# A MAXIMAL THEOREM FOR HOLOMORPHIC SEMIGROUPS ON VECTOR-VALUED SPACES

GORDON BLOWER, IAN DOUST, AND ROBERT J. TAGGART

Abstract. Suppose that $1 < p \leq \infty$, $(\Omega, \mu)$ is a $\sigma$-finite measure space and $E$ is a closed subspace of a Lebesgue–Bochner space $L^p(\Omega; X)$, consisting of functions on $\Omega$ that take their values in some complex Banach space $X$. Suppose also that $-A$ is injective and generates a bounded holomorphic semigroup $\{T_z\}$ on $E$. If $0 < \alpha < 1$ and $f$ belongs to the domain of $A^\alpha$ then the maximal function $\sup_z \|T_z f\|_X$, where the supremum is taken over any given sector contained in the sector of holomorphy, belongs to $L^p$. A similar result holds for generators that are not injective. This extends earlier work of Blower and Doust [BD].

## 1. Introduction

Suppose that $\{T_t\}_{t\geq 0}$ is a $C_0$-semigroup of bounded linear operators on a Banach space $E$. In the case that $E$ is a space of functions $f$ from a measurable set $\Omega$ to a normed space $X$, an important tool in the analysis of such a semigroup is the maximal function $Mf$ where

$$Mf(\omega) = \operatorname*{ess-sup}_{t\geq 0} \|T_t f(\omega)\|_X .$$

The classical theorems of Stein [St] and Cowling [Co] apply to symmetric diffusion semigroups on $E$, where $E = L^p(\Omega)$ and $1 < p < \infty$, and show that in this case $\|Mf\|_p \leq c \|f\|_p$ for all $f$ in $L^p(\Omega)$.

Taggart [Ta] extended this to the vector-valued context where $E = L^p(\Omega; X)$ and $X$ satisfies a geometric condition weak enough to include, for example, many of the classical reflexive function spaces.

Under much weaker hypotheses, Blower and Doust [BD] showed that in the scalar-valued case, if the semigroup $\{T_t\}_{t>0}$ can be extended to a bounded holomorphic semigroup on a sector of the complex plane, then $Mf$ lies in $L^p(\Omega)$, at least for $f$ in a large submanifold of $L^p(\Omega)$.

In this paper we show that the result of [BD] may be extended to the vector-valued case where $E$ is a subspace of $L^p(\Omega; X)$ and $X$ is any complex Banach space. Moreover, the result also holds when the supremum used to define the maximal function is taken over sectors contained in the sector of holomorphy of the semigroup (c.f., the results of [Co] and [Ta]). We will deduce both of these extensions by modifying the original argument of [BD].

The paper is organised as follows. In Section 2 we introduce some notation and state the main theorem of the paper. As with the result of [BD], the theorem is proved by representing the semigroup in terms of fractional powers of its generator

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and obtaining good $L^p$ bounds for parts of this representation. Some of the arguments of [BD] which made use of Yosida approximants to the semigroup's generator have been replaced by direct appeals to the functional calculus for sectorial operators. Salient facts about the functional calculus are presented in Section 3, while the representation of the semigroup and corresponding bounds are established in Section 4. In Section 5 we complete the proof of the theorem.

As an application, we show how the main theorem can be used to deduce almost everywhere pointwise convergence for the semigroup on a large submanifold of $L^p$. The precise details are stated in Section 2 and proved at the end of the paper. For examples of semigroups to which the theorem applies, see [BD, Sections 1 and 4].

## 2. Notation and theorem

We begin by introducing some notation. Given $\theta$ in $[0,\pi)$, let $S_\theta^0$ and $S_\theta$ denote the open and closed sectors of $\mathbb{C}$ given by

$$S_\theta^0 = \{\zeta \in \mathbb{C} \setminus \{0\} : |\arg \zeta| < \theta\}$$

and

$$S_\theta = \{\zeta \in \mathbb{C} \setminus \{0\} : |\arg \zeta| \leq \theta\} \cup \{0\},$$

where $\arg \zeta$ denotes the principal argument of a nonzero complex number $\zeta$. Note that $S_0 = [0, \infty)$.

Throughout, suppose that $X$ is a complex Banach space and that $(\Omega, \mu)$ is a positive $\sigma$-finite measure space. When $1 \leq q \leq \infty$, let $L^q(\Omega; X)$ denote the Lebesgue–Bochner space of strongly measurable functions $f : \Omega \to X$ whose norm is given by

$$\|f\|_{L^q(\Omega;X)} = \left( \int_\Omega \|f(\omega)\|_X^q \, \mathrm{d}\mu(\omega) \right)^{1/q}$$

if $q < \infty$ and $\|f\|_{L^\infty(\Omega;X)} = \text{ess-sup}_{\omega\in\Omega} \|f(\omega)\|_X$. We write $L^q(\Omega)$ for $L^q(\Omega; \mathbb{C})$.

**Definition 2.1.** Suppose that $E$ is a Banach space and $0 < \theta < \pi/2$. A family $\{T_z : z \in S_\theta^0\}$ of bounded linear operators acting on $E$ is said to be a *bounded holomorphic semigroup* of angle $\theta$ on $E$ if

i) $T_z T_w = T_{z+w}$ whenever $z, w \in S_\theta^0$,
ii) for each $\theta'$ in $[0, \theta)$ there exists a positive constant $K_{\theta'}$ such that $\|T_z\| \leq K_{\theta'}$ whenever $z \in S_{\theta'}$,
iii) the mapping $z \mapsto T_z$ is a holomorphic function from $S_\theta^0$ into the space of bounded linear operators on $E$, and
iv) for each $f$ in $E$, $T_z f \to f$ in $E$ as $z \to 0$ with $z$ in $S_\theta^0$.

The setting for our main result is as follows. Suppose that $E$ is a closed subspace of $L^p(\Omega; X)$, where $1 < p \leq \infty$, and suppose that $\{T_z : z \in S_\theta^0\}$ is a bounded holomorphic semigroup of angle $\theta$ on $E$, for some $\theta$ in $(0, \pi/2)$. Let $-A$ denote the infinitesimal generator of this semigroup. (See [RS2, Section X.8] or [Da] for definitions of these terms.) When $0 < \alpha < 1$ we can define the fractional power $A^\alpha$ of $A$ and, if $A$ is injective, the power $A^{-\alpha}$ (see Remark 3.3). Given $f$ in $D(A^\alpha) \cap D(A^{-\alpha})$, define $\|f\|_{p,\alpha}$ by

$$\|f\|_{p,\alpha} = \|A^\alpha f\|_{L^p(\Omega;X)} + \left\|A^{-\alpha} f\right\|_{L^p(\Omega;X)}.$$

Whenever $0 \leq \theta' < \theta$ and $f \in E$, define the maximal function $M_{\theta'} f$ by

$$(M_{\theta'} f)(\omega) = \sup \{\|(T_z f)(\omega)\|_X : z \in S_{\theta'}\} \qquad \forall \omega \in \Omega.$$

Since $\{T_z : z \in S_\theta^0\}$ is a holomorphic semigroup, the maximal function $M_{\theta'} f$ is well-defined and measurable (see [St, pp. 72–73] for further details).

The following is the main result of the paper.

**Theorem 2.2.** *Suppose $E$, $A$ and $\{T_z : z \in S_\theta^0\}$ are as above, that $A$ is injective and that $0 < \alpha < 1$. If $f \in D(A^\alpha) \cap D(A^{-\alpha})$ and $0 \leq \theta' < \theta$ then $M_{\theta'} f \in L^p(\Omega)$ and there is a constant $C(A, \alpha, \theta')$ such that*

$$\|M_{\theta'} f\|_{L^p(\Omega)} \leq C(A, \alpha, \theta') \, \|f\|_{p,\alpha} \, .$$

The proof of the theorem will be deferred to Section 5. We conclude the present section with a remark and corollary.

*Remark* 2.3. The constant $C(A, \alpha, \theta')$ is bounded by

$$\frac{C_\eta \sec(\theta' + \eta) \sec(\alpha\pi/2)}{\pi\alpha}$$

for any $\eta$ such that $\pi/2 - \theta < \eta < \pi/2 - \theta'$, where $C_\eta$ is the constant appearing in the resolvent bound (3.1) for $A$ on $L^p(\Omega; X)$. Note that if the semigroup acts on a range of $L^p$ spaces then these quantities may vary with $p$.

The following corollary shows that the injectivity hypothesis of Theorem 2.2 may be discarded provided that one modifies the maximal function appropriately. Moreover, this modified maximal function can be used to deduce almost everywhere pointwise convergence for the semigroup. To be precise, assume the setting introduced prior to the statement of Theorem 2.2. For each positive number $s$ and each function $f$ in $E$, define the maximal function $M_{\theta'}^s f$ by

$$(M_{\theta'}^s f)(\omega) = \sup\{\left\|e^{-zs} (T_z f)(\omega)\right\|_X : z \in S_{\theta'}\} \qquad \forall \omega \in \Omega.$$

**Corollary 2.4.** *Suppose $E$, $A$ and $\{T_z : z \in S_\theta^0\}$ are as in the setting of Theorem 2.2 but that $A$ is not injective. If $0 < \alpha < 1$, $s > 0$, $f \in D(A^\alpha)$ and $0 \leq \theta' < \theta$ then $M_{\theta'}^s f \in L^p(\Omega)$ and there is a constant $C(A, \alpha, \theta', s)$ such that*

$$\begin{aligned} &\|M_{\theta'}^s f\|_{L^p(\Omega)} \\ &\qquad \leq C(A, \alpha, \theta', s) \left( \|(sI + A)^\alpha f\|_{L^p(\Omega;X)} + \left\|(sI + A)^{-\alpha} f\right\|_{L^p(\Omega;X)} \right). \end{aligned}$$

*Moreover, for $\mu$-almost all $\omega$ in $\Omega$, $T_z f(\omega) \to f(\omega)$ as $z \to 0$ in $S_{\theta'}$.*

The proof of the corollary is given at the end of Section 5.

## 3. Functional calculus calculus for sectorial operators

In this section we summarise for use in Section 4 a few pertinent facts about the holomorphic functional calculus for sectorial operators.

**Definition 3.1.** Suppose that $0 \leq \vartheta < \pi$ and that $E$ is any Banach space. We say that an operator $A$ in $E$ is *sectorial of type* $\vartheta$ if $A$ is closed, $\sigma(A) \subseteq S_\vartheta$ and for each $\eta$ in $(\vartheta, \pi)$ there exists a constant $C_\eta$ such that

$$\left\|(\zeta I - A)^{-1}\right\| \leq C_\eta |\zeta|^{-1} \qquad \forall \zeta \in \mathbb{C} \setminus S_\eta. \tag{3.1}$$

We recall the following important characterisation of generators of bounded holomorphic semigroups. Details may be found in [Da] or [RS2].

**Theorem 3.2.** *Suppose that $E$ is a Banach space and $0 < \theta < \pi/2$. If $\{T_z : z \in S_\theta^0\}$ is a bounded holomorphic semigroup on $E$ with generator $-A$, then $A$ is a densely defined sectorial operator in $E$ of type $\pi/2 - \theta$. Conversely, if $A$ is a sectorial operator in $E$ of type $\pi/2 - \theta$ then $-A$ is the generator of a bounded holomorphic semigroup $\{T_z : z \in S_\theta^0\}$ on $E$.*

We now describe a holomorphic functional calculus for sectorial operators. Suppose that $0 < \vartheta < \nu < \pi$. Let $\psi$ denote the complex-valued function defined on $\mathbb{C}$ by

$$\psi(\zeta) = \zeta/(1+\zeta)^2 \qquad \forall \zeta \in \mathbb{C}.$$

Denote by $H(S_\nu^0)$ the space of all holomorphic functions on $S_\nu^0$. Following the notation of [CDMY], we define the following subspaces of $H(S_\nu^0)$:

$$H^\infty(S_\nu^0) = \bigl\{ f \in H(S_\nu^0) : \sup_{\zeta \in S_\nu^0} |f(\zeta)| < \infty \bigr\},$$

$$\Psi(S_\nu^0) = \bigl\{ f \in H(S_\nu^0) : f\psi^{-s} \in H^\infty(S_\nu^0) \text{ for some } s > 0 \bigr\}$$

and

$$\mathscr{F}(S_\nu^0) = \bigl\{ f \in H(S_\nu^0) : f\psi^{s} \in H^\infty(S_\nu^0) \text{ for some } s > 0 \bigr\}.$$

Note that

$$\Psi(S_\nu^0) \subset H^\infty(S_\nu^0) \subset \mathscr{F}(S_\nu^0) \subset H(S_\nu^0).$$

It is well known (see [CDMY, Section 2] and [Ha, Chapter 2]) that if $A$ is an injective sectorial operator of type $\vartheta$ on a Banach space $E$, then $A$ has an $\mathscr{F}(S_\nu^0)$ functional calculus. Moreover, if $f \in \Psi(S_\nu^0)$ then $f(A)$, defined by the contour integral

$$f(A) = \frac{1}{2\pi i} \int_\gamma (\zeta I - A)^{-1} f(\zeta)\, \mathrm{d}\zeta, \tag{3.2}$$

is a bounded operator on $E$. Here the integral converges absolutely in the uniform topology and the contour $\gamma$ is given by

$$\gamma(t) = \begin{cases} -te^{i\eta} & \text{if } -\infty < t \le 0 \\ te^{-i\eta} & \text{if } 0 < t < \infty, \end{cases}$$

where $\eta$ is any angle strictly between $\vartheta$ and $\nu$. It can be shown that the definition of $f(A)$ is independent of the choice of angle $\eta$ in this range.

*Remark* 3.3. The functional calculus defined above allows one to define fractional powers for injective sectorial operators, and in particular for injective generators of holomorphic semigroups. If $0 < |\alpha| < 1$ and $\zeta \in S_\nu$, then we define the fractional power $\zeta^\alpha$ by

$$\zeta^\alpha = \exp(\alpha \ln|\zeta| + i\alpha \arg \zeta).$$

Note that the function $\zeta \mapsto \zeta^\alpha$ belongs to $\mathscr{F}(S_\nu^0)$. Thus if $A$ has an $\mathscr{F}(S_\nu^0)$ functional calculus then the operator $A^\alpha$ may be defined by $A^\alpha = g(A)$, where $g(\zeta) = \zeta^\alpha$. The fractional powers of $A$ (and of the sectorial operator $sI + A$ for positive $s$) have the following properties:

(1) $D(A) \subset D(A^\beta) \subset D(A^\alpha)$ whenever $0 < \alpha < \beta < 1$,
(2) $D(A^\alpha) = D((sI + A)^\alpha)$ whenever $0 < \alpha < 1$ and $s > 0$, and
(3) if $A$ is invertible and $0 < \alpha < 1$ then $A^{-\alpha}$ is bounded.

See [Ha, Chapter 3] for further details.

## 4. A representation for the semigroup

Suppose that $t \in \mathbb{R}$, $0 < \alpha < 1$, $\varphi \in \mathbb{R}$, $\zeta \in \mathbb{C}$ and $|\arg(e^{i\varphi}\zeta)| < \pi/2$. By Fourier inversion,

$$\begin{aligned} e^{-|t|e^{i\varphi}\zeta} &= \frac{1}{\pi} \int_{\mathbb{R}} \frac{e^{i\varphi}\zeta}{(e^{i\varphi}\zeta)^2 + u^2}\, e^{itu}\, \mathrm{d}u \\ &= \frac{e^{i\varphi}}{\pi} \left( \int_{|u|<1} e^{itu} F_{u,\varphi}(\zeta)\, \zeta^{-\alpha}\, \mathrm{d}u + \int_{|u|>1} e^{itu} G_{u,\varphi}(\zeta)\, \zeta^{\alpha}\, \mathrm{d}u \right) \end{aligned}$$

where $0 < \alpha < 1$,

$$F_{u,\varphi}(\zeta) = \frac{\zeta^{1+\alpha}}{(e^{i\varphi}\zeta)^2 + u^2} \qquad \text{and} \qquad G_{u,\varphi}(\zeta) = \frac{\zeta^{1-\alpha}}{(e^{i\varphi}\zeta)^2 + u^2}\,.$$

This observation and the $\mathscr{F}(S_\nu^0)$ functional calculus leads to the following lemma.

**Lemma 4.1.** *Suppose that $A$ and $\theta$ are as in the hypothesis of Theorem 2.2. If $0 < \alpha < 1$, $f \in D(A^\alpha)$ and $z \in S_\theta$ then*

$$T_z f = \frac{e^{i\varphi}}{\pi} \int_{|u|<1} e^{itu} F_{u,\varphi}(A) A^{-\alpha} f \,\mathrm{d}u + \frac{e^{i\varphi}}{\pi} \int_{|u|>1} e^{itu} G_{u,\varphi}(A) A^{\alpha} f \,\mathrm{d}u,$$

*where $t = |z|$ and $\varphi = \arg(z)$.*

The proofs of both lemmata in this section make frequent use of the following fact. If $|\phi| < \pi/2$ then

$$\sup\left\{ \frac{t^2+u^2}{|(te^{i\phi})^2 + u^2|} : t > 0, u > 0 \right\} = \sec\phi.$$

This may be deduced using planar trigonometry. We turn now to the proof of Lemma 4.1.

*Proof.* Suppose that $z = te^{i\varphi} \in S_\theta^0$ and choose $\nu$ such that $\pi/2 - \theta < \nu < \pi/2 - |\varphi|$. By the hypotheses on $A$ and Theorem 3.2, $A$ has an $\mathscr{F}(S_\nu^0)$ functional calculus. If $\zeta \in S_\nu^0$ then

$$|F_{u,\varphi}(\zeta)| \le \sec(\varphi + \nu)\, \frac{|\zeta|^{1+\alpha}}{|\zeta|^2 + u^2}$$

and hence $F_{u,\varphi} \in \Psi(S_\nu^0)$ for all nonzero $u$ in $\mathbb{R}$. In fact, if

$$\tilde{F}_{z,\varepsilon}(\zeta) = \int_{\varepsilon<|u|<1} e^{itu} F_{u,\varphi}(\zeta)\,\mathrm{d}u \quad \text{and} \quad \tilde{F}_z(\zeta) = \int_{|u|<1} e^{itu} F_{u,\varphi}(\zeta)\,\mathrm{d}u$$

whenever $0 < \varepsilon < 1$ then

$$\begin{aligned} |\tilde{F}_{z,\varepsilon}(\zeta)| &\le 2\sec(\varphi+\nu) \int_\varepsilon^1 \frac{|\zeta|^{1+\alpha}}{|\zeta|^2+u^2}\,\mathrm{d}u \\ &= 2\sec(\varphi+\nu)\,|\zeta|^\alpha \big( \arctan(1/|\zeta|) - \arctan(\varepsilon/|\zeta|) \big). \end{aligned}$$

Hence $\tilde{F}_{z,\varepsilon} \in \Psi(S_\nu^0)$ and $\tilde{F}_{z,\varepsilon}$ converges to $\tilde{F}_z$ uniformly on compact subsets of $S_\nu$ as $\varepsilon \to 0^+$. This convergence implies that $\tilde{F}_z$ is holomorphic in $S_\nu$ and, combining this with the bounds on $\tilde{F}_{z,\epsilon}$, one concludes that $\tilde{F}_z \in \Psi(S_\nu^0)$.

One can similarly show that $G_{u,\varphi}$ and $\tilde{G}_z$, where

$$\tilde{G}_z(\zeta) = \int_{|u|>1} e^{itu} G_{u,\varphi}(\zeta)\,\mathrm{d}u,$$

both belong to $\Psi(S_\nu^0)$.

Finally, since $T_z f = e^{-zA} f$ for all $f$ in $E$ and

$$e^{-z\zeta} = \frac{e^{i\varphi}}{\pi} \Big( \tilde{F}_z(\zeta)\,\zeta^{-\alpha} + \tilde{G}_z(\zeta)\,\zeta^{\alpha} \Big) \qquad \forall \zeta \in S_\nu$$

by Fourier inversion, the lemma follows from the $\mathscr{F}(S_\nu^0)$ functional calculus for $A$. □

For the proof of Theorem 2.2 we require good bounds for $F_{u,\varphi}(A)$ and $G_{u,\varphi}(A)$, which will be expressed in terms of two operators $\Gamma_{u,\eta}$ and $\Delta_{u,\eta}$ defined below.

Suppose that $u$ is a nonzero real number, $\pi/2 - \theta < \eta < \pi$ and $f \in L^p(\Omega; X)$. Recall that the map $\zeta \mapsto (\zeta I - A)^{-1}$ is holomorphic from $\rho(A)$ into the space of bounded linear operators on $E$. Hence the map $t \mapsto F_{u,0}(t)\,\big\|(te^{i\eta}I - A)^{-1} f\big\|_X$ is continuous from $(0,\infty)$ to $L^p(\Omega)$. Therefore, for each finite interval $[a,b]$ in $(0,\infty)$, the integral

$$\int_a^b F_{u,0}(t)\left\|(te^{i\eta}I-A)^{-1}f\right\|_X \,\mathrm{d}t \tag{4.1}$$

can be defined as an $L^p(\Omega)$ limit of Riemann sums. Moreover, the triangle inequality and resolvent bounds (3.1) for $A$ show that

$$\begin{aligned}\left\|\int_a^b F_{u,0}(t)\left\|(te^{i\eta}I-A)^{-1}f\right\|_X \,\mathrm{d}t\right\|_{L^p(\Omega)} &\le C_\eta \left\|f\right\|_{L^p(\Omega;X)}\int_a^b \frac{t^\alpha}{t^2+u^2}\,\mathrm{d}t\\ &\le C_\eta K_{u,\alpha}\left\|f\right\|_{L^p(\Omega;X)}\end{aligned}$$

for some finite constant $K_{u,\alpha}$ independent of the interval $[a,b]$ in $(0,\infty)$. Hence the integral

$$\int_0^\infty F_{u,0}(t)\left\|(te^{i\eta}I-A)^{-1}f\right\|_X \,\mathrm{d}t$$

can be defined as an $L^p(\Omega)$ limit of integrals of the form (4.1) and is itself in $L^p(\Omega)$.

In a similar manner, we define the scalar-valued functions $\Gamma_{u,\eta}f$ and $\Delta_{u,\eta}f$ by the contour integrals

$$\Gamma_{u,\eta}f=\frac{1}{2\pi}\int_0^\infty F_{u,0}(t)\left(\left\|(te^{i\eta}I-A)^{-1}f\right\|_X+\left\|(te^{-i\eta}I-A)^{-1}f\right\|_X\right)\,\mathrm{d}t$$

and

$$\Delta_{u,\eta}f=\frac{1}{2\pi}\int_0^\infty G_{u,0}(t)\left(\left\|(te^{i\eta}I-A)^{-1}f\right\|_X+\left\|(te^{-i\eta}I-A)^{-1}f\right\|_X\right)\,\mathrm{d}t.$$

By definition, both functions belong to $L^p(\Omega)$.

**Lemma 4.2.** *Suppose that* $0<\theta'<\pi/2-\eta<\theta$*,* $f\in L^p(\Omega;X)$ *and* $u$ *is a nonzero real number. Then*

$$\left\|(F_{u,\varphi}(A)f)(\omega)\right\|_X \le \sec(\theta'+\eta)(\Gamma_{u,\eta}f)(\omega)$$

*and*

$$\left\|(G_{u,\varphi}(A)f)(\omega)\right\|_X \le \sec(\theta'+\eta)(\Delta_{u,\eta}f)(\omega)$$

*for* $\mu$*-almost every* $\omega$ *in* $\Omega$ *and for all* $\varphi$ *in* $[-\theta',\theta']$*. Moreover,*

$$\left\|\Gamma_{u,\eta}f\right\|_{L^p(\Omega)} \le \frac{C_\eta}{2}\sec(\alpha\pi/2)|u|^{-1+\alpha}$$

*and*

$$\left\|\Delta_{u,\eta}f\right\|_{L^p(\Omega)} \le \frac{C_\eta}{2}\sec(\alpha\pi/2)|u|^{-1-\alpha},$$

*where* $C_\eta$ *is the constant appearing in the resolvent bound (3.1) for* $A$*.*

*Proof.* If $\nu$ is chosen such that $\pi/2-\theta<\eta<\nu<\pi/2-\theta'$ then $F_{u,\varphi}\in\Psi(S_\nu^0)$. Hence the function $F_{u,\varphi}(A)f$ can be represented as a contour integral of the form (3.2). Now if $\{g_\varepsilon\}$ is a convergent net in $L^p(\Omega;X)$ then

$$\left\|\lim_\varepsilon g_\varepsilon\right\|_X=\lim_\varepsilon\left\|g_\varepsilon\right\|_X,$$

where convergence on the left is in $L^p(\Omega; X)$ while convergence on the right in is $L^p(\Omega)$. So we may move the $X$-norm through the limit of Riemann sums representing $F_{u,\varphi}(A)f$ to obtain

$$\begin{aligned}\|F_{u,\varphi}(A)f\|_X &\le \frac{1}{2\pi}\int_0^\infty |F_{u,\varphi}(te^{i\eta})| \left\|(te^{i\eta}I - A)^{-1}f\right\|_X \,\mathrm{d}t \\ &\quad + \frac{1}{2\pi}\int_0^\infty |F_{u,\varphi}(te^{-i\eta})| \left\|(te^{-i\eta}I - A)^{-1}f\right\|_X \,\mathrm{d}t \\ &\le \sec(\varphi+\eta)\Gamma_{u,\eta}f \\ &\le \sec(\theta'+\eta)\Gamma_{u,\eta}f.\end{aligned}$$

By resolvent bounds for $A$, we also have

$$\begin{aligned}\|\Gamma_{u,\eta}f\|_{L^p(\Omega)} &\le \frac{C_\eta}{\pi}\|f\|_{L^p(\Omega)}\int_0^\infty \frac{t^{1-\alpha}}{t^2+u^2}\frac{\mathrm{d}t}{t} \\ &\le \frac{C_\eta}{2}\sec(\alpha\pi/2)|u|^{1-\alpha}\|f\|_{L^p(\Omega)}.\end{aligned}$$

The bounds for $\|G_{u,\varphi}(A)f\|_X$ and $\|\Delta_{u,\eta}f\|_{L^p(\Omega)}$ are verified in a similar fashion. $\square$

## 5. Proof of the maximal theorem and its corollary

In this final section we present a proof of Theorem 2.2 and Corollary 2.4.

*Proof of Theorem 2.2.* Assume the setting and hypotheses of Theorem 2.2. Suppose that $f \in D(A^\alpha)$ and define $v : \Omega \times S_{\theta'} \to X$ by

$$v(\omega, z) = T_z f(\omega) \qquad \forall\, (\omega, z) \in \Omega \times S_{\theta'}\,.$$

Note that $M_{\theta'}f \in L^p(\Omega)$ if and only if $v \in L^p(\Omega; L^\infty(S_{\theta'}; X))$, where

$$\|v\|_{L^p(L^\infty)} = \left(\int_\Omega \operatorname*{ess-sup}_{z\in S_{\theta'}} \|v(\omega,z)\|_X^p \,\mathrm{d}\mu(\omega)\right)^{1/p}$$

and where we have written $L^p(L^\infty)$ for $L^p(\Omega; L^\infty(S_{\theta'}; X))$.

Our aim now is to embed $L^p(L^\infty)$ inside the dual of a suitable Banach space $Z$ and to then show that

$$\|v\|_{L^p(L^\infty)} = \sup\left\{|\langle g, v\rangle| \,:\, \|g\|_Z \le 1\right\} \tag{5.1}$$

is finite.

Each operator $T_z$ of the semigroup acts on the closed subspace $E$ of $L^p(\Omega; X)$. Thus in particular $v(\omega, z)$ can be considered as an element of $X^{**}$ for each $\omega$ and $z$. Writing $Y$ for $X^*$, we note that the standard duality theory for Lebesgue-Bochner spaces (see [DU, Chapter IV]) says that $L^\infty(S_{\theta'}; Y^*) \subseteq L^1(S_{\theta'}; Y)^*$ isometrically, and so

$$L^p(\Omega; L^\infty(S_{\theta'}; Y^*)) \subseteq L^p(\Omega; L^1(S_{\theta'}; Y)^*).$$

But if $\frac{1}{p} + \frac{1}{q} = 1$, then

$$L^p(\Omega; L^1(S_{\theta'}; Y)^*) \subseteq L^q(\Omega; L^1(S_{\theta'}; Y))^*$$

isometrically, and hence

$$L^p(\Omega; L^\infty(S_{\theta'}; X)) \subseteq L^p(\Omega; L^\infty(S_{\theta'}; X^{**})) \subseteq L^q(\Omega; L^1(S_{\theta'}; Y))^*. \tag{5.2}$$

As above we shall write $L^q(L^1)$ for $L^q(\Omega; L^1(S_{\theta'}; Y))$. From (5.1) and (5.2) it follows that

$$\begin{aligned}
&\|v\|_{L^p(L^\infty)} \\
&= \sup\left\{ \Big| \int_\Omega \langle g(\omega,\cdot), v(\omega,\cdot)\rangle \, d\mu(\omega) \Big| \, : \, \|g\|_{L^q(L^1)} \le 1 \right\}. \\
&= \sup\left\{ \Big| \int_\Omega \int_{S_{\theta'}} \langle g(\omega,z), v(\omega,z)\rangle_{\langle Y,X\rangle} \, \mathrm{d}z \, \mathrm{d}\mu(\omega) \Big| \, : \, \|g\|_{L^q(L^1)} \le 1 \right\}.
\end{aligned}$$

Suppose then that $g \in L^q(\Omega; L^1(S_{\theta'}; Y))$ and $\|g\|_{L^q(L^1)} \le 1$. Writing $z$ as $te^{i\varphi}$ and using Lemma 4.1, we find that

$$\begin{aligned}
\langle g, v\rangle &= \int_\Omega \int_{S_{\theta'}} \langle g(\omega,z), T_z f(\omega)\rangle_{\langle Y,X\rangle} \, \mathrm{d}z \, \mathrm{d}\mu(\omega) \\
&= \frac{1}{\pi} \int_{|u|<1} \int_\Omega \int_{S_{\theta'}} \langle g(\omega,z), e^{i\varphi} e^{itu} F_{u,\varphi}(A) A^{-\alpha} f(\omega)\rangle_{\langle Y,X\rangle} \, \mathrm{d}z \, \mathrm{d}\mu(\omega) \, \mathrm{d}u \\
&\quad + \frac{1}{\pi} \int_{|u|>1} \int_\Omega \int_{S_{\theta'}} \langle g(\omega,z), e^{i\varphi} e^{itu} G_{u,\varphi}(A) A^{\alpha} f(\omega)\rangle_{\langle Y,X\rangle} \, \mathrm{d}z \, \mathrm{d}\mu(\omega) \, \mathrm{d}u,
\end{aligned}$$

where the use of Fubini's theorem is justified by estimates in Section 4. The modulus of the first of these terms may be estimated using Hölder's inequality and Lemma 4.2 as follows:

$$\begin{aligned}
&\Big| \int_{|u|<1} \int_\Omega \int_{S_{\theta'}} \langle g(\omega,z), e^{i\varphi} e^{itu} F_{u,\varphi}(A) A^{-\alpha} f(\omega)\rangle_{\langle Y,X\rangle} \, \mathrm{d}z \, \mathrm{d}\mu(\omega) \, \mathrm{d}u \Big| \\
&\le \int_{-1}^1 \int_\Omega \int_{S_{\theta'}} \|g(\omega,z)\|_Y \left\| F_{u,\varphi}(A) A^{-\alpha} f(\omega) \right\|_X \, \mathrm{d}z \, \mathrm{d}\mu(\omega) \, \mathrm{d}u \\
&\le \sec(\theta'+\eta) \int_{-1}^1 \int_\Omega \int_{S_{\theta'}} \|g(\omega,z)\|_Y \, (\Gamma_{u,\eta} A^{-\alpha} f)(\omega) \, \mathrm{d}z \, \mathrm{d}\mu(\omega) \, \mathrm{d}u \\
&\le \sec(\theta'+\eta) \int_{-1}^1 \int_\Omega \|g(\omega,\,\cdot\,)\|_{L^1(S_{\theta'};Y)} \, (\Gamma_{u,\eta} A^{-\alpha} f)(\omega) \, \mathrm{d}\mu(\omega) \, \mathrm{d}u \\
&\le \sec(\theta'+\eta) \int_{-1}^1 \|g\|_{L^q(L^1)} \left\| \Gamma_{u,\eta} A^{-\alpha} f \right\|_{L^p(\Omega;X)} \, \mathrm{d}u \\
&\le \frac{C_\eta}{2} \sec(\theta'+\eta) \sec(\alpha\pi/2) \left\| A^{-\alpha} f \right\|_{L^p(\Omega;X)} \int_{-1}^1 |u|^{-1+\alpha} \, \mathrm{d}u \\
&\le C_\eta \sec(\theta'+\eta) \sec(\alpha\pi/2) \, \alpha^{-1} \left\| A^{-\alpha} f \right\|_{L^p(\Omega;X)},
\end{aligned}$$

where $\theta' < \pi/2 - \eta < \theta$. A similar calculation shows that

$$\begin{aligned}
&\Big| \int_{|u|>1} \int_\Omega \int_{S_{\theta'}} \langle g(\omega,z), e^{i\varphi} e^{itu} G_{u,\varphi}(A) A^{\alpha} f(\omega)\rangle_{\langle Y,X\rangle} \, \mathrm{d}z \, \mathrm{d}\mu(\omega) \, \mathrm{d}u \Big| \\
&\quad \le C_\eta \sec(\theta'+\eta) \sec(\alpha\pi/2) \, \alpha^{-1} \|A^\alpha f\|_{L^p(\Omega;X)}.
\end{aligned}$$

It follows therefore that

$$|\langle g, v\rangle| \le \frac{C_\eta \sec(\theta'+\eta)\sec(\alpha\pi/2)}{\pi\alpha} \|f\|_{p,\alpha}$$

and hence

$$\|M_{\theta'} f\|_{L^p(\Omega)} \le \frac{C_\eta \sec(\theta'+\eta)\sec(\alpha\pi/2)}{\pi\alpha} \|f\|_{p,\alpha}.$$

This completes the proof of Theorem 2.2. □

*Proof of Corollary 2.4.* Adopt now the setting and hypotheses of Corollary 2.4. Since $-A$ is the generator of the holomorphic semigroup $\{T_z : z \in S^0_\theta\}$, it is sectorial of type $\pi/2-\theta$. Hence the operator $sI + A$ is has a bounded inverse on $E$ and is also sectorial of type $\pi/2-\theta$. By Theorem 3.2, $-(sI + A)$ is the generator of a holomorphic semigroup, which we denote by $\{e^{-z(sI+A)} : z \in S^0_\theta\}$. Now apply Theorem 2.2 to this semigroup, noting that $e^{-z(sI+A)} = e^{-zs}T_z$ for each $z$ in $S^0_\theta$ and that, by Remark 3.3, it is sufficient that $f \in D(A^\alpha)$. This proves that the maximal function $M^s_{\theta'} f$ is in $L^p(\Omega)$ with the stated bounds.

We now prove almost everywhere pointwise convergence. For convenience, write $z \to 0^+$ as short hand for $z \to 0$ with $z$ in $S_{\theta'}$. Fix positive $s$. Since the semigroup is holomorphic, $T_{z+r}f$ converges pointwise almost everywhere to $T_r f$ as $z \to 0^+$ whenever $r > 0$ (see [St, p. 72]). Hence, for almost every $\omega$ in $\Omega$,

$$\begin{aligned}
&\limsup_{z\to 0^+} \|(T_z f)(\omega) - f(\omega)\|_X \\
&\quad\le \limsup_{z\to 0^+} |e^{sz}| \left\|(e^{-sz}T_z(f - T_r f)(\omega)\right\|_X + \|(T_r f)(\omega) - f(\omega)\|_X \\
&\qquad + \limsup_{z\to 0^+} \|(T_{z+r} f)(\omega) - T_r f(\omega)\|_X \\
&\quad\le e^s M^s_{\theta'}(f - T_r f)(\omega) + \|T_r f(\omega) - f(\omega)\|_X\,.
\end{aligned}$$

Now, given positive $\epsilon$, choose positive $r$ such that $\|f - T_r f\|_{L^p(\Omega;X)} < \epsilon$ and $\|(sI + A)^\alpha f - T_r(sI + A)^\alpha f\|_{L^p(\Omega;X)} < \epsilon$. By the boundedness of the maximal function and the operator $(sI + A)^{-\alpha}$, there exist constants $C$ and $c$, depending only on $A$, $\alpha$, $\theta'$ and $s$, such that

$$\begin{aligned}
&\left\|\limsup_{z\to 0^+} \|T_z f - f\|_X\right\|_{L^p(\Omega)} \\
&\quad\le C\left(\|(sI + A)^\alpha(f - T_r f)\|_{L^p(\Omega;X)} + \left\|(sI + A)^{-\alpha}(f - T_r f)\right\|_{L^p(\Omega;X)}\right) \\
&\qquad + \|T_r f - f\|_{L^p(\Omega;X)} \\
&\quad< C\left(\|(sI + A)^\alpha f - T_r(sI + A)^\alpha f\|_{L^p(\Omega;X)} + c\,\|f - T_r f\|_{L^p(\Omega;X)}\right) + \epsilon \\
&\quad< (C + cC + 1)\epsilon.
\end{aligned}$$

Here we have used the fact that $T_r$ and $(sI + A)^\alpha$ commute on $D(A^\alpha)$ by the functional calculus (see [Ha, Theorem 1.3.2]). It follows that

$$\limsup_{z\to 0^+} \|(T_z f)(\omega) - f(\omega)\|_X = 0$$

for almost all $\omega$ and hence that $T_z f$ converges pointwise almost everywhere to $f$ as $z \to 0^+$. This completes the proof of Corollary 2.4. □

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Gordon Blower, Department of Mathematics and Statistics, Lancaster University, Lancaster LA1 4YF, United Kingdom.
*E-mail address*: g.blower@lancaster.ac.uk

Ian Doust, School of Mathematics and Statistics, UNSW Sydney, NSW, 2052, Australia.
*E-mail address*: i.doust@unsw.edu.au

Robert Taggart, Mathematical Sciences Institute, The Australian National University, ACT 0200, Australia
*E-mail address*: robert.taggart@anu.edu.au