* = F^* - F_k^*$ for $1 \leq k \leq m$. Then one has

$$\triangle_{F^*}(fg) = \sum_{k=1}^{m} \triangle_{F_k} f \triangle_{\bar{F}_{k-1}^*} g.$$

The proof of Lemma 3.2. The Hermite interpolation operator p_{F^*} can be represented in the forms

$$p_{F^*} f = \sum_{k=1}^{m} p_{F_k^*} f - p_{F_{k-1}^*} f \text{ and } p_{F^*} g = \sum_{k=1}^{m} p_{\bar{F}_{k-1}^*} g - p_{\bar{F}_k^*} g. \tag{1}$$

Part a) of Lemma 3.2, Theorem 3.3, Definition 3.1 and Corollary 3.1 imply the identities

$$\triangle_{F^*}(fg) = \triangle_{F^*}\left(p_{F^*} f p_{F^*} g\right) \text{ and } p_{F^*}\left(p_{F^*} f p_{F^*} g\right) = p_{F^*}(fg) \tag{2}$$

Thus, we have

$$p_{F^*} f p_{F^*} g = \sum_{i,j=1}^{m} (p_{F_i^*} f - p_{F_{i-1}^*} f)(p_{\bar{F}_{j-1}^*} g - p_{\bar{F}_j^*} g) =$$

$$= \sum_{i \leq j} (p_{F_i^*} f - p_{F_{i-1}^*} f)(p_{\bar{F}_{j-1}^*} g - p_{\bar{F}_j^*} g) + \sum_{i > j} p_{F_i^*} f - p_{F_{i-1}^*} f)(p_{\bar{F}_{j-1}^*} g - p_{\bar{F}_j^*} g) = I_0 + I_1. \tag{3}$$

Comparing (3) with the last assertion of Theorem 3.3 and the uniqueness of the Hermite interpolation polynomial $p_{F^*}\left(p_{F^*} f p_{F^*} g\right)$, we see that

$$I_0 = \sum_{i \leq j} (p_{F_i^*} f - p_{F_{i-1}^*} f)(p_{\bar{F}_{j-1}^*} g - p_{\bar{F}_j^*} g) = p_{F^*}\left(p_{F^*} f p_{F^*} g\right) \text{ and } I_1 = \omega_{F^*} q, \tag{4}$$

where q is a polynomial. Indeed, all summands in I_1 have the factor ω_{F^*} according to Theorem 3.3. We finish the proof by comparing the coefficients near x^{m-1} in the both sides of the first identity in (4). $\square$

This explicit formula for a divided difference in Definition 3.3 leads to a simple proof of both the merging and general continuity of a divided difference of a minimally smooth function with the aid of the classical B. Taylor's expansions with the remainder terms in the forms of G. Peano and J.L. Lagrange.

Part *a*) of the next theorem is due to G. Peano, while Part *b*) follows from the corresponding result of J.L. Lagrange

Theorem 3.4. *a*) *For an interval $I \subset \mathbb{R}$, $x \in I$ and $n \in \mathbb{N}$, let g possess the derivatives $g^{(n)}(x)$ and $g^{(n-1)}(y)$ for every $y \in I$. Then one has, for $y \in I$*

$$g(y) = \sum_{k=0}^{n} \frac{g^{(k)}(x)}{k!}(y-x)^k + (y-x)^n r_P(y),$$

where r_P is continuous on I with $r_P(x) = 0$, and there exists $r_P^{(n-1)}(y)$ for $y \in I \setminus \{x\}$.
b) *For an interval $I \subset \mathbb{R}$, $x \in I$ and $n \in \mathbb{N}$, let g possess the bounded derivative $g^{(n+1)}$ on I. Then one has, for $y, z \in I$,*

$$g(y) = \sum_{k=0}^{n} \frac{g^{(k)}(z)}{k!}(y-z)^k + (y-z)^n r_L(y,z),$$

where $\lim_{y-z\to 0} \sup_{z\in I} r_L(y,z) = 0$.

Corollary 3.2. *For $m_0, m_1 \in \mathbb{N}$ and $n = m_0 + m_1 - 1$, let g satisfy either Part a), or Part b) of Theorem 3.4. Then we have the representation*

$$\frac{1}{(m_0-1)!}\left(\frac{g}{(\cdot - x)^{m_1}}\right)^{(m_0-1)}(y) + \frac{1}{(m_1-1)!}\left(\frac{g}{(\cdot - y)^{m_0}}\right)^{(m_1-1)}(x) = \frac{g^{(n)}}{n!}(x) + r,$$

where r possesses either the properties of r_P in Part a) of Theorem 3.4 (except for the existence of $r_P^{(n-1)}(y)$ for $y \in I \setminus \{x\}$), or r_L in Part b) of Theorem 3.4 correspondingly.

The proof of Corollary 3.2. For $0 \le l \le n$, Theorem 3.4 justifies the expansion

$$\frac{g^{(l)}(y)}{l!} = \sum_{k=l}^{n} \frac{g^{(k)}(x)}{k!}(y-x)^{k-l} + (y-x)^{n-l} r_l. \tag{1}$$

With the aid of the Leibnitz rule and (1), we obtain the identities

$$\begin{aligned}
&\frac{(y-x)^n}{(m_0-1)!}\left(\frac{g}{(\cdot - x)^{m_1}}\right)^{(m_0-1)}(y) \\
&= \sum_{l=0}^{m_0-1} \frac{g^{(l)}(y)}{l!}(y-x)^l (-1)^{m_0-1-l}\binom{m_0+m_1-2-l}{m_0-1-l} \\
&= \sum_{k=0}^{n} \frac{g^{(k)}(x)}{k!}(y-x)^k \sum_{l=0}^{\min(k,m_0-1)} \binom{k}{l}(-1)^{m_0-1-l}\binom{m_0+m_1-2-l}{m_0-1-l} \\
&\quad + (y-x)^n \sum_{l=0}^{m_0-1} r_l (-1)^{m_0-1-l}\binom{m_0+m_1-2-l}{m_0-1-l}.
\end{aligned} \tag{2}$$

Multiplying the Taylor expansions for $(1+z)^k$ and $(1+z)^{-m_1}$ on the (open) unit disc, we see that

$$\sum_{i=0} c_{k,i} z^i = (1+z)^{k-m_1}$$

and

$$c_{k,m_0-1} = \sum_{l=0}^{\min(k,m_0-1)} \binom{k}{l} (-1)^{m_0-1-l} \binom{m_0+m_1-2-l}{m_0-1-l}. \tag{3}$$

The computation of c_{k,m_0-1} (using Theorem 3.4, for example) gives us the identity

$$c_{k,m_0-1} = \begin{cases} (-1)^{m_0-1}\binom{n-k-1}{m_1-k-1} & \text{if } 0 \le k \le m_1-1; \\ 0 & \text{if } m_1 \le k < n; \\ 1 & \text{if } k = n. \end{cases} \tag{4}$$

In turn, the Leibniz rule also provides

$$\frac{(y-x)^n}{(m_1-1)!} \left(\frac{g}{(\cdot - y)^{m_0}} \right)^{(m_1-1)} (x) = (-1)^{m_0} \sum_{k=0}^{m_1-1} \frac{g^k(x)}{k!} (y-x)^k \binom{n-1-k}{m_1-k-1}. \tag{5}$$

The addition of (3) and (5) with the aid of (4) and Theorem 3.4 finish the proof with

$$r = \sum_{l=0}^{m_0-1} r_l (-1)^{m_0-1-l} \binom{n-1-l}{m_0-1-l}. \tag{6}$$

□

The next two theorems provide sufficiently sharp conditions imposed on a function f for the continuous dependence of its divided difference with respect to two cases of the simultaneous convergence of (multiple) nodes. When p is a polynomial, it is shown by de Boor (see Proposition 21 in [3]) that a shorter and simpler proof of both theorems at once is available.

Theorem 3.5 (Merging convergence)**.** *For $n \in \mathbb{N}$, $\{n_j\}_{j=1}^n \subset \mathbb{N}$ and $\{m_{j,i}\}_{j=1,i=0}^{n,n_j} \subset \mathbb{N}$, let $F^* = \{(x_{j,i}, m_{j,i})\}_{j=1,i=0}^{n,n_j}$ and $F_0^* = \{(x_{j,0}, m_j)\}_{j=1}^n$ with $m_j = \sum_{i=0}^{n_j} m_{j,i}$ for $1 \le j \le n$. Assume also that $\{x_{j,i}\}_{i=0} \subset I_j$, where $I_j \subset \mathbb{R}$ is an interval, and $G = \cup_{j=1}^n I_j$ corresponds to F_0^* in the sense of Definition 3.1. Then, for every $f \in C^{F_0^* - F_0 \times \{1\}}(G)$, we have*

$$\lim_{F^* \to F_0^*} \triangle_{F^*} f = \triangle_{F_0^*} f,$$

where $F^* \to F_0^*$ means that $x_{j,i} \to x_{j,0}$ for every $1 \le j \le n$ and $0 \le i \le n_j$.

The proof of Theorem 3.5. Let us define

$$\omega(x) = \prod_{j=1}^n \prod_{i=0}^{n_j} (x - x_{j,i}),\ \omega_0(x) = \prod_{j=1}^n (x - x_{j,0})^{m_j} \text{ and}$$

$$\omega_{j,i}(x) = \frac{\omega}{(x - x_{j,i})^{m_{j,i}}},\ \omega_{0,j}(x) = \frac{\omega_0(x)}{(x - x_{j,0})^{m_j}}.$$

Then we have, thanks to Definition 3.2,

$$\triangle_{F_0^*} f = \sum_{j=1}^n \frac{1}{(m_j-1)!} \left(\frac{f}{\omega_j} \right)^{(m_j-1)} (x_{j,0}) \text{ and}$$

$$\triangle_{F^*} f = \sum_{j=1}^n \sum_{i=0}^{n_j} \frac{1}{(m_{j,i}-1)!} \left(\frac{f}{\omega_{j,i}} \right)^{(m_{j,i}-1)} (x_{j,i}). \tag{1}$$

Thanks to Part b) of Lemma 3.1, the summands

$$I_0 = \sum_{i=0}^{n_{j_0}} \frac{1}{(m_{j_0,i}-1)!} \left(\frac{f}{\omega_{j_0,i}} \right)^{(m_{j_0,i}-1)} (x_{j_0,i}) \text{ and } J_0 = \frac{1}{(m_{j_0}-1)!} \left(\frac{f}{\omega_j} \right)^{(m_{j_0}-1)} (x_{j_0,0}) \tag{2}$$

depend continuously on $\{x_{j,i}\}_{j\neq j_0}^{1\le i\le n_j}$. This means that it is enough to prove that the first expression I_0 converges to J_0 if $x_{j_0,i}$ converges to $x_{j_0,0}$ for every $1 \le i \le n_{j_0}$.

Without loss of generality, we assume that $j_0 = 1$. Corollary 3.2 (the Lagrange reminder version) permits us to start calculating I_0 with

$$\begin{aligned} I_2 &= \frac{1}{(m_{1,1}-1)!}\left(\frac{f}{\omega_{1,1}}\right)^{(m_{1,1}-1)}(x_{1,1}) + \frac{1}{(m_{1,1}-1)!}\left(\frac{f}{\omega_{1,2}}\right)^{(m_{1,2}-1)}(x_{1,2}) \\ &= \frac{1}{(m_{1,1}+m_{1,2}-1)!}\left(\frac{f}{\omega_{F^*-\{(x_{1,2},m_{1,1}),(x_{1,2},m_{1,1})\}}}\right)^{(m_{1,1}+m_{1,2}-1)}(x_{1,2}) \\ &\quad + r_{L2}(x_{1,1}-x_{1,2},x_{1,2}). \end{aligned} \tag{3}$$

Continuing in the same manner and calculating

$$I_{j+1} = I_j + \frac{1}{(m_{1,j+1}-1)!}\left(\frac{f}{\omega_{1,j+1}}\right)^{(m_{1,j+1}-1)}(x_{1,j+1}),$$

we obtain

$$I_{n_1} = \frac{1}{(m_{1,1}-1)!}\left(\frac{f}{\omega_{F^*-\sum_{i=1}^{n_1}\{(x_{1,i},m_{1,i})\}}}\right)^{(m_1-m_{1,0}-1)}(x_{1,n_1}) + \sum_{i=2}^{n_1} r_{Li}(x_{1,i-1}-x_{1,i},x_{1,i}). \tag{4}$$

Eventually we apply Corollary 3.2 with the Peano remainder to establish

$$\begin{aligned} I_0 &= I_j + \frac{1}{(m_{1,0}-1)!}\left(\frac{f}{\omega_{1,0}}\right)^{(m_{1,0}-1)}(x_{1,0}) \\ &= \frac{1}{(m_1-1)!}\left(\frac{f}{\omega_{0,1}}\right)^{(m_1-1)}(x_{1,0}) + r_{P0}(x_{1,n_1}-x_{1,0}) + \sum_{i=2}^{n_1} r_{Li}. \end{aligned} \tag{5}$$

Since the remainder terms in (5) converge to 0 when $x_{1,i} \to x_{1,0}$ for $0 < i \le n_1$, we have just proved that the left expression in (2) converges to the right one. This finishes the proof of the theorem. □

Theorem 3.6 (General convergence)**.** *For $n \in \mathbb{N}$, $\{n_j\}_{j=1}^n \subset \mathbb{N}$ and $\{m_{j,i}\}_{j=1,i=1}^{n,n_j} \subset \mathbb{N}$, let $F^* = \{(x_{j,i}, m_{j,i})\}_{j=1,i=1}^{n,n_j}$ and $H^* = \{(x_j, m_j)\}_{j=1}^n$ with $m_j = \sum_{i=1}^{n_j} m_{j,i}$ for $1 \le j \le n$. Assume also that $\{x_{j,i}\}_{i=0} \subset I_j$, where $I_j \subset \mathbb{R}$ is an interval, and $G = \cup_{j=1}^n I_j$ corresponds to H^* in the sense of Definition* 3.1*. Then, for every f with the bounded $f^{(m_j)}$ on I_j for every $1 \le j \le n$, we have*

$$\lim_{F^*\to H^*} \triangle_{F^*} f = \triangle_{F_0^*} f,$$

where $F^* \to H^*$ means that $x_{j,i} \to x_j$ if for every $1 \le j \le n$ and $1 \le i \le n_j$.

The proof of Theorem 3.6. It is almost literal repetition of the proof of Theorem 3.5 with only two exceptions: we should always use the Lagrange remainder version of Corollary 3.2, and at the very last step we have to show that

$$\lim_{x_{j_0,n_{j_0}}\to x_{j_0}} \frac{1}{(m_{j_0}-1)!}\left(\frac{f}{\omega_{0,j_0}}\right)^{(m_{j_0,n_{j_0}}-1)}(x_{j_0,n_{j_0}}) = \frac{1}{(m_{j_0}-1)!}\left(\frac{f}{\omega_{j_0}}\right)^{(m_{j_0}-1)}(x_{j_0})$$

for $1 \le j_0 \le n$ with the aid of Part *a*) of Theorem 3.4. □

The next theorem provides minimally necessary smoothness conditions for the validity of Gel'fond's formula in the real case.

Theorem 3.7 (General Gel'fond's formula)**.** *For $n \in \mathbb{N}$ and a multiplicity relation $F^* = \{(x_j, m_j)\}_{j=1}^n$. Then, for every $f \in C^{F^*}(G)$ and $x \in G$, we have*

$$f(x) = p_{F^*} f(x) + \omega_{F^*}(x) \triangle_{F_x^*} f,$$

where $F_x^* = F^* + \{(x,1)\}$.

Remark 3.2. It is interesting that Gel'fond's formula reflects, in particular, a pure algebraic phenomenon. If $\mathcal{P}$ is the ring of polynomials, and p is an arbitrary polynomial of degree m, then the ideal $p\mathcal{P}$ is described by the conditions

$$q \in p\mathcal{P} \Longleftrightarrow q^{(k)}(z_j) = 0 \text{ for } 0 \le k < m_j,\ 1 \le j \le n, \text{ where } p = c\omega_{F^*},$$

while the dimension of $\mathcal{P}/p\mathcal{P}$ is $m-1$. Similarly, in the case of the ring $H(\Omega)$ for an open $\Omega \subset \mathbb{C}$, it shows that the ideal $pH(\Omega)$ for an arbitrary polynomial p (one always has $p = p_0\omega_{F^*}$ where F is the part of its roots that is in Ω) is described by the conditions

$$f \in p\mathcal{P} \Longleftrightarrow f^{(k)}(z_j) = 0 \text{ for } 0 \le k < m_j,\ 1 \le j \le n, \text{ where } F^* = \{(z_j, m_j)\}_{j=1}^n,$$

and the dimension of $H(\Omega)/pH(\Omega)$ is equal to $m(F^*) - 1$.

<u>The proof of Theorem 3.7.</u> Let $\omega_j = \omega_{F^* - \{(x_j, m_j)\}}$. Noticing that $\omega_{F_x^*}(y) = (y-x)\omega_{F^*}(y)$, we see, with the aid of Definition 3.3 and the Leibnitz rule that

$$\triangle_{F_x^*} f = \frac{f(x)}{\omega_{F^*}(x)} + \sum_{j=1}^{n} \frac{1}{(m_j-1)!}\left(\frac{f}{\omega_j(\cdot - x)}\right)^{(m_j-1)} =$$

$$= \frac{f(x)}{\omega_{F^*}(x)} - \frac{1}{\omega_{F^*}(x)} \sum_{j=1}^{n} \sum_{k=0}^{m_j-1} \left(\frac{f}{\omega_j}\right)^{(k)} \frac{(x-x_j)^k}{k!} \quad (1)$$

The proof is finished by multiplying both sides of (1) by $\omega_{F^*}(x)$ and by noticing that both sides of the resulted identity are well-defined according to Definition 3.3 also for $x \in F$ and continuous thanks to Theorem 3.5. □

The composition rule for divided differences is well-known but can be deduced from the Leibnitz rule as shown in the proof of the next corollary. A different proof can be found, for example, in [3].

Corollary 3.3 (Composition rule). *For multiplicity relations F^* and $H^* \ne \emptyset^*$, let $D^* = F^* + H^*$ and $f \in C^{D^*}$. Then one has*

$$\triangle_{D^*} f = \triangle_{H^*} \phi_{F,f} \text{ where } \phi_{F,f}(x) = \triangle_{F^* + \{(x,1)\}} f.$$

<u>The proof of Corollary 3.3.</u> Since $\triangle_{D^*} p_{F^*} f = 0$, one applies Lemma 3.2 to Gel'fond's formula (Theorem 3.7) to obtain

$$\triangle_{D^*} f = \triangle_{D^*} \left(\omega_{F^*} \phi_{F^*,f}\right) = \triangle_{H^*} \phi_{F,f}$$

because

$$\triangle_{Q^*} \omega_{f^*} = \begin{cases} 0 & \text{if } Q^* \le F^*, \\ 1 & \text{if } F^* \le Q^* \text{ and } m\left(Q^* - F^*\right) = 1, \\ 0 & \text{if } F^* \le Q^* \text{ and } m\left(Q^* - F^*\right) > 1. \end{cases} \quad (1)$$

□

Remark 3.3. Noting that $\frac{1}{\omega_{F^*}}(x) = \triangle_{F^*}\left(\frac{1}{x-\cdot}\right)$, one can write the following well-known identity, useful for the integration of rational or meromorphic functions,

$$\frac{f(x)}{\omega_{F^*}(x)} = \triangle_{F^*}\left(\frac{f(x)}{x-\cdot}\right) = \triangle_{F^*}\left(\frac{f(\cdot)}{x-\cdot}\right) + \triangle_{F^*}\left(\frac{f(x)-f(\cdot)}{x-\cdot}\right). \quad (1)$$

The rigorous proof of this identity can be reduced to the Lagrange case ($m_j = 1$ for every j), thanks to Theorem 3.5. The superposition rule (Corollary 3.3) shows that the second summands in the right-hand sides of (1) and Gel'fond's formula (Theorem 3.7) coincide, immediately implying the following representation for the Hermite interpolation polynomial of f:

$$\frac{p_{F^*} f(x)}{\omega_{F^*}(x)} = \triangle_{F^*}\left(\frac{f(\cdot)}{x-\cdot}\right).$$

3.3. **Lagrange-Hermite interpolation basis and right inverses.** In this subsection we interpret results of the previous section from the geometric point of view.

The following theorem is classical and traditionally proved with the aid of either Taylor expansions or l'Hôpital's rule [2, 24]. we deduce it as an immediate corollary of Part *a*) of Lemma 3.1 and a particular case of Theorem 3.3.

Theorem 3.8 (Hermite interpolation basis)**.** *For $n \in \mathbb{N}$, let $F^* = \{(x_j, m_j)\}_{j=1}^n$ be a multiplicity relation. The Hermite interpolation basis of the space of polynomials of degree less then $m(F^*)$, that is the system $\{p_{j,k}\}_{1\le j\le n}^{0\le k<m_j}$ of polynomials satisfying*

$$p_{i,l}^{(k)}(x_j) = \delta_{ij}\delta_{kl} \text{ for } 1 \le j \le n \text{ and } 0 \le k < m_j,$$

consists of the polynomials

$$p_{j,k}(x) = \omega_j(x)\frac{(x-x_j)^k}{k!}\sum_{i=0}^{m_j-k-1}\left(\frac{1}{\omega_j}\right)^{(i)}\frac{(x-x_j)^i}{i!}.$$

The proof of Theorem 3.8. To find $p_{j,k}$, it is sufficient to take $f(x) = \frac{(x-x_j)^k}{k!}$ in the formula (1) in the proof of Theorem 3.3, and then use the Leibnitz rule to extract the common multiplier from the resulting Taylor sum. □

Gel'fond's formula (Theorem 3.7) and Theorem 3.2 applied to polynomials and $H^{\mathbb{N}}(G)$-functions have the following geometric meaning.

Corollary 3.4. *a*) *Let $\mathcal{P} = \cup_{i\in\mathbb{N}}\mathcal{P}_i$ be the space of real or complex polynomials, and let F^* be a multiplicity relation (over $\mathbb{C}$ or $\mathbb{R}$). Then the operator $T_{F^*}:\ p(x) \mapsto \triangle_{F^*+\{(x,1)\}}p$ acts from $\mathcal{P}_l$ onto $\mathcal{P}_{l-m(F^*)}$ for $l \ge m(F^*)-1$, $\ker T_{F^*} = \mathcal{P}_{m(F^*)-1}$, and T_{F^*} also possesses the right inverse $M_{\omega_{F^*}}$, that is the pointwise multiplier by the polynomial ω_{F^*}. Moreover, $p_{F^*}^2 = p_{F^*} = I - M_{\omega_{F^*}}T_{F^*}$.*

b) *For $F \subset G \subset \mathbb{C}$ for some open $\mathbb{C}\backslash D \subset G$ with a closed disc $D \subset \mathbb{C}$, let F^* be a multiplicity relation (over $\mathbb{C}$ or $\mathbb{R}$). Then the operator $T_{F^*}:\ f(x) \mapsto \triangle_{F^*+\{(x,1)\}}f$ acts from $H^l(G)$ onto $H^{(l-m(F^*)+1)_+-1}(G)$ for $l \ge m(F^*)-1$, $\ker T_{F^*} = \mathcal{P}_{m(F^*)-2}$, and T_{F^*} also possesses the right inverse $M_{\omega_{F^*}}$, that is the pointwise multiplier by the polynomial ω_{F^*}. Moreover, $p_{F^*}^2 = p_{F^*} = I - M_{\omega_{F^*}}T_{F^*}$.*

Remark 3.4. It is easily checked that if ω_{F^*} is a polynomial with real coefficients (i.e. F^* is invariant with respect to the complex conjugation), then p_{F^*} and $T_{F^*} = \triangle_{F^*+\{(\cdot,1)\}}$ map the real polynomials onto the real polynomials.

In the case of the bounded holomorphic functions we can add the following quantitative estimate. For $z \in \mathbb{C}$ and $r > 0$, let $D(z,r)$ be the disc with the centre z and the radius r and $C(z,r)$ be its boundary circle.

Theorem 3.9. *For $n \in \mathbb{N}$, let $F^* = \{(z_j, m_j)\}_{j=1}^n$ be a multiplicity relation with $F \subset \Omega$ for a bounded open*

$$\bigcup_{j=1}^n D(z_j, d_j) \subset \Omega \subset \mathbb{C} \text{ where } d_j = \min_{i\ne j}|z_i - z_j|.$$

Then the operator $T_{F^*}:\ f(z) \mapsto \triangle_{F^*+\{(z,1)\}}f$ acts from $H(\Omega)$ onto itself, $\ker T_{F^*} = \mathcal{P}_{m(F^*)-1}(\mathbb{C})$, *and it also possesses the right inverse $M_{\omega_{F^*}}$, that is the pointwise multiplier by the polynomial ω_{F^*}. Moreover, $p_{F^*}^2 = p_{F^*} = I - M_{\omega_{F^*}}T_{F^*}$, and we have the estimate*

$$\|p_{F^*}|\mathcal{L}(H_\infty(\Omega))\| \le \sum_{j=1}^n\sum_{l=1}^{m_j} d_j^{l-m}\phi_j(l)\left\|\omega_{F^*-\{(z_j,l)\}}\right| H_\infty(\Omega)\| ,$$

where

$$\phi_j(l) = \begin{cases} \frac{(m-l)^{m-l}}{(m_j-l)^{m_j-l}(m-m_j)^{m-m_j}} & \text{if } 1 \le l < m_j; \\ 1 & \text{if } l = m_j. \end{cases}$$

The proof of Theorem 3.9. Theorem 3.3 and Definition 3.3 show that $\ker T_{F^*} = \mathcal{P}_{m(F^*)-1}(\mathbb{C})$. This, in turn, implies the identity $p_{F^*}^2 = p_{F^*}$. Gel'fond's formula (Theorem 3.7) provides $p_{F^*} = I - M_{\omega_{F^*}} T_{F^*}$ and the observation that $M_{\omega_{F^*}}$ is the right inverse for T_{F^*} thanks to Theorem 3.2 (Bezout). To see that $T_{F^*} f$ is holomorphic function, we can either use the induction and the composition rule (Corollary 3.3), or Theorem 3.6 (the proof works even easier for analytic functions), or Gel'fand's representation for T_{F^*} (see §3.1).

To estimate the quantity

$$I_{j,k} = \frac{1}{k!}\left(\frac{f}{\omega_j}\right)^{(k)} \text{ for } 0 \le k < m_j$$

we use its Cauchy representation

$$I_{j,k} = \frac{1}{2\pi i} \oint\limits_{C(z_j,r)} \frac{f(\zeta)}{\omega_j(\zeta)(\zeta - z_j)^{k+1}} d\zeta \text{ for } r \in (0, r_j). \tag{1}$$

Namely, one has

$$|I_{j,k}| \le \frac{\|f|H_\infty(\Omega)\|}{r^k \min_{C(z_j,r)} |\omega_j(z)|} \le \frac{\|f|H_\infty(\Omega)\|}{g_{j,k}(r)} \text{ where } g(r) = r^k (r_j - r)^{m-m_j}. \tag{2}$$

Mean arithmetic-geometric inequality clearly suggests that

$$\max_{[0,r_j)} g_{j,k}(r) = \frac{(m - m_j + k)^{m-m_j+k}}{k^k (m - m_j)^{m-m_j}} r_j^{m-m_j+k} \text{ for } 0 < k < m_j$$

and

$$\sup_{[0,r_j)} g_{j,0}(r) = r_j^{m-m_j+k}. \tag{3}$$

To finish the proof of the estimate for the norm of p_{F^*}, we use the triangle inequality and $(2, 3)$:

$$\|p_{F^*} f|H_\infty(\Omega)\| \le \sum_{j=1}^{n} \sum_{k=0}^{m_j-1} |I_{j,k}| \left\| \omega_{F^* - \{(z_j, m_j-k)\}} \right| H_\infty(\Omega) \right\| \le$$

$$\le \|f|H_\infty(\Omega)\| \sum_{j=1}^{n} \sum_{l=1}^{m_j} d_j^{l-m} \phi_j(l) \left\| \omega_{F^* - \{(z_j,l)\}} \right| H_\infty(\Omega) \right\|. \tag{4}$$

□

3.4. **Arithmetics of polynomials and wavelet theory.** In this subsection we provide examples demonstrating that the Hermite interpolation theory considered in the previous subsection delivers very simple constructive proofs for two classical results about the ring of polynomials and two results from the wavelet theory.

Theorem 3.10. *For $n \in \mathbb{N}$, let $\{p_i\}_{i=1}^n$ be a system of complex or real polynomials without any common nontrivial divisors (i.e. $F^*_{\min} = \emptyset^*$), and $p_i = a_{i,m(F_i^*)} \omega_{F_i^*}$ for $1 \le i \le n$. Then, for every polynomial p, there exists a system $\{q_i\}_{i=1}^n$ of polynomials satisfying*

$$\sum_{i=1}^{n} q_i p_i = p.$$

The system $\{q_i\}_{i=1}^n$ is unique if, and only if, the degree of p is less than $m(F^*_{\max})$ and either $n = 2$, or $n = m(F^*_{\max})$, $\{F_i\}_{i=1}^n = \{F_{\max} \setminus \{y\}\}_{y \in F_{\max}}$.

The proof of Theorem 3.10. Without loss of generality, we may assume that $a_{i,m(F_i^*)} = 1$ for $1 \leq i \leq n$. According to Corollary 3.4, every polynomial p can be represented in the form

$$p = p_{F^*_{\max}} p + \omega_{F^*_{\max}} T_{F^*_{\max}} p$$
$$= \sum_{y \in F_{\max}} \omega_y \sum_{k=0}^{\mu_{F^*_{\max}}(y)-1} \frac{(\cdot - y)^k}{k!} \left(\frac{f}{\omega_y}\right)^{(k)} + \omega_{F^*_{\max}} T_{F^*_{\max}} p = \sum_{y \in F_{\max}} \omega_y q_y. \quad (1)$$

where

$$\omega_y = \omega_{F^*_{\max} - \{(y, \mu_{F^*_{\max}}(y))\}},$$

and this representation is unique if, and only if, $p \in \mathrm{Ker}(T_{F^*_{\max}}) = \mathcal{P}_{m(F^*_{\max})-1}$. To finish the existence part of the proof of the theorem, it is left to note that, for every $y \in F_{\max}$, there exists, at least, one $F_j \leq F^*_{\max} - \{(y, \mu_{F^*_{\max}}(y))\}$ meaning that

$$\omega_y = \omega_{F_j^*} \omega_{F^*_{\max} - \{(y, \mu_{F^*_{\max}}(y))\} - F_j}. \quad (2)$$

The proof of the uniqueness part is finished by observing that, for $n > 2$, there exist, at least, two relations

$$F_j \leq F^*_{\max} - \{(y, \mu_{F^*_{\max}}(y))\} \text{ and } F_i \leq F^*_{\max} - \{(y, \mu_{F^*_{\max}}(y))\}$$

for some $y \in F_{\max}$ if $\{F_i\}_{i=1}^n \neq \{F_{\max} \setminus \{y\}\}_{y \in F_{\max}}$. □

Remark 3.5. *a*) The case $n = 2$ and $p = 1$ of the previous theorem plays an important role in the classical wavelet theory (see Theorem 6.1.1 on page 169 in [7]).
b) For example, the method of Cohen, Daubechies and Feauveau of constructing wavelets utilizes the explicit expressions (see page 171 in [7]) for the general form of the solutions q_1 and q_2 in a particular case $p_1(y) = y^m$, $p_2(y) = (1-y)^m$ and $p = 1$ that can be treated with the aid of Theorem 3.10 (even Theorem 3.3) without resorting to the symmetry argument.

Theorem 3.11. *For $n \in \mathbb{N}$, let $\{p_i\}_{i=1}^n$ and $\{r_i\}_{i=1}^n$ be systems of complex or real polynomials, such that every couple p_i, p_j with $i \neq j$ have no common nontrivial divisors, and the degree of r_i is strictly less than the degree of p_i for $1 \leq i \leq n$. Then there exists a polynomial p of the degree strictly less than $m(F^*_{\max})$, where $p_i = a_{i,m(F_i^*)} \omega_{F_i^*}$ for $1 \leq i \leq n$, satisfying $p = q_i p_i + r_i$ for $1 \leq i \leq n$ and a system $\{q_i\}_{i=1}^n$ of polynomials.*

The proof of Theorem 3.11. As in the classical approach, this theorem is proved by induction in n with the aid of the previous theorem in the case $n = 2$. □

The last example of natural applications shows that the discrete Fourier transform is a particular case of the Lagrange interpolation case $m_j = 1$.

For $n \in \mathbb{N}$, let $L^* = L \times \{1\}$ with $L = \{z_j\}_{j=1}^n$ and $z_j = e^{ij\pi/n}$. If $p(z) = \sum_{k=0}^{n-1} a_k z^k$, one has $p_{L^*}p = p$, i.e.,

$$\begin{aligned} p(z) &= \sum_{j=1}^n p(z_j)\frac{\omega_j(z)}{\omega_j(z_j)} \\ &= \sum_{j=1}^n p(z_j)\frac{\sum_{k=0}^{n-1} z^k z_j^{n-k-1}}{nz^{n-1}} \\ &= \sum_{k=0}^{n-1} z^k \frac{1}{n}\sum_{j=1}^n p(z_j) z_j^{-k} \\ &= \sum_{k=0}^{n-1} z^k a_k. \end{aligned}$$

The inverse discrete Fourier transform corresponds to calculating the values $\{p(z_j) : 1 \le j \le n\}$ relying on the knowledge of the coefficients $\{a_k\}_{k=0}^{n-1}$.

3.5. **Probability distributions, Steklov and B-splines and ordinary differences.** The most natural measures on a convex envelope of a finite number of vectors are those defined in terms of barycentres. In this subsection we first apply Theorem 3.5 on merging convergence to show, in particular, that the projections of the measures from some class have B-spline densities. Then we demonstrate that the Steklov splines, corresponding to the projections of the uniform measures, play the same role in the theory of the ordinary differences as the B-splines in the theory of divided differences.

Definition 3.4 (Barycentric distributions)**.** For $n \in \mathbb{N}$ and $m \in \mathbb{N}^n$, let $\{z_j\}_{j=1}^n$ be elements of a linear space X. We define a relation $F^* = \sum_{j=1}^n \{(z_j, m_j)\}$. In particular, if all $z_j \ne z_k$ for $j \ne k$, we can write $F^* = \{(z_j, m_j)_{j=1}^n\}$. For a vector e in the (linear) dual X', let $F_e^* = e(F^*) = \sum_{j=1}^n \{(\langle z_j, e\rangle), m_j)\}$.

Assume that S_n is the following simplex in $\mathbb{R}^n$ defined by

$$S_n = \{x \in \mathbb{R}^n : \ x_j \ge 0 \text{ and } \sum_{j=1}^n x_j = 1\} \tag{S_n}$$

and endowed with the probability measure

$$d\mu_{S,\vec{m}}(x) = \frac{(m-1)!}{n^{1/2}}\frac{x^{\vec{m}-\vec{1}}}{(\vec{m}-\vec{1})!}d\mu_{n-1}(x), \text{ where } \frac{x^{\vec{m}-\vec{1}}}{(\vec{m}-\vec{1})!} = \prod_{j=1}^n \frac{x_j^{m_j-1}}{(m_j-1)!} \text{ and}$$

μ_{n-1} is the $n-1$-dimensional Lebesgue measure on S_n. Let $\zeta_{\vec{m}}$ be the corresponding S_n-valued stochastic variable with the probability $\mu_{S,\vec{m}}$. We say that a conv$F$-valued stochastic variable ξ_{F^*} has *$\vec{m}$-barycentric distribution* (and belongs to the class $\mathcal{Z}_{\vec{m}}$) if

$$\xi_{F^*} = \langle \vec{z}, \zeta_{\vec{m}}\rangle = \sum_{j=1}^n \zeta_{\vec{m}j} z_j. \tag{$\mathcal{Z}_{\vec{m}}$}$$

The probability measure corresponding to ξ_{F^*} will be called *$\vec{m}$-barycentric* on conv$F$.

Definition 3.5 (Peano kernel/B-spline)**.** For a (scalar) multiplicity relation F^* with $F \subset \mathbb{R}$, the *Peano kernel* (or B-spline) is

$$b_{F^*}(t) = (m(F^*) - 1)\triangle_{F^*}(\cdot - t)^{m(F^*)-2}.$$

Remark 3.6. The name Peano kernel is explained by the following corollary of the Taylor expansion with the remainder in integral form. For $n \in \mathbb{N}$, let $F^* = \{(x_j, m_j)\}_{j=1}^n$ be a multiplicity relation with $F \subset \mathbb{R}$, and let f be a function on $[\min_j x_j, \max_j x_j$ with the integrable $f^{(m(F^*)-1)}$. Then one has

$$\triangle_{F^*} f = \frac{1}{(m(F^*)-1)!} \int_{\mathbb{R}} b_{F^*}(x) f^{(m(F^*)-1)}(x)dx. \qquad (P)$$

Indeed, this identity clearly holds in the Lagrangian case ($m(F^*) = |F|$) and, then we can use the merging convergence continuity property (Theorem 3.5) to validate the identity for an arbitrary F^* because $b_{H^*}(\tau)$ converges to $b_{F^*}(\tau)$ when H^* is merging to F^* thanks to the same Theorem 3.5.

Another representation for a divided difference of a smooth function is the content of Part *a*) of the next lemma slightly generalising Exercise 55 on page 193 in [2]. The Lagrangian case ($F^* = F \times \{1\}$) of Part *a*) was established by Genocchi [14], while Hermite [15] establishes a representation for his remainder term that implies Part *a*) in its full generality.

Lemma 3.3. *For $n \in \mathbb{N}$ and $m \in \mathbb{N}^n$, let $\vec{x} \in \mathbb{R}^n$, $\vec{m} \in \mathbb{N}^n$ with $m = \sum_{j=1}^n m_j$ and $F^* = \sum_{j=1}^n \{(x_j, m_j)\}$. Assume also that f is a function on $[\min_j x_j, \max_j x_j]$ with integrable $f^{(m(F^*)-1)}$. Then we have*

$$a)\ \triangle_{F^*} f = \int_0^1 \int_0^{t_{n-1}} \dots \int_0^{t_3} \int_0^{t_2} f^{(m-1)}\left(\phi_x(t)\right) \tilde{\rho}_{\vec{m}}(t)\, dt_1 dt_2 \dots dt_{n-1}$$

where

$$\phi_x(t) = t_1 x_1 + \sum_{j=2}^{n-1} (t_j - t_{j-1}) x_j + (1 - t_{n-1}) x_n$$

and

$$\tilde{\rho}_{\vec{m}}(t) = \frac{t_1^{m_1-1}}{(m_1-1)!} \prod_{j=2}^{n-1} \frac{(t_j - t_{j-1})^{m_j-1}}{(m_j-1)!} \frac{(1-t_{n-1})^{m_n-1}}{(m_n-1)!};$$

$$b)\ \triangle_{F^*} f = \frac{1}{(m-1)!} \int_{S_n} f^{(m-1)}\left(\langle \vec{x}, \vec{y} \rangle\right) d\mu_{S,\vec{m}}(\vec{y})$$

$$= \frac{1}{(m-1)!} \int_{S_n} f^{(m-1)} \left(\sum_{j=1}^n x_j y_j \right) d\mu_{S,\vec{m}}(\vec{y}).$$

The proof of Lemma 3.3. Let us note that Part *a*) implies Part *b*) with the aid of the change of variables $y_1 = t_1$ and $y_j = t_j - t_{j-1}$ for $1 < j < n$ followed by further mapping of the new domain $\{y_j \geq 0,\ \sum_{j=1}^{n-1} y_j \leq 1\}$ in $\mathbb{R}^{n-1}$ onto S_n (inverse of the orthogonal projector).

The Lagrangian case ($m = n$) of Part *a*) followed from the Newton-Leibniz formula (i.e. the case $m = n = 2$) and a particular case of the composition rule (Corollary 3.3):

$$\triangle_{F^*} f = \triangle_{\{x_n, x_{n-1}\} \times \{1\}} \triangle_{F^* - \{x_n, x_{n-1}\} \times \{1\} + \{(\cdot, 1)\}}.$$

This means that, for $H^* = H \times \{1\}$ and distinct $\{y_i\}_{i=1}^m = H \supset F$ with $\min_i y_i = \min_j x_j$ and $\max_i y_i = \max_j x_j$, we have

$$\triangle_{H^*} f = \int_0^1 \int_0^{t_{m-1}} \dots \int_0^{t_3} \int_0^{t_2} f^{(m-1)}\left(\phi_x(t)\right) \rho_{\vec{m}}(t)\, dt_1 dt_2 \dots dt_{n-1}, \qquad (1)$$

where $\phi_y(t) = t_1 y_1 + \sum_{j=2}^{m-1}(t_j - t_{j-1})y_j + (1 - t_{m-1})y_m$ and

$$\rho_{\vec{m}}(t) = \frac{t_1^{m_1}}{m_1!} \prod_{j=2}^{n-1} \frac{(t_j - t_{j-1})^{m_j}}{m_j!} \frac{(1-t_{n-1})^{m_n}}{m_n!}.$$

It is clear that the integral in the right-hand side of (1) has the form

$$I(\vec{y}) = \int_{\min_i y_i}^{\max_i y_i} f^{m-1}(\tau) \rho_{\vec{y}}(\tau) d\tau$$

with the density $\rho_{\vec{y}}$ depending continuously on τ, and, thus, continuous in $\vec{y}$ itself. Using the merging convergence continuity (Theorem 3.5) with H^* merging to F^*, we see that (1) holds for $H = F$. Now we use $m - n$ times the identity

$$\int_a^b (b-\tau)^{k-1}(\tau - a)^{l-1} d\tau = \frac{(k-1)!(l-1)!}{(k+l-1)!}(b-a)^{k+l-1}$$

to establish Part a) with $x_j \neq x_k$ for $k \neq j$. One more application of Theorem 3.5 permits us to establish Part a) with $\{x_j\}_{j=1}^n$ allowed to coincide ($\vec{x} \in \mathbb{R}^n$). □

Theorem 3.12. *For a linear space X, $e \in X'$, $n \in \mathbb{N}$, $\vec{m} \in \mathbb{N}^n$ and $\vec{z} \in X^n$, let $F^* = \sum_{j=1}^n \{(z_j, m_j)\}$, and let the* conv$F$*-valued stochastic variable ξ_{F^*} have the $\vec{m}$-barycentric distribution. Then the scalar-valued stochastic variable $e(\xi_{F^*})$ possesses the density $b_{F_e^*} = (m(F^*)-1)\triangle_{F_e^*}(\cdot - t)^{m(F^*)-2}$, where $F_e^* = e(F^*)$. In particular, $b_{F_e^*}$ is strictly positive on* $(\min\ e(F), \max\ e(F)\rangle)$.

Remark 3.7. In a view of the direct proof of Theorem 3.13 corresponding to the Steklov splines below, it would be interesting to find a direct proof of Theorem 3.12.

The proof of Theorem 3.12. Taking $z_{je} = e(z_j)$, we see that Remark 3.6 and Theorem 3.11 imply the identity

$$\int_{S_n} f^{(m-1)}\left(\langle \vec{z}_e, \vec{y}\rangle\right) d\mu_{S,\vec{m}}(\vec{y}) = (m(F^*)-1)!\triangle_{F_e^*} f = \int_{\mathbb{R}} b_{F^*}(\tau) f^{(m(F_e^*)-1)}(\tau) d\tau \quad (1)$$

for every f with integrable $f^{(m(F^*)-1)}$, particularly, $f^{(m(F^*)-1)} = \chi_I$, where I is an arbitrary subinterval of $[\min e(F^*), \max e(F^*)]$. The conclusion that $b_{F_e^*}$ is the density follows. In turn, geometric considerations imply the strict positivity of the density, that is $b_{F_e^*}$, on $(\min_j\langle e, z_j\rangle, \min_j\langle e, z_j\rangle)$. This finishes the proof. □

Before the appearance of the Sobolev averaging (with the hat-function or other C^∞-functions), their role was played by the Steklov averages, i.e. the products of the averaging operators of the form

$$S_h g(x) = \frac{1}{h}\int_0^h g(x+\tau) d\tau \text{ for } h \in \mathbb{R}\setminus\{0\} \text{ and } g \in L_{1,loc}(\mathbb{R}).$$

The Steklov splines correspond to the densities of the Steklov averages.

Definition 3.6 (Steklov differences and splines)**.** For $n \in \mathbb{N}$, a finite subset $F \subset \mathbb{R}(\mathbb{C})$ and a multiplicity relation F^*, we define the *Steklov difference*

$$\triangle^{F^*} = \prod_{h \in F} D_h^{\mu_{F^*}(h)}, \text{ where} \qquad (\triangle^{F^*})$$

$$D_h = \begin{cases} D & \text{if } h = 0, \\ \frac{\triangle_h}{h} & \text{if } h \neq 0. \end{cases}$$

Here D is the differentiation, and $\triangle_h$ is the ordinary difference of the first order $\triangle_h g(x) = g(x+h) - g(x)$. Let us define the *Steklov F^*-spline* by the formula

$$s_{F^*}(z) = \triangle^{F^*} \frac{(\cdot - z)_+^{m(F^*)-1}}{(m(F^*)-1)!}. \qquad (s_{F^*}).$$

Note that the Steklov spline s_{F^*} coincides with a B-spline b_{H^*} only in the extreme case $F^* = \{(h, m)\}$, $H^* = H \times \{1\}$ and $m(H^*) = m(F^*)+1$ of the consecutive equidistant nodes of the Lagrangian H^* with the step h, i.e. the classical B-splines are also the classical Steklov splines.

The next theorem is the counterpart of Theorem 3.12 for Steklov splines and uniform distributions.

Theorem 3.13. *a) For a linear space X, $e \in X'$, $n \in \mathbb{N}$, $\vec{m} \in \mathbb{N}^n$ and $\vec{z} \in X^n$, let $F^* = \sum_{j=1}^n \{(z_j, m_j)\}$, and let $\{\xi_{j,k}\}_{1 \le j \le n}^{1 \le k \le m_j}$ be a system of independent stochastic variables with the identical uniform distribution on $[0,1]$ (on $\mathbb{R}$ with the density $\chi_{[0,1]}$), and*

$$\eta_{F^*} = \sum_{j=1}^n \sum_{k=1}^{m_j} \xi_{j,k} z_j.$$

Then the scalar-valued stochastic variable $e(\xi_{F^*})$ possesses the Steklov F^*-spline density $s_{F_e^*}$, where $F_e^* = e(F^*)$. In particular, $s_{F_e^*}$ is strictly positive on

$$\left(\min_{H^* \le F_e^*} \sum_{h \in H^*} \mu_{H^*}(h) h, \max_{H \le F_e^*} \sum_{h \in H^*} \mu_{H^*}(h) h \right).$$

b) For $m \in \mathbb{N}$, the Euclidean space $\mathbb{R}^m$ with an orthonormal basis $\{e_j\}_{j=1}^m$, $h \in \mathbb{R}^m$ and a system $\{\xi_j\}_{j=1}^m$ of independent stochastic variables with the identical uniform distribution on $[0,1]$ (on $\mathbb{R}$ with the density $\chi_{[0,1]}$), let

$$\xi = \sum_{j=1}^m \xi_j e_j \text{ and } \xi_h = (h, \xi)_{\mathbb{R}^m} = \sum_{j=1}^m \xi_j h_j.$$

Then ξ_h possesses the Steklov spline density s_{F^*} for $F^* = \sum_{j=1}^m \{(h_j, 1)\}$:

$$\rho_{\xi_h}(x) = \prod_{j=1}^m D_{h_j} \left(\frac{(\cdot - x)_+^{m-1}}{(m-1)!} \right).$$

Remark 3.8. Let us note that, in the Lagrangian case $F^* = F \times \{1\}$ of linearly independent F spanning X and with the appropriate choices of e, Part a) describes the volume of the intersections of the parallelepiped $\sum_{z \in F} [0,1] z$ with the family of the parallel hyperplanes described by e.

The proof of Theorem 3.13. Part a) follows from Part b) with a linearly renumbered $h_{j,k} = \delta_{l,j} z_l$ thanks to the identity

$$e(\xi_{F^*}) = \sum_{j=1}^n \sum_{k=1}^{m_j} \xi_{j,k} h_{j,k}. \qquad (1)$$

Let us first consider the case $h_j = (h, e_j) \neq 0$ for every j of Part a) and assume that $\psi_{F^*}(H^*) = \sum_{z\in H} \mu_{H^*}(z)z$ for $H^* \leq F^*$. The distribution function of ξ_h is equal to the volume $V(x)$ of

$$\{v \in [0,1]^m : (h,v) < x\}.$$

Recalling that the (oriented) volume of the simplex $\operatorname{conv}(\{0\} \cup \{v_j e_j\}_{j=1}^m)$ with some $v_j \neq 0$ for every j is equal to $(m!)^{-1} \prod_{j=1}^m v_j$, we employ the inclusion-exclusion formula to obtain the representation

$$V(x) = \sum_{H^* \leq F^*} (-1)^{m(H^*)} \prod_{z\in H} \frac{1}{z} \binom{\mu_{F^*}(z)}{\mu_{H^*}(z)} \frac{(x - \psi_{F^*}(H^*))_+^m}{m!}. \tag{2}$$

Differentiating we establish the density

$$\rho_{\xi_h}(x) = \sum_{H^* \leq F^*} (-1)^{m(H^*)} \prod_{z\in H} \frac{1}{z} \binom{\mu_{F^*}(z)}{\mu_{H^*}(z)} \frac{(x - \psi_{F^*}(H^*))_+^{m-1}}{(m-1)!}. \tag{3}$$

Eventually, thanks to the identities

$$(x-a)_+^l = (x-a)^l + (-1)^{l+1}(a-x)_+^l \text{ and } \prod_{z\in F} \frac{\triangle_z^{\mu_{F^*}(z)}}{z} (\cdot - x)^{m(F^*)-1} = 0, \tag{4}$$

we can rewrite (3) in the desirable equivalent form (recall that $m = m(F^*)$)

$$\rho_{\xi_h}(x) = \prod_{z\in F} \frac{\triangle_z^{\mu_{F^*}(z)}}{z} \frac{(\cdot - x)_+^{m-1}}{(m-1)!}. \tag{5}$$

If $0 \in F$, then the sum $\sum_{j=1}^m h_j \xi_j$ contains only $l = m(F^*) - \mu_{F^*}(0)$ summands, and we have (5) with l instead of m. To finish the proof one only has to note that

$$\frac{(y-x)_+^{m(F^*)-\mu_{F^*}(0)-1}}{(m(F^*) - \mu_{F^*}(0) - 1)!} = D_0^{\mu_{F^*}(0)} \frac{(\cdot - x)_+^{m(F^*)-1}}{(m(F^*)-1)!}(y) \tag{6}$$

and that D_w^k commute for different $w \in \mathbb{R}$ and $k \in \mathbb{N}$. □

The following lemma is the counterpart of Part b) of Lemma 3.3 and Remark 3.6.

Lemma 3.4. *For $n \in \mathbb{N}$ and $m \in \mathbb{N}^n$, let $\vec{h} \in \mathbb{R}^n$, $\vec{m} \in \mathbb{N}^n$ with $F^* = \sum_{j=1}^n \{(x_j, m_j)\}$. Assume also that f is a function on*

$$\left(\min_{H^* \leq F_e^*} \sum_{h\in H^*} \mu_{H^*}(h)h, \max_{H \leq F_e^*} \sum_{h\in H^*} \mu_{H^*}(h)h \right)$$

with integrable $f^{(m(F^*)-1)}$. Let also $Q = [0,1]^{m(F^*)}$ be the unit cube in

$$\mathbb{R}^{m(F^*)} = \prod_{j=1}^n \mathbb{R}^{m_j},$$

and let $t = \{t_{j,k}\}_{1\leq j\leq n}^{1\leq k\leq m_j}$ be the variable describing the points of Q endowed with the Lebesgue measure. Then we have

$$a)\ \triangle^{F^*} f = \int_{\mathbb{R}} s_{F^*}(\tau) f^{(m(F^*))}(\tau) d\tau;$$

$$b)\ \triangle^{F^*} f = \int_Q f^{(m)} \left(\sum_{j=1}^n \sum_{k=1}^{m_j} x_j t_{j,k} \right) dt.$$

The proof of Lemma 3.4. As the statement of Remark 3.6, Part a) is an immediate consequence of the Taylor expansion formula with the remainder in integral form:

$$f(y)=\sum_{k=0}^{m-1}\frac{f^{(k)}(x)}{k!}(y-x)^k+\int_x^y f^{(m)}(\tau)\frac{(y-\tau)_+^{m-1}}{(m-1)!}d\tau. \quad (1)$$

Part b) is implied by either Part b) of Theorem 3.13, or by the repeated application of the Newton-Leibniz formula accompanied by the homogeneous change of variables. □

4. Algebraic operators

The notion of the algebraic operator was introduced by Dirac [8]. Particular important cases of algebraic operators are involutions of order n ($A^n = I$), idempotents of order n ($A^{n+1} = A$, for example, projectors), nilpotent operators of order n ($A^n = 0$) and the operators with a finite-dimensional range (of finite rank) $\mathcal{F}(X,Y)$.

Definition 4.1 (Algebraic operator)**.** Let X be linear space, and let A be linear operator $A:\ X \to X$ with $D(A) = X$. The operator A is *algebraic* if there exists a polynomial q satisfying $q(A) = 0$. The polynomial q is *minimal* for B if it is the polynomial of the minimal degree with this property. We assume it normalized by $a_n = 1$ (a_n is the senior coefficient of q), i.e. $q = \omega_{F^*}$ for some multiplicity relation F^* with $F \subset \mathbb{C}$.

Let also $L(X)$ be the ring of *all linear operators* from X into itself. If R is a subring of $L(X)$ (linear operators), and J is an ideal in R, then A is *almost algebraic* if $A + J$ is algebraic in the quotient R/J.

Remark 4.1. Let us note that the Taylor expansion for the minimal polynomial q implies that

$$(A-\lambda I)^{-1}=-\frac{1}{q(\lambda)}\sum_{k=1}^{m(F^*)}\frac{q^{(k)}(\lambda)}{k!}(A-\lambda I)^{k-1} \text{ if } q(\lambda)\neq 0,$$

i.e. $\lambda \notin F$.

A particular class of almost algebraic operators ($R = \mathcal{L}(X,Y)$ and $J = \mathcal{F}(X,Y)$ is the ideal of the finite rank operators) was investigated by S.M. Nikol'skii in 1943 in connection with his celebrated characterisation of Fredholm (i.e. of index zero) operators as invertible elements of $\mathcal{L}(X,Y)/\mathcal{F}(X,Y)$. In turn, the latter characterization shows that every Fredholm operator is a sum of an invertible and an algebraic operator! In 1947, related abstract notions were studied by Khalilov. The developed abstract theory of algebraic and almost algebraic operators appeared in [24].

In 1948, Kaplansky characterized algebraic operators as operators with the bounded maximal dimension of the linear envelope of the orbits $[\{A^k x\}_{k\in\mathbb{N}}]$ (the maximal dimension is equal to the degree of the minimal polynomial) and applied this result to some cases of the Kurosch problem partially solved (in different settings) by Jacobson [16] in 1945 and Malcev [21] in 1943. Kaplansky's characterization shows, in particular, that all operators of finite rank are algebraic.

In this section we establish a variant of a Jordan form representation for an algebraic operator strengthening some of the main tools (Properties 2.1 – 2.3 on pp. 69-70 and Theorems $A.II.5.1$, $A.II.7.2$ and $A.II.7.3$ in [24]) of the abstract theory developed by D. Przeworska-Rolewicz and S. Rolewicz in [24], and then show that bounded algebraic operators share the properties of the classes of projectors, idempotents and involutions. In particular, Theorem 4.3 characterizes the class of functions $f \in C^{F^*}$, such that $\mathrm{Im}(f(A))$ is closed for a given bounded algebraic operator A.

Parts a) and b) of the next Theorem are properties 2.1 and 2.2 from [24] (p. 69), while Part c) is the (ultimate) improvement of Property 2.3 on page 70 and Part b) of Theorem $A.II.5.1$ in [24] (i.e.the inclusion $\mathrm{Im}(p_x) \subset \mathrm{Ker}((A - xI)^{\mu_{F^*}(x)})$), while the first half of Part d) is the improvement of Part c) of Theorem $A.II.5.1$ from [24]. Part g) is classical (see [29] or [24]).

We assume that $p(A) = \sum_{k=0}^{n} a_k A^k$ and $A^0 = I$ if $p(z) = \sum_{k=0}^{n} a_k z^k$.

Theorem 4.1 (Jordan form)**.** *For a linear space X, let $A \in L(X)$ be an algebraic operator with the minimal polynomial ω_{F^*}. Assume also that, for every $x \in F$, the operator $P_x \in L(X)$ is $P_x = p_x(A)$, where*

$$p_x(z) = \omega_{F^*-\{(x,\mu_{F^*}(x))\}}(z) \sum_{k=0}^{\mu_{F^*}(x)} \frac{1}{k!} \left(\frac{1}{\omega_{F^*-\{(x,\mu_{F^*}(x))\}}} \right)^{(k)} (x)(z-x)^k.$$

Then we have the following properties:

a) $I = \sum_{x \in F} P_x$;

b) $P_x P_y = P_y P_x = \delta_x(y) P_x$ *for* $x, y \in F$;

c) $\mathrm{Im}(P_x) = \mathrm{Ker}((A - xI)^{\mu_{F^*}(x)}) = \mathrm{Ker}((A - xI)^l)$ *for* $l > \mu_{F^*}(x)$ *and* $x \in F$;

d) $X = \bigoplus_{x \in F} \mathrm{Ker}((A - xI)^{\mu_{F^*}(x)})$ *and*

$$\mathrm{Ker}((A - xI)^{\mu_{F^*}(x)}) = \bigcap_{y \in F \setminus \{x\}} \mathrm{Im}((A - yI)^{\mu_{F^*}(y)}) \text{ for } x \in F;$$

e) $\mathrm{Im}((A - xI)^{\mu_{F^*}(x)}) = \bigoplus_{y \in F \setminus \{x\}} \mathrm{Ker}((A - yI)^{\mu_{F^*}(y)}) = \mathrm{Im}((A - xI)^l)$
for $l > \mu_{F^*}(x)$ *and* $x \in F$;

f) $\mathrm{Ker}((A - xI)^l) \neq \mathrm{Ker}((A - xI)^{l+1})$ *and* $\mathrm{Im}((A - xI)^l) \neq \mathrm{Im}((A - xI)^{l+1})$
for $0 < l < \mu_{F^*}(x)$, $x \in F$;

g) $\mathrm{Ker}((A - xI)^k) \subset \mathrm{Ker}((A - xI)^l)$ *and* $\mathrm{Ker}((A - xI)^k) \cap \mathrm{Ker}((A - yI)^l) = \{0\}$
for $k, l \in \mathbb{N}$, $k \leq l$, $x \neq y$.

Moreover, Part g) holds for a non-algebraic A, as well as the inclusion $\mathrm{Im}((A - xI)^l) \subset \mathrm{Im}((A - xI)^k)$. If, in addition, X is Banach and A is bounded, then the projectors P_z for $z \in F$ are bounded and the sums in d) and e) are topological.

The proof of Theorem 4.1. Let us assume that $F^* \leq H^*$ and define $p_{H^*,x}$ and $P_{H^*,x}$ as we have defined p_x and P_x substituting F^* with H^*. We also assume that $F = H$ because $P_{H^*,y} = 0$ if $y \in H \setminus F$. Thanks to Theorem 3.3, the observation

$$\sum_{x \in H} p_{H^*,x}(z) = 1 \text{ implies } \sum_{x \in H} P_{H^*,y} = I \tag{1}$$

showing the validity of Part a) (case $H^* = F^*$).

Part b) with $P_{H^*,x}$ instead of P_x follows from (1) and the observation

$$p_{H^*,x} p_{H^*,y} = q_0 \omega_{H^*} = q_1 \omega_{F^*} \text{ if } x \neq y. \tag{2}$$

To establish the first equality in c) (with H^* instead of F^*), we expand

$$\omega_{H^*-\{(x,\mu_{H^*}(x))\}} \text{ and } 1/\omega_{H^*-\{(x,\mu_{H^*}(x))\}}$$

into Taylor series in a neighborhood of x:

$$\omega_{H^*-\{(x,\mu_{H^*}(x))\}}(z)=\sum_{k=0}^{\infty}b_k(z-x)^k$$

where $b_k=0$ for $k>s_x=m(H^*)-\mu_{H^*}(x)$ and

$$\frac{1}{\omega_{H^*-\{(x,\mu_{H^*}(x))\}}(z)}=\sum_{k=0}^{\infty}c_k(z-x)^k. \tag{3}$$

Multiplying we establish the convolution identity

$$\delta_{0n}=\sum_{k=0}^{n}c_kb_{n-k}. \tag{4}$$

For $v\in \operatorname{Ker}((A-xI)^{\mu_{H^*}(x)})$ one has $(A-xI)^lv=0$ for $l\geq \mu_{H^*}(x)$. Using this observation and, then, (4), we obtain

$$p_{H^*,x}(A)v=\sum_{l=0}^{m(H^*)-1}(A-xI)^lv\sum_{k=0}^{l}c_kb_{l-k}=v, \tag{5}$$

meaning that $\operatorname{Ker}((A-xI)^{\mu_{H^*}(x)})\subset \operatorname{Im}(P_{H^*,x})$. The opposite inclusion follows from the identity

$$(A-xI)^{\mu(H^*)(x)}p_{H^*,x}(A)=q_2(A)\omega_{H^*}(A)=0. \tag{6}$$

The inclusion and the cases $x\notin H=F$ or $y\notin H=F$ of the identity in Part g) (see Remark 4.1) are trivial. When $x,y\in H=F$, the identity follows from the existence of the polynomials q_3,q_4 provided by Theorem 3.10 and satisfying

$$q_3(A)(A-xI)^k+q_4(A)(a-yI)^l=I.$$

Now the proved identities of Parts $a)-c)$ imply the identity

$$X=\bigoplus_{x\in H}\operatorname{Ker}((A-xI)^{\mu_{H^*}(x)}), \tag{7}$$

which, particularly, holds when $H^*=F^*$ (i.e. the first identity in Part d)). Comparing this particular case with (7) with the aid of Part g), we finish the proof of c).

To obtain e), we observe, with the aid of the commutativity of the polynomial products defining $p_{H^*,y}$ (and c) with H^* instead of F^*), that

$$\operatorname{Ker}((A-yI)^{\mu_{H^*}(y)})=\operatorname{Im}(P_{H^*,y})\subset \operatorname{Im}((A-xI)^{\mu_{F^*}(x)}) \text{ for } y\in H\setminus\{x\}. \tag{8}$$

At the same time, the second equality in c) shows that

$$\operatorname{Ker}((A-xI)^l)\cap \operatorname{Im}((A-xI)^{\mu_{F^*}(x)})=\{0\} \text{ for } x\in H=F. \tag{9}$$

Comparing now the first equality in c) with (8) and (9) in the same way as we compared (7) with the both statements of Part g), we see that

$$\operatorname{Im}((A-xI)^{\mu_{H^*}(x)})=\bigoplus_{y\in F\setminus\{x\}}\operatorname{Ker}((A-yI)^{\mu_{F^*}(y)}). \tag{10}$$

Now comparing (10) with the trivial inclusion after Part g), we finish the proof of e).

Now the second identity in Part d) follows from the first one in Part e).

To finish the proof of the theorem, it is enough to note that the boundedness of P_z on a Banach X implies that the sums in d) and e) are topological. □

Theorem 3.3, Corollaries 3.1 and 3.4 and the next Lemma provide the correctness for the following definition of the C^{F^*}-functional calculus for a linear algebraic operator. The idea is based on Remark 3.1.

Definition 4.2 (C^{F^*}-functional calculus)**.** For a linear space X and a linear algebraic $A \in L(X)$ ($D(A) = X$) with the minimal polynomial $\omega_{F^*}(A) = 0$, let the C^{F^*}-*functional calculus* $\mathcal{F}_{C^{F^*}} : C^{F^*}(G) \to L(X)$ be defined by $\mathcal{F}_{C^{F^*}} : f \mapsto p_{F^*} f(A)$.

Since p_{F^*} in Theorems 3.1 and 3.3 does not depend on $G = \cup_{x\in F} I_x \supset F$, this symbol will be often omitted or chosen conveniently.

Lemma 4.1. *For a linear space X, let $A \in L(X)$ be an algebraic operator with the minimal polynomial ω_{F^*}. Assume also that $f, g \in C^{F^*}(G)$. Then one has*

$$a)\ p_{F^*} f(A) p_{F^*} g(A) = p_{F^*}(fg)(A); \tag{4.1}$$

$$b)\ (p_{F^*} f(A))^{-1} = p_{F^*}(1/f)(A) \text{ if } f(x) \neq 0 \text{ for } x \in F. \tag{4.2}$$

The proof of Lemma 4.1. Part a) follows from Corollaries 3.1 and 3.4:

$$p_{F^*} f(A) p_{F^*} g(A) = p_{F^*}(fg)(A) + r(A)\omega_{F^*}(A) = p_{F^*}(fg)(A),$$

where r is some polynomial. In turn, Part a) implies b). □

Theorem 4.1 and the methods of its proof permits us to establish a counterpart of Theorems $A.II.7.2$ and $A.II.7.3$ from [24] providing necessary and sufficient conditions for the solvability of the equation $f(A)x = y$ with given $f \in C^{F^*}$ and $y \in X$.

Theorem 4.2. *For $y \in X$, $f \in C^{F^*}$ and an algebraic $A \in L(X)$ with the minimal polynomial ω_{F^*}, let F_0^* be the maximal multiplicity relation satisfying $f^{(k)}(z) = 0$ for $1 \leq k \leq \mu_{F_0}$ and $F_1^* = F^* - F_0^*$. Then there exists $x \in X$ satisfying $f(A)x = y$ if, and only if,*

a) $\omega_{F_1^*}(A)y = 0$, and

b) for every $z \in F_0 \cap F_1$, there exists $x_z \in X$ satisfying

$$(A - zI)^{\mu_{F_0^*}} x_z = \omega_{F_{1z}^*}(A) \sum_{k=0}^{\mu_{F_1^*}(z)-1} \left(\frac{(\cdot - z)^{\mu_{F_0^*}(z)}}{f\omega_{F_{1z}^*}} \right)^{(k)} \frac{(A - zI)^k}{k!} y,$$

where $F_{1z}^* = F_1^* - \{z, \mu_{F_1^*}(z)\}$.

Every solution x, if it exists, has the form $x = \sum_{z\in F} x_z$, where x_z are as in b) if $z \in F_0 \cap F_1$, x_z is an arbitrary element of $\mathrm{Ker}((A - zI)^{\mu_{F^*}(z)})$ *if $z \notin F_1$, and, if $z \in F_1 \setminus F_0$, x_z is uniquely described by*

$$x_z = \omega_{F_{1z}^*}(A) \sum_{k=0}^{\mu_{F_1^*}(z)-1} \left(\frac{1}{f\omega_{F_{1z}^*}} \right)^{(k)} \frac{(A - zI)^k}{k!} y$$

The proof of Theorem 4.2. The necessity of a) follows from the observation that $p_{F^*} f(A) = \omega_{F_0^*}(A) q(A)$ and, hence,

$$\omega_{F_1^*}(A)y = q(A)\omega_{F^*}(A)y = 0. \tag{1}$$

Let $P_z = p_z(A)$ be the projectors from Theorem 4.1. The commutativity $p_z(A)g(A) = g(A)p_z(A)$ for every $g \in C^{F^*}$ shows the equivalence of the equations

$$g(A)x = y \iff g(A)x_z = P_z y \text{ and } z = \sum_{z\in F} x_z. \tag{2}$$

In particular, $g(A)\mathrm{Ker}((A - zI)^{\mu_{F^*}(z)}) \subset \mathrm{Ker}((A - zI)^{\mu_{F^*}(z)})$. Noting that the minimal polynomial for the restriction of A to $\mathrm{Ker}((A - zI)^{\mu_{F^*}(z)})$ is $\omega_{\{(z,\mu_{F^*}(z))\}}(w) = (w - z)^{\mu_{F^*}(z)}$, we see with the aid of Part b) of Lemma 4.1 that $g(A)$ is invertible on $X_z = \mathrm{Ker}((A - zI)^{\mu_{F^*}(z)})$ if, and only if, $g(z) \neq 0$ and, in this case,

$$g(A)^{-1}_{|X_z} = \sum_{k=0}^{\mu_{F^*}(z)-1} \frac{(1/g)^{(k)}}{k!} (A - zI)^k. \tag{3}$$

Now (3) shows that $\omega_{F_1^*-\{(z,\mu_{F_1^*}(z))\}}$ is invertible on X_z. Together with the equivalence (2) with $g=\omega_{F_1^*}$ and the condition a), this shows that

$$y \in X_{F_1^*} = \bigoplus_{z\in F_1} \operatorname{Ker}((A-zI)^{\mu_{F_1^*}(z)}), \tag{4}$$

and, in fact, we have, for every $g \in C^{F^*}$,

$$g(A)v = p_{F_1^*} g(A)v \text{ for every } v \in X_{F_1^*}. \tag{5}$$

Combining (4) and (5), we also observe that

$$P_z v = p_{F_1^*,z}(A)v = \omega_{F_1^*,z}(A) \sum_{k=0}^{\mu_{F_1^*}(z)} (1/\omega_{F_1^*,z})^{(k)} \frac{(A-zI)^k}{k!} v \text{ for } v \in X_{F_1^*}. \tag{6}$$

Eventually, the observation (3) with $g(w)=g_z(w)=\frac{f(w)}{(w-z)^{\mu_{F_0^*}(z)}}$ implies, with the aid of (6) and (4) the equivalence

$$f(A)x_z = P_z y \Longleftrightarrow \sum_{k=0}^{\mu_{F_1^*}(z)-1} \frac{(1/g_z)^{(k)}}{k!}(A-zI)^k P_{F_1^*,z} y = (A-zI)^{\mu_{F_0^*}(z)} x_z. \tag{7}$$

Now the "B. Taylor calculus" in the form of Part a) of Lemma 4.1 with $\{(z,\mu_{F^*}(z))\}$ instead of F^* (or the less conceptual considerations led to the proof of the identity (5) in the proof of Theorem 4.1) shows that for $v \in \operatorname{Ker}((A-zI)^{\mu_{F_1^*}(z)})$, one has

$$p_{\{(z,\mu_{F_1^*}(z))\}} \frac{1}{g_z} p_{\{(z,\mu_{F_1^*}(z))\}} \frac{1}{\omega_{F_1^*,z}} v = p_{\{(z,\mu_{F_1^*}(z))\}} \left(\frac{1}{g_z \omega_{F_1^*}}\right) v. \tag{8}$$

Part b) for $z \in F_0$ is exactly (7) simplified with the aid of (8), i.e.

$$(A-zI)^{\mu_{F_0^*}} x_z = \omega_{F_{1z}^*}(A) \sum_{k=0}^{\mu_{F_1^*}(z)-1} \left(\frac{(\cdot - z)^{\mu_{F_0^*}(z)}}{f\omega_{F_{1z}^*}}\right)^{(k)} \frac{(A-zI)^k}{k!} y \text{ for } z \in F_1, \tag{9}$$

while (7) itself is equivalent to (2) with $g=f$. Note that, for $z \in F_1 \setminus F_0$, (9) gives the explicit unique value of x_z since $\mu_{F_0^*}(z)=0$.

The solution x has the form

$$x = \sum_{z\in F} x_z, \text{ where}$$

x_z can be any element of X_z if $z \notin F_1$ (meaning that $f(A)X_z=\{0\}$ because of $\omega_{F_0^*}(A)X_z=\{0\}$) and is defined by (9) if $z \in F_1$. □

Corollary 4.1. *For a multiplicity relation F^*, $F_z^* = F^* - \{(z,\mu_{F^*}(z))\}$ for $z \in F$ and a linear space X, let $A \in L(X)$ be an algebraic operator with the minimal polynomial ω_{F^*}. Let also $\vec{q} = \{q_z\}_{z\in F}$ be a family of rational or holomorphic in an open $G \supset F$ functions satisfying $q_z(z) \neq 0$ for every $z \in F$. Then, for $f \in C^{F^*}$, we have*

$$p_{\tilde{q},F^*} f(A) = p_{F^*} f(A),$$

where $\tilde{q} = \{q_z \omega_{F_z^*}\}_{z\in F}$ and $p_{\tilde{q},F^*} f$ is as in AH-decomposition (Theorem 3.1).

The proof of Corollary 4.1. We employ the tool that has been used twice in the proof of Theorem 4.2. Thanks to Part d) of Theorem 4.1 and Theorem 4.2 $\operatorname{Im}(\omega_{F_z^*}(A)) = \operatorname{Ker}((A-zI)^{\mu_{F^*}(z)})$ meaning that it is enough to show that

$$\begin{aligned} p_{\{(z,\mu_{F^*}(z))\}}(f/q_z)(A)q_z(A) &= p_{\{(z,\mu_{F^*}(z))\}}(f/q_z)(A)p_{\{(z,\mu_{F^*}(z))\}}q_z(A) \\ &= p_{\{(z,\mu_{F^*}(z))\}} f(A) \end{aligned} \tag{1}$$

for $z \in F$. The second identity in (1) is provided by Part a) of Lemma 4.1, while the first follows from either the Bezout theorem (Theorem 3.2) or the definition of C^{F^*}-calculus:

$$g(A) = p_{F^*} f(A).$$

□

The rest of this section is devoted to metric properties of functions of bounded algebraic operators.

Definition 4.3 (A priori constant)**.** For Banach spaces X, Y and a closed operator $A: X \supset D(A) \to Y$ with $\overline{\mathrm{Im}(A)} = \mathrm{Im}(A)$, an *a priori constant* $C_A(A)$ is the infimum of the constants C satisfying the following property:
for every $y \in \mathrm{Im}(A)$, there exists $x \in D(A)$, such that $y = Ax$ and $\|x\|_X \le C\|y\|_Y$.

Note that the definition is correct thanks to the Banach open-mapping theorem. Moreover, if $\mathrm{Im}(A) \subset D(B)$, we have the inequality

$$C_A(BA) \le C_A(B) C_A(A). \qquad (**)$$

The next theorem of this section demonstrates in a quantitative manner that polynomials $p(A)$ (and even C^{F^*} functions $f(A)$) of a bounded algebraic operator possess closed ranges if some particular polynomials of the form $(A - zI)^k$ do.

Theorem 4.3. *For an algebraic $A \in \mathcal{L}(X)$ with the minimal polynomial ω_{F^*}, let $F_0^* \le F^*$ be a multiplicity relation and $F_1^* = F^* - F_0^*$. Then $\overline{\mathrm{Im}(f(A))} = \mathrm{Im}(f(A))$ for every $f \in C^{F^*}$ satisfying $f^{(k)}(z) = 0$ for $1 \le k \le \mu_{F_0}$ if, and only if,*

$$\overline{\mathrm{Im}((A - zI)^{\mu_{F_0^*}(z)})} = \mathrm{Im}((A - zI)^{\mu_{F_0^*}(z)}) \text{ for } z \in F_0. \qquad (***)$$

If this is the case, we also have the estimate

$$C_A(f(A)) \le$$

$$\sum_{z \in F_1} c_A\left((A - zI)^{\mu_{F_0^*}(z)}\right) \left\| \omega_{F^*_{1z}}(A) \sum_{k=0}^{\mu_{F_1^*}(z)-1} \left(\frac{(\cdot - z)^{\mu_{F_0^*}(z)}}{f \omega_{F^*_{1z}}} \right)^{(k)} \frac{(A - zI)^k}{k!} \right\|_{\mathcal{L}(X)} .$$

Remark 4.2. a) Let us note that the condition $(***)$ cannot be omitted thanks to the following example of a nilpotent operator of an order n.

Let X be an infinite-dimensional Banach space and $A \in \mathcal{L}(X)$ be a non-algebraic compact operator. Considering the operator

$$S_A: X^n \to X^n: (x_1, x_2, \dots, x_n) \mapsto (0, Ax_1, Ax_2, \dots, Ax_{n-1}),$$

we see that $S_A^n = 0$ but $\overline{\mathrm{Im}(S_A^k)}$ is never closed for $1 \le k < n$ because A^k is compact while $\mathrm{Im}(A^k)$ is infinite-dimensional (otherwise A^k would be algebraic according to Kaplansky's criterium mentioned above).

Since every infinite-dimensional Banach space contains a basic sequence, the operator

$$Ax = \sum_{k \in \mathbb{N}} \alpha_k f_k(x) e_k, \text{ where } \alpha_k \ge 0, \sum_{k \in \mathbb{N}} \alpha_k < \infty$$

and $\{e_k\}_{k \in \mathbb{N}} \subset X$ is a normalized basic sequence with a bi-orthogonal normalized system $\{f_k\}_{k \in \mathbb{N}} \subset X^*$, is compact but not algebraic thanks to the following criterium due to Kaplansky (or Theorem 4.4 below).

b) Kaplansky [18] has established the following characterisation of bounded algebraic operators. For a Banach space X, an operator $A \in \mathcal{L}(X)$ is algebraic if, and only if, for every $x \in X$, one has

$$\dim\left[\{A^k\}_{k \in \mathbb{N}_0}\right] < \infty, \text{ i.e. } A \text{ is locally algebraic.}$$

The proof of Theorem 4.3. The estimate in the statement of the theorem follows from Theorem 4.2 and Definition 4.3. Indeed, it is enough to set $x_z = 0$ for $z \in F_0 \setminus F_1$ and use the representation for x_z with $z \in F_1$.

Part *d*) and the last sentence of Theorem 4.1, imply, as in the proof of Theorem 4.2, the equivalence

$$\overline{\operatorname{Im}(f(A))} = \operatorname{Im}(f(A)) \Longleftrightarrow \overline{\operatorname{Im}(f(A)P_z)} = \operatorname{Im}(f(A)P_z) \text{ for } z \in F.$$

Now the proof of formula (3) from the proof of Theorem 4.2 shows, for $f = \omega_{F_0^*}$, the further equivalence

$$\overline{\operatorname{Im}(f(A)P_z)} = \operatorname{Im}(f(A)P_z) \Longleftrightarrow \overline{\operatorname{Im}((A-zI)^{\mu_{F_0^*}(z)})} = \operatorname{Im}((A-zI)^{\mu_{F_0^*}(z)}),$$

for $z \in F$, finishing the proof of the theorem. □

The following theorem extends the last Kaplansky's characterization of bounded algebraic operators to closed operators.

Theorem 4.4. *For some $n \in \mathbb{N}_0$, a Banach space X, and a closed operator $A: X \supset D(A) \to X$ with $\rho(A) \neq \emptyset$, assume that for every $x \in D_n(A)$, there exists a polynomial $p_x \in \mathcal{P}$ satisfying $p_x(A)x = 0$. Then A is a bounded algebraic operator.*

The proof of Theorem 4.4. For $w \in \rho(A)$ and $x \in D_n(A)$, let $T = (A - wI)^{-1} \in \mathcal{L}(X)$ and p be a polynomial of degree m, such that $p_x(A)x = 0$ (meaning also that $x \in D_{\max(m_x,n)}$). Then $D_l(A) = \operatorname{Im}(T^l)$ for $l \in \mathbb{N}$ and

$$p_x(A) = T^{-m_x} \sum_{k=0}^{m_x} \frac{p^{(m_x-k)}(w)}{(m_x-k)!} T^k = T^{-m_x} \bar{p}_x(T). \tag{1}$$

Hence, for every $y \in X$, there exists a polynomial

$$q_y(z) = \bar{p}_x(z), \text{ where } x = T^n, \text{ satisfying } q_y(T)y = 0. \tag{2}$$

Thanks to the second criterium due to Kaplansky [18] (that is Part *b*) of Remark 4.2), operator T is algebraic because it is locally algebraic (see (2)) and bounded, meaning that $q(T) = 0$ (and $T^{-m}q(T)x = 0$ for $x \in D_m(A)$) for some polynomial $q(z) = \sum_{k=0}^m c_k z^k$ of degree m. This immediately implies that

$$\sum_{k=0}^{m} c_{m-k}(A - wI)^k x = 0 \text{ for } x \in D_m(A).$$

Since A is algebraic, we see that it is also bounded thanks to Theorem 4.6 below due to A. Taylor [29]. □

The last but one theorem of this section is an immediate corollary with respect to Theorem 3.9 and its proof, where the key estimates for the related coefficients are established.

Theorem 4.5. *For $n \in \mathbb{N}$, let $F^* = \{(z_j, m_j)\}_{j=1}^n$ be a multiplicity relation with $F \subset \Omega$ for a bounded open*

$$\bigcup_{j=1}^{n} D(z_j, d_j) \subset \Omega \subset \mathbb{C}, \text{ where } d_j = \min_{i \neq j} |z_i - z_j|.$$

Assume also that X is a Banach space and an algebraic $A \in \mathcal{L}(X)$ with the minimal polynomial ω_{F^*}. Then the restriction $\mathcal{F}_{H_\infty(\Omega)}$ of the functional calculus $\mathcal{F}_{C^{F^*}}$ is a bounded $H_\infty(\Omega)$-calculus with

$$\left\| \mathcal{F}_{H_\infty(\Omega)} \right| \mathcal{L}\left(H_\infty(\Omega), \mathcal{L}(X)\right) \| \leq \sum_{j=1}^{n} \sum_{l=1}^{m_j} d_j^{l-m} \phi_j(l) \left\| \omega_{F^* - \{(z_j,l)\}} \right| H_\infty(\Omega) \|,$$

where

$$\phi_j(l) = \begin{cases} \frac{(m-l)^{m-l}}{(m_j-l)^{m_j-l}(m-m_j)^{m-m_j}} & \text{if } 1 \le l < m_j; \\ 1 & \text{if } l = m_j. \end{cases}$$

We finish this section with the next remarkable characterization of the closed algebraic operators with non-empty resolvent set due to A. Taylor (Theorem 12.2 in [29]) that we use in the proof of Theorem 4.4.

Theorem 4.6 ([29]). *For $n \in \mathbb{N}_0$, a Banach space X, a polynomial p of degree n and a closed operator $A: X \supset D(A) \to X$ with $\rho(A) \neq \emptyset$, let*

$$p(A)x = 0 \text{ for every } x \in D(A^n) = D(p(A)). \text{ Then } A \in \mathcal{L}(X).$$

5. Continuous $H^{\mathbb{N}}$-calculus for closed operators

In this section we define an example of $H^{\mathbb{N}}$-calculus for closed operators and investigate its properties (including the uniqueness) relying heavily on the definitions and results established by A. Taylor in [29] providing related natural examples of algebraic operators. Then we provide a representation for this functional calculus permitting to "ignore" a finite number of isolated points of spectrum. The section ends with a correct definition of a mixed $H^{\mathbb{N}}$-calculus for the operators that are (double)sectorial and possess an additional bounded spectral set outside the (double)sector. Eventually a representation is provided, when the additional spectral set is a finite number of isolated points of spectrum that are poles of the resolvent operator.

In the first definition we outline the constructive definition of $H^{\mathbb{N}}$-calculus for a closed (unbounded) operator with non-empty resolvent set provided by A. Taylor [29]. More precisely, he defined $H(\Omega \cup \{\infty\})$-calculus and polynomial calculus ($\mathcal{P}$-calculus), along with their products $H(\Omega \cup \{\infty\})\mathcal{P}$ and $\mathcal{P}H(\Omega \cup \{\infty\})$ and showed their correctness and the majority of their properties, including even a representation for $f(A) + p(A)$ with $f \in H(\Omega \cup \{\infty\})$ and $p \in \mathcal{P}$. We use his definitions and results to define an $H^{\mathbb{N}}$-calculus relying on the identity $H^{\mathbb{N}}(\Omega) = H^{-1}(\Omega) + \mathcal{P}$ and outline the correctness (except for the convergence property 6) in Definition 2.4). Then we show the presence of the convergence property and the uniqueness that are traditionally investigated in connection with the $H^{\mathbb{N}}$-calculus for (bi)sectorial operators relying on the well-known approaches (see [22, 1, 6]). For the history and further references related to the definitions of functional calculi one can consult [11] and [1].

Definition 5.1 (An $H^{\mathbb{N}}$-calculus). For a Banach space X and a closed operator $A: X \supset D(A) \to X$, let $\rho(A) \neq \emptyset$. Assume also that $\sigma(A) \subset \Omega \subset \mathbb{C}_\infty$ for an open $\Omega \neq \mathbb{C}_\infty$ with the bounded boundary $\partial\Omega$. The *calculus* $\mathcal{F}_T: f \mapsto f(A)$ is defined as follows.

If $A \in \mathcal{L}(X)$, i.e. $\sigma(A) = \sigma(A) \cap \mathbb{C}$ is bounded, for every $f \in H(\Omega) \supset H^{\mathbb{N}}(\Omega)$, one defines

$$f(A) = \frac{1}{2\pi i} \oint_\gamma f(\zeta) R_A(\zeta) d\zeta, \text{ where } R_A(\zeta) = (\zeta I - A)^{-1} \text{ and } \gamma = \partial G \quad (1)$$

is positively oriented contour for a Cauchy domain $G \supset \sigma(A)$ enveloping $\sigma(A)$ in Ω that we assume fixed for the rest of the definition.

If $\sigma(A)$ is not bounded and $f \in H^{-1}(\Omega)$, we define $f(A)$ using the same representation (1). If $p(z) = \sum_{k=0}^n a_k z^k$ is a polynomial of degree $n \in \mathbb{N}$ ($a_n \neq 0$), the operator

$$p(A) = \sum_{k=0}^n a_k A^k \text{ is well-defined on } D_n(A) = D(p(A)) \quad (2)$$

and closed (see Theorem 6.1 from [29]).

For $n \in \mathbb{N}$ and $z_0 \in \mathbb{C} \setminus \Omega$, we define the *Cauchy projector* $P_{\mathcal{P}_n} : H^n(\Omega) \to \mathcal{P}_n : f \mapsto \sum_{k=0}^n c_k(f)(z-z_0)^k$ by

$$c_k(f) = \frac{-1}{2\pi i}\oint_\gamma \frac{f(\zeta)d\zeta}{(\zeta - z_0)^{k+1}} = \frac{-1}{2\pi i}\oint_{\gamma_\infty} \frac{f(\zeta)d\zeta}{(\zeta - z_0)^{k+1}},$$

where $\gamma_\infty = \partial G_\infty$ and G_∞ is the unbounded component of G. Note that

$$\mathrm{Ker}(P_{\mathcal{P}_n}) = H^{-1}(\Omega).$$

Eventually, for $n \in \mathbb{N}_0$ and $f \in H^n(\Omega) \setminus H^{n-1}(\Omega)$, we define $\mathcal{F}_T : f \mapsto f(A)$ by

$$f(A) : D_n(A) \to \mathcal{C}(X) : x \mapsto P_{\mathcal{P}_n} f(A)x + (f - P_{\mathcal{P}_n} f)(A)x.$$

Let us note that $\mathcal{F}_T$ just defined does not depend on the particular choice of G thanks to the operator-valued Cauchy theorem.

To prove the first theorem in this section, we need the following counterpart of (1) from Definition 5.1 established by A. Taylor (Theorem 6.4 in [29]).

Theorem 5.1 ([29])**.** *Let $f \in H^n(\Omega)$. Then, for every $x \in D_n(A)$, one has*

$$\mathcal{F}_T f x = f(A)x = \frac{1}{2\pi i}\oint_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}(A - z_0 I)^{n+1} R_A(\zeta) x d\zeta,$$

where the integral exists thanks to the representation

$$\left(\frac{A - z_0 I}{\zeta - z_0}\right)^{n+1} R_A(\zeta)x = R_A(\zeta)x - \sum_{k=0}^n \left(\frac{A - z_0 I}{\zeta - z_0}\right)^k x \text{ for } \zeta \in \gamma.$$

Theorem 5.2 ($H^{\mathbb{N}}(\Omega)$-calculus)**.** *The functional calculus operator $\mathcal{F}_T$ from Definition* 5.1 *is an $H^{\mathbb{N}}(\Omega)$-calculus and a bounded $H^0_\infty(\Omega)$-calculus. (as described in Definition* 2.4*). Moreover, it is also the unique $H^{-1}(\Omega)$-calculus (and, hence, $H^{\mathbb{N}}(\Omega)$-calculus) and bounded $H^0_\infty(\Omega)$-calculus satisfying*

$$\mathcal{F}(\cdot - z)^{-1} = R_A(z) \text{ for } z \in \rho(A). \tag{R_A}$$

The proof of Theorem 5.2. The conditions $1) - 4)$ of Definition 2.4 are trivially satisfied, while the validity of 7) follows from the independence of G mentioned above. Theorem 5.1 and (1) from Definition 5.1 imply the following representation used in [29] as the definition of $H(\Omega \cup \{\infty\})$-calculus:

$$f(A) = f(\infty)I + \frac{1}{2\pi i}\oint_\gamma f(\zeta)R_A(\zeta)d\zeta \text{ for } f \in H^0(\Omega). \tag{1}$$

With the aid of the triangle inequality, this representation is followed by the validity of 8) (Def. 2.4):

$$\big\| \mathcal{F}_T |\, \mathcal{L}(H^0_\infty(\Omega), \mathcal{L}(X))\big\| \le 1 + \frac{1}{2\pi}\int_\gamma \|R_A(\zeta)\|_{\mathcal{L}(X)}|d\zeta|. \tag{2}$$

Now we see that, for $f \in H^n(\Omega)$, $(I - P_n f)(A)$ is bounded, while $P_n f(A)$ is closed thanks to Theorem 6.1 from [29], meaning that $f(A) \in \mathcal{C}(X)$.

Since $\gamma = \partial G$ is compact, the condition 6) (Def. 2.4) is satisfied too. Namely, if $f_k \to f$ uniformly on compact subsets for $\{f_k\}_{k\in\mathbb{N}} \subset H^n(\Omega)$ then $(f_k - P_n f)(A) \overset{\mathcal{L}(X)}{\longrightarrow} (f - p_n f)(A)$ due to (2) and the coefficients of $P_n f_k(A)$ converge to the coefficients of $P_n f(A)$.

To check the multiplication invariance 5) (Def. 2.4) in the case $f \in H^{-1}(\Omega)$ and $g \in H^n(\Omega)$, we use Theorem 5.1, the Hilbert identity and the Cauchy theorem and formula for holomorphic $\mathbb{C}$-valued functions. Indeed, assume that $\sigma(A) \subset G_1 \subset G \subset \Omega$ are two Cauchy domains with $\gamma = \partial G$ and $\gamma_1 = \partial G_1$ satisfying $\bar{G_1} \subset G$ and $\bar{G} \subset \Omega$. Then we have, for $x \in D_n(A)$ and $z_0 \in \rho(A) \setminus \Omega$,

$$g(A)f(A)x = \frac{-1}{4\pi^2}\oint_\gamma\oint_{\gamma_1} f(w)\frac{g(\zeta)}{(\zeta-z_0)^{n+1}}(A-z_0I)^{n+1}R_A(\zeta)R_A(w)xdwd\zeta =$$
$$= \frac{-1}{4\pi^2}\oint_\gamma\oint_{\gamma_1} f(w)\frac{g(\zeta)}{(\zeta-z_0)^{n+1}}(A-z_0I)^{n+1}\frac{R_A(\zeta)}{\zeta-w}xdwd\zeta +$$
$$+ \frac{-1}{4\pi^2}\oint_{\gamma_1}\oint_{\gamma} f(w)\frac{g(\zeta)}{(\zeta-z_0)^{n+1}}(A-z_0I)^{n+1}\frac{R_A(w)}{w-\zeta}xd\zeta dw =$$
$$= \frac{-1}{4\pi^2}\oint_\gamma\oint_{\gamma_1} f(w)\frac{g(\zeta)}{(\zeta-z_0)^{n+1}}(A-z_0I)^{n+1}\frac{R_A(\zeta)}{\zeta-w}xdwd\zeta =$$
$$= \frac{1}{2\pi i}\oint_\gamma \frac{f(\zeta)g(\zeta)}{(\zeta-z_0)^{n+1}}(A-z_0I)^{n+1}R_A(\zeta)xd\zeta = (gf)(A)x. \quad (3)$$

The last identity in (3) holds due to Theorem 5.1 and the inclusion $gf \subset H^n(\Omega)$. Note that

$$\{R_A(\zeta)R_A(w)x, R_A(w)x, R_A(\zeta)x\} \subset D_{n+1}(A) \text{ for } x \in D_n(A),$$

while $f(A)x \in D_{n+1}(A)$ thanks to the definition of $f(A)$ for $f \in H^{-1}(\Omega)$. Similarly we establish $f(A)g(A)x = (fg)(A)x = (gf)(A)$ for $x \in D_n(A)$.

In the general case $f \in H^m(\Omega)$ and $g \in H^n(\Omega)$ we obtain with the aid of Definition 5.1 that for $x \in D_{m+n}(A)$,

$$f(A)g(A)x = P_mf(A)P_ng(A)x + P_mf(A)(g-P_ng)(A)x + (f-p_mf)(A)g(A)x =$$
$$= (P_mfP_ng)(A)x + (P_mf(g-P_ng))(A)x + ((f-p_mf)g)(A)x = (fg)(A)x, \quad (4)$$

where $P_mf(A)P_ng(A)x = (P_mfP_ng)(A)x$ as polynomials.

The identity (R_A) follows from the identity

$$(\zeta-z_0)^{-1}R_A(\zeta) = \left((\zeta-z_0)^{-1} - R_A(\zeta)\right)R_A(z_0) \text{ for } z_0 \in \rho(A), \quad (5)$$

the Cauchy theorem applied to $R_A(\zeta)$, the Cauchy formula applied to $(\zeta-z_0)^{-1}$ and the choice of G with $z_0 \notin \bar{G}$.

To establish the uniqueness, we note that, thanks to the continuity property related to the approximation of this integral with its Riemann sums, the integral defining $\mathcal{F}_T$ commutes with another calculus $\mathcal{F}$ satisfying (R_A), implying the uniqueness. □

Let us discuss the definition of an $H^{\mathbb{N}}$-calculus in the mixed case of an operator that has a spectral set outside a (double)sector. Such an operator can appear as a perturbation of a (double)sectorial operator (see Definition 5.3).

Definition 5.2 (Sectors and classes)**.** The *sector and double sector* are defined by

$$S^1_\theta := \{z \in \mathbb{C}\setminus\{0\} : |\arg z| < \theta\} \text{ for } \theta \in [0,\pi), \text{ and}$$

$$S^2_\theta := -S^1_\theta \cup S^1_\theta \text{ for } \theta \in [0,\pi/2).$$

Let S_θ be either S^1_θ or S^2_θ, and let $\Omega \subset \mathbb{C}$ be $\Omega_0 \cup S_\theta$ with $\overline{\Omega_0} \cap \overline{S_\theta} = \emptyset$, where Ω_0 is bounded and open.

As earlier the symbol $H(\Omega)$ denotes the space of all holomorphic functions on Ω endowed with the topology of the uniform convergence on compact subsets of Ω. For $\beta \in \mathbb{R}$, let also $H^\beta(\Omega) \subset H(\Omega)$ be the subspace of functions satisfying

$$\limsup_{|z|\to 0} |f(z)||z/(\alpha^2+z^2)|^{-\beta} < \infty \text{ and } \limsup_{|z|\to\infty} |f(z)||z/(\alpha^2+z^2)|^{-\beta} < \infty$$

for some $\alpha > 0$ with $\pm i\alpha \notin \Omega_0$. Assume also that $H_\infty(\Omega)$ is the Banach space of the bounded holomorphic functions on Ω endowed with the L_∞-norm, and

$$\Psi(\Omega) = \bigcup_{\beta>0} H^\beta(\Omega) \text{ and } F(\Omega) = \bigcup_{\beta\le 0} H^\beta(\Omega).$$

Definition 5.3 ($F(\Omega)$-calculus)**.** For a Banach space X and Ω defined in Definition 5.2, let $A:\ X \supset D(A) \to X$ be a closed operator with $\overline{D(A)} = X$ and (unbounded) spectrum $\sigma(A)$ containing a bounded spectral set (i.e. compact and open subset of $\sigma(A)$ in the induced topology) σ_0 satisfying

$$\sigma_0 \subset \Omega_0,\ \sigma(A) \setminus \sigma_0 \subset \Omega \setminus \Omega_0 \text{ and } \left\|(A-\lambda I)^{-1}\right\|_{\mathcal{L}(X)} \le \frac{C(\nu)}{|\lambda|} \tag{1}$$

for every $\lambda \in \Omega \setminus (\Omega_0 \cup S_\nu)$ and $\nu \in (\theta_0, \theta]$ for some $\theta_0 \in (0, \theta)$.

Let us define $\mathcal{F}:\ F(\Omega) \to \mathcal{C}(X):\ f \mapsto f(A)$ by the formula

$$f(A) = \frac{1}{2\pi i} \oint_{\gamma_0} f(\zeta) R_A(\zeta) d\zeta + \mathcal{F}_M f \tag{$\mathcal{F}(F)$}$$

for some contour $\gamma_0 = \partial G_0$ enveloping σ_0 in Ω_0 and the functional calculus operator $\mathcal{F}_M:\ F(S_\theta) \to \mathcal{C}(X)$ introduced by A. M^{c}Intosh in [22] and further investigated in [1, 6].

Note that the first summand in ($\mathcal{F}(F)$) does not depend on a particular choice of γ_0.

Theorem 5.3 ($\mathcal{F}_D + \mathcal{F}_M$)**.** *The functional calculus operator $\mathcal{F}$ from Definition* 5.3 *is correctly defined and satisfies the conditions of Definition* 2.4*. Moreover, it is also unique $H_\infty(\Omega)$-calculus (and, hence, $H^{\mathbb{N}}(\Omega)$-calculus) and bounded $H^0_\infty(\Omega)$-calculus satisfying*

$$\mathcal{F}(\cdot - z)^{-1} = R_A(z) \textit{ for } z \in \rho(A). \tag{R_A}$$

The proof of Theorem 5.3. The right-hand side of ($\mathcal{F}(F)$) is a closed operator as a sum of a bounded and a closed operators. Without loss of generality we may assume that Ω_0 is bounded. There exists a bounded open $\Omega_1 \supset \overline{\Omega_0}$ with $\overline{\Omega_1} \cap \overline{S_\theta} = \emptyset$. Choosing $\Omega' = \Omega_0 \cup \mathbb{C} \setminus \overline{\Omega_1}$, we can use Theorem 5.1 on $H^{\mathbb{N}}(\Omega')$-calculus and the results from [29]. In particular, we see with the aids of the results from §8 in [29] that $P_{\sigma_0} = \chi_{\Omega_0}(A)$ is a spectral projector with $X = X_{\sigma_0} \oplus X'_{\sigma_0}$ for $X_{\sigma_0} = \mathrm{Im}(()P_{\sigma_0})$ and $X'_{\sigma_0} = \mathrm{Ker}(P_{\sigma_0})$, that the restriction $A_{\sigma_0} = AP_{\sigma_0} \in \mathcal{L}(X_{\sigma_0})$, and that

$$R_A(z)_{|X'_{\sigma_0}} = R_{A_{|X'_{\sigma_0}}} \text{ and } R_A(z)_{|X_{\sigma_0}} = R_{A_{|X_{\sigma_0}}}. \tag{1}$$

The definition of $\mathcal{F}_M$ in [1] (pages 89-90) also shows that

$$\mathcal{F}_M f P_{\sigma_0} = 0. \tag{2}$$

The observations (1) and (2) imply that it is enough to check the conditions of Definition 2.4 for the restrictions of A onto X_{σ_0} and X'_{σ_0} because

$$f(A) = f\left(A_{|X_{\sigma_0}}\right) P_{\sigma_0} + f\left(A_{|X'_{\sigma_0}}\right)(I - P_{\sigma_0}),\ \text{where } f\left(A_{|X'_{\sigma_0}}\right)(I - P_{\sigma_0}) = \mathcal{F}_M f \tag{3}$$

for a (double)sectorial operator $A_{|X'_{\sigma_0}}$ (Indeed, $\sigma\left(A_{|X'_{\sigma_0}}\right) = \sigma(A) \cap S_\theta$ and the resolvent bound is preserved by its restriction to X'_{σ_0}.) Since the validity of the conditions of Definition 2.4 holds for the restrictions thanks to Theorem 5.2 above and Lecture 2 in [1] respectively, the proof of the theorem is complete. □

Theorem 5.4. *For $n \in \mathbb{N}$, let $f \in H^n(\Omega)$. Assume also that A is as in Definition* 5.3*, and that, for a multiplicity relation F^*, $\sigma_0 = F$ consists of a finite number* *of isolated poles $z \in F$ of the orders $\mu_{F^*}(z)$ of the resolvent $R_A(\zeta)$. Let also $z_0 \in \rho(A) \setminus \Omega$. Then*

$$\mathcal{F} f x = p_{F^*}\left(f(\phi_\alpha^{-m(F^*)}\right)(A)\,(\phi_\alpha(A))^{m(F^*)}\,x + \\ + \mathcal{F}_M\left(((\phi_\alpha^{m(F^*)}\omega_{F^*}\triangle_{F^*+\{(\cdot,1)\}}\left(f\phi_\alpha^{-m(F^*)}\right)\right) x \text{ for } x \in D_n(A),$$

where $\phi_\alpha(z) = \frac{z}{\alpha^2+z^2}$ for some $\alpha > 0$ with $\{\pm i\alpha\} \subset \rho(A) \setminus \Omega$.

The proof of Theorem 5.4. As was shown by A. Taylor in Theorem 10.8 in [29] with the aid of the Laurent expansion of the resolvent $R_A(\zeta)$ around an isolated pole λ of $R_A(\zeta)$ of order l, one has

$$X = \mathrm{Ker}((A-\lambda I)^l) \bigoplus \mathrm{Im}((A-\lambda I)^l), \text{ and } \overline{\mathrm{Im}((A-\lambda I)^l)} = \mathrm{Im}((A-\lambda I)^l). \quad (1)$$

Hence we see that the subspace X_{σ_0} (see the proof of Theorem 5.3) has the form X_{F^*}:

$$X_{\sigma_0} = \mathrm{Im}(P_{\sigma_0}) = X_{F^*} = \bigoplus_{z\in F} \mathrm{Ker}((A-zI)^{\mu_{F^*}(z)}) \text{ and } X = X_{\sigma_0} \bigoplus X'_{\sigma_0}. \quad (2)$$

Therefore, the restriction of A onto X_{F^*} is an algebraic operator with the minimal polynomial ω_{F^*}. Applying the general Gel'fond's formula (Theorem 3.7) to the holomorphic (on Ω_0) function $f\phi_\alpha^{-m(F^*)} \in \Psi(\Omega)$, we obtain, with the aid of Theorem 5.3 and its proof, that

$$f\left(A_{|X_{\sigma_0}}\right) P_{\sigma_0} = p_{F^*}\left(f(\phi_\alpha^{-m(F^*)}\right)(A)\,(\phi_\alpha(A))^{m(F^*)}\,P_{\sigma_0} \text{ and} \\ f\left(A_{|X'_{\sigma_0}}\right)(I-P_{\sigma_0}) = p_{F^*}\left(f(\phi_\alpha^{-m(F^*)}\right)(A)\,(\phi_\alpha(A))^{m(F^*)}\,(I-P_{\sigma_0}) + \\ + \mathcal{F}_M\left(((\phi_\alpha^{m(F^*)}\omega_{F^*}\triangle_{F^*+\{(\cdot,1)\}}\left(f\phi_\alpha^{-m(F^*)}\right)\right)(I-P_{\sigma_0}). \quad (3)$$

These identities imply the statement of the theorem thanks to (2) in the proof of Theorem 5.3. □

References

[1] *D. Albrecht, X. Duong, A. M^c Intosh*, Operator theory and harmonic analysis. // (In: "Instructional workshop on Analysis and Geometry", Part III (Canberra, 1995)), Proc. Centre Math. Appl. Austral. Nat. Univ. 1996. V. 34. P. 77–136.

[2] *I. S. Berezin, N. P. Zhidkov*, Methods of calculation. V. I and II. Moscow: Gos. Izd. Fiz.-Mat. Lit. (Fizmatgiz) 1959. P. 1–464. V. 2. 1–620. // English transl. by O. M. Blunn: Computing methods. V. I, II. Oxford-London-Edinburgh-New York-Paris-Frankfurt: Pergamon Press Ltd. 1965. V. 1. P. 1–464+i–xxxiv. V. 2. P. 1–679+i–xv.

[3] *C. de Boor*, Divided Differences. // Surveys Approx. Theory. 2008. V. 1. P. 46–69.

[4] *A. Buchheim*, An extension of a theorem of Professor Sylvester's relating to matricies. // Phil. Mag. Ser. 5. 1886. V. 22. Iss. 135. P. 173–174.

[5] *L. Chakalov*, On a certain presentation of the Newton divided differences in interpolation theory and its applications (Bulgarian). // Annuaire Univ. Sofia (Fiz. Mat. Fakultet) 1938. V. 34. P. 353–394.

[6] *M. Cowling, I. Doust, A. M^c Intosh, A. Yagi*, Banach space operators with a bounded H^∞-calculus. // J. Aust. Math. Soc. (Ser. A) 1996. V. 60. P. 51–89.

[7] *I. Daubechies*, Ten lectures on wavelets. (In: "CBMS-NSF Regional Conference Series in Applied Mathematics", V. 61.) Philadelphia: Society for Industrial and Applied Mathematics. 1992. P. 1–357+i–xx.

[8] *P. A. M. Dirac*, The Principles of Quantum Mechanics. 3rd edition. Oxford: Clarendon Press. 1947. P. 1–311+i–xii.

[9] *N. Dunford*, Spectral theory. I. Convergence to projections. // Trans. Amer. Math. Soc. 1943. V. 54. P. 185–217.

[10] *N. Dunford*, Spectral theory. // Bull. Amer. Math. Soc. 1943. V. 49. P. 637–651.

[11] *N. Dunford, J.T. Schwartz*, Linea operators. Part I: General Theory. (In: "Pure and Applied Mathematics. A series of books and monographs, v. VII") With assist. W.D. Bade and R.G. Burtle. New York: Interscience Publishers, Inc. (New-York-London-Sydney: John Wiley & Sons, Inc.) 1958. P. 1–243+i–x.

[12] *G. Frobenius*, Über die Entwicklung analytischer Functionen in Reihen, die nach gegebenen Functionen fortschreiten. // J. reine und angew. Math. 1871. V. 73. P. 1–30.

[13] *A. O. Gel'fond*, The calculus of finite differences. Moscow-Leningrad: Gos. Izd. Tech.-Teor. Lit. (Gostekhizdat) 1952. P. 1–479. // 3rd corr. ed.: Moscow: "Nauka". 1967. P. 1–375. // German transl. by H. Limberg and G. Tesch: Differenzenrechnung. Berlin: VEB Deutscher Verlag der Wissenschaften. 1958. P. 1–336+i–viii. // French transl. by G. Rideau: Calcul des différences finies. Dunod, Paris: Collect. Univ. Math., XII. 1963. P. 1–378.// Authorized English transl. from the 3rd Russian ed.: Calculus of finite differences. (Intern. Mon. Adv. Math. Phys.) 6-U.B., Jawahar Nagar, Delhi-7: Hindustan Publishing Corp. 1971. P. 1–451+i–vi.

[14] *A. Genocchi*, Relation entre la différence et la dérivée d'un même ordre quelconque. // Archiv Math. Phys. Ser. I. 1869. V. 49. P. 342–345.

[15] *Ch. Hermite*, Sur la formule d'interpolation de Lagrange. // J. reine und angew. Math. 1877. V. 84. P. 70–79.

[16] *N. Jacobson*, Structure theory for algebraic algebras of bounded degree. // Ann. of Math. 2nd Ser. 1945. V. 46. P. 695–707.

[17] *I. Kaplansky*, Rings with a polynomial identity. // Bull. Amer. Math. Soc. 1948. V. 54. P. 575–580.

[18] *I. Kaplansky*, Infinite abelian groups. Ann Arbor: University of Michigan Press. 1954. P. 1–91.

[19] *Z. I. Khalilov*, Linear singular equations in normed rings. (Russian) // Doklady Acad. Sci. USSR. 1947. V. 58. P. 1613–1616.

[20] *G. Kowalewski*, Interpolation und genäherte Quadratur. Berlin: B. G. Teubner. 1932.

[21] *A. Malcev*, On the representations of infinite algebras. // Mat. Sbornik. N.S. 1943. V. 13(55). P. 263–286.

[22] *A. M^cIntosh*, Operators which have an H_∞ functional calculus. // Proc. Centre Math. Anal. Austral. Nat. Univ. 1986. V. 14. P. 210–231.

[23] *T. Popoviciu*, Sur quelques propriétés des fonctions d'une ou de deux variables réelles.// Dissertation presented to the Faculté des Sciences de Paris. 1933. Cluj: Institutul de Arte Grafice "Ardealul" (Romania).

[24] *D. Przeworska-Rolewicz, S. Rolewicz*, Equations in linear spaces. (In: "Monografie Matematyczne", T. 47.) Warszawa: Państwowe Wydawnictwo Naukowe. 1968. P. 1–380.

[25] *R. F. Rinehart*, The equivalence of definitions of a matric function. // Amer. Math. Month. 1955. V. 62. No. 6. P. 395–414.

[26] *J. F. Steffensen*, Note on divided differences. // Danske Vid. Selsk. Math.-Fys. Medd. 1939. V. 17. No. 3. P. 1–12.

[27] *J. J. Sylvester*, On the equation to the secular inequalities in the planetary theory. // Phil. Mag. Ser. 5. 1883. V. 16. Iss. 100. P. 267–269.

[28] *A. E. Taylor*, Analysis in complex Banach spaces. // Bull. Amer. Math. Soc. 1943. V. 49. P. 652–669.

[29] *A. E. Taylor*, Spectral theory of closed distributive operators. // Acta Math. 1951. V. 84. P. 189–224.

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