CALDERÓN INVERSE PROBLEM FOR THE SCHRÖDINGER OPERATOR ON RIEMANN SURFACES

COLIN GUILLARMOU AND LEO TZOU

Abstract. On a fixed smooth compact Riemann surface with boundary (M_0, g), we show that the Cauchy data space (or Dirichlet-to-Neumann map $\mathcal{N}$) of the Schrödinger operator $\Delta + V$ with $V \in C^\infty(M_0)$ determines uniquely the potential V.

1. Introduction

The problem of determining the potential in the Schrödinger operator by boundary measurement goes back to Calderón [7]. Mathematically, it amounts to ask if one can detect some data from boundary measurement in a domain (or manifold) Ω with boundary. The typical model to have in mind is the Schrödinger operator $P = \Delta_g + V$ where g is a metric and V a potential, then we define the Cauchy data space by

$$\mathcal{C} := \{(u|_{\partial\Omega}, \partial_n u|_{\partial\Omega}) \in C^\infty(\partial\Omega) \times C^\infty(\partial\Omega)\,;\, u \in \ker P\}$$

where ∂_n is the interior pointing normal vector field to $\partial\Omega$.

The first natural question is the following *full data* inverse problem: does the Cauchy data space determine uniquely the metric g and/or the potential V? In a sense, the most satisfying known results are when the domain $\Omega \subset \mathbb{R}^n$ is already known and g is the Euclidean metric, then the recovery of V has been proved in dimension $n > 2$ by Sylvester-Uhlmann [19] and very recently in dimension 2 by Bukgheim [5]. A related question is the conductivity problem which consists in taking $V = 0$ and replacing Δ_g by $-\mathrm{div}\sigma\nabla$ where σ is a field of positive definite symmetric matrices. An elementary observation shows that the problem of recovering an sufficiently smooth isotropic conductivity (i.e. $\sigma = \sigma_0\mathrm{Id}$ for a function σ_0) is contained in the problem above of recovering a potential V. For domain of $\mathbb{R}^2$, Nachman [17] used the $\bar{\partial}$ techniques to show that the Cauchy data space determines the conductivity. Recently a new approach developed by Astala and Päivärinta in [2] improved this result to assuming that the conductivity is only a L^∞ scalar function. This was later generalized to L^∞ anisotropic conductivities by Astala-Lassas-Päivärinta in [3]. We notice that there still are rather few results in the direction of recovering the Riemannian manifold (Ω, g) when $V = 0$, for instance the surface case by Lassas-Uhlmann [16] (see also [4, 12]), the real-analytic manifold case by Lassas-Taylor-Uhlmann [15] (see also [10] for the Einstein case), the case of manifolds admitting limiting Carleman weights and in a same conformal class by Dos Santos Ferreira-Kenig-Salo-Uhlmann [8].

Date: Received 9 November 2009 / Accepted 13 April 2010.

2010 *Mathematics Subject Classification.* 35R30.

Key words and phrases. Calderon problem, Dirichlet-to-Neumann operator.

The second natural, but harder, problem is the *partial data* inverse problem: if Γ_1 and Γ_2 are open subsets of $\partial\Omega$, does the partial Cauchy data space

$$\mathcal{C}_{\Gamma_1,\Gamma_2} := \{(u_{\partial\Omega}, \partial_n u|_{\Gamma_1}) \in C^\infty(\partial\Omega) \times C^\infty(\Gamma_1)\,;\, u \in \ker P;\, u|_{\partial\Omega} \in C_0^\infty(\Gamma_2)\}$$

determine the domain Ω, the metric, the potential? For a fixed domain of $\mathbb{R}^n$, the recovery of the potential if $n > 2$ with partial data measurements was initiated by Bukhgeim-Uhlmann [6] and later improved by Kenig-Sjöstrand-Uhlmann [14] to the case where Γ_1 and Γ_2 are respectively open subsets of the 'front' and 'back' ends of the domain. We refer the reader to the references for a more precise formulation of the problem. In dimension 2, the recent works of Imanuvilov-Uhlmann-Yamamoto [13] solves the problem for fixed domains of $\mathbb{R}^2$ in the case when $\Gamma_1 = \Gamma_2$.

In this work, we address the same question when the background domain is a known Riemann surface with boundary. We prove the following recovery result under full data measurement:

Theorem 1.1. *Let (M_0, g) be a smooth compact Riemann surface with boundary and let Δ_g be its positive Laplacian. Let $V_1, V_2 \in C^\infty(M_0)$ be two real potentials and let $\mathcal{C}_1, \mathcal{C}_2$ be the respective Cauchy data spaces. If $\mathcal{C}_1 = \mathcal{C}_2$ then $V_1 = V_2$.*

Notice that when $\Delta_g + V_i$ do not have L^2 eigenvalues for the Dirichlet condition, the statement above can be given in terms of Dirichlet-to-Neumann operators. Since $\Delta_{\hat{g}} = e^{-2\varphi}\Delta_g$ when $\hat{g} = e^{2\varphi}g$ for some function φ, it is clear that in the statement in Theorem 1.1, we only need to fix the conformal class of g instead of the metric g (or equivalently to fix the complex structure on M). Observe also that Theorem 1.1 implies that, for a fixed Riemann surface with boundary (M_0, g), the Dirichlet-to-Neumann map for the operator $u \to -\mathrm{div}_g(\gamma\nabla^g u)$ determines the isotropic conductivity γ if $\gamma \in C^\infty(M_0)$ in the sense that two conductivities giving rise to the same Dirichlet-to-Neumann are equal. This is a standard observation by transforming the conductivity problem to a potential problem with potential $V := (\Delta_g \gamma^{\frac{1}{2}})/\gamma^{\frac{1}{2}}$. So our result also extends that of Henkin-Michel [12] in the case of isotropic conductivities.

The method to reconstruct the potential follows [5, 13] and is based on the construction of special complex geometric optic solutions of $(\Delta_g + V)u = 0$, more precisely solutions of the form $u = e^{\Phi/h}(a + r(h))$ where $h > 0$ is a small parameter, Φ and a are holomorphic functions on (M, g) and $r(h)$ is an error term small as $h \to 0$. The idea of [5] to reconstruct $V(p)$ for $p \in M$ is to take Φ with a non-degenerate critical point at p and then use stationary phase as $h \to 0$. In our setting, one of our main contribution is the construction of the holomorphic Carleman weights Φ which is quite a lot more complicated since we are working on a Riemann surface instead of a domain of $\mathbb{C}$. We also need to prove a Carleman estimate on the surface for this degenerate weight.

In [11], we actually prove a much more general result: on a Riemann surface, we show that we can identify a $C^{1,\alpha}$ potential from partial data measurement. Since the proof in the present paper is quite simple compared to the partial data case, we thought that it still has some interest; for instance, we believe it is easier than the proof of Henkin-Michel [12] and allow to deal with a more general assumption (potential instead of isotropic conductivity). The smoothness assumption in the paper is only taken for simplicity of exposition, it will be clear to the reader that this can be relaxed.

2. Harmonic and Holomorphic Morse Functions on a Riemann Surface

2.1. **Riemann surfaces.** Let (M_0, g_0) be a compact connected smooth Riemannian surface with boundary ∂M_0. The surface M_0 can be considered as a subset of a compact Riemannian surface (M, g), for instance by taking the double of M_0 and extending smoothly the metric g_0 to M. The conformal class of g on the closed surface M induces a structure of closed Riemann surface, i.e. a closed surface equipped with a complex structure via holomorphic charts $z_\alpha : U_\alpha \to \mathbb{C}$. The Hodge star operator $\star$ acts on the cotangent bundle T^*M, its eigenvalues are $\pm i$ and the respective eigenspace $T^*_{1,0}M := \ker(\star + i\mathrm{Id})$ and $T^*_{0,1}M := \ker(\star - i\mathrm{Id})$ are sub-bundle of the complexified cotangent bundle $\mathbb{C}T^*M$ and the splitting $\mathbb{C}T^*M = T^*_{1,0}M \oplus T^*_{0,1}M$ holds as complex vector spaces. Since $\star$ is conformally invariant on 1-forms on M, the complex structure depends only on the conformal class of g. In holomorphic coordinates $z = x + iy$ in a chart U_α, one has $\star(udx + vdy) = -vdx + udy$ and

$$T^*_{1,0}M|_{U_\alpha} \simeq \mathbb{C}dz, \quad T^*_{0,1}M|_{U_\alpha} \simeq \mathbb{C}d\bar{z}$$

where $dz = dx + idy$ and $d\bar{z} = dx - idy$. We define the natural projections induced by the splitting of $\mathbb{C}T^*M$

$$\pi_{1,0} : \mathbb{C}T^*M \to T^*_{1,0}M, \quad \pi_{0,1} : \mathbb{C}T^*M \to T^*_{0,1}M.$$

The exterior derivative d defines the De Rham complex $0 \to \Lambda^0 \to \Lambda^1 \to \Lambda^2 \to 0$ where $\Lambda^k := \Lambda^k T^*M$ denotes the real bundle of k-forms on M. Let us denote $\mathbb{C}\Lambda^k$ the complexification of Λ^k, then the ∂ and $\bar{\partial}$ operators can be defined as differential operators $\partial : \mathbb{C}\Lambda^0 \to T^*_{1,0}M$ and $\bar{\partial} : \mathbb{C}\Lambda_0 \to T^*_{0,1}M$ by

$$\partial f := \pi_{1,0} df, \quad \bar{\partial} := \pi_{0,1} df,$$

they satisfy $d = \partial + \bar{\partial}$ and are expressed in holomorphic coordinates by

$$\partial f = \partial_z f \, dz, \quad \bar{\partial} f = \partial_{\bar{z}} f \, d\bar{z}.$$

with $\partial_z := \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\bar{z}} := \frac{1}{2}(\partial_x + i\partial_y)$. Similarly, one can define the ∂ and $\bar{\partial}$ operators from $\mathbb{C}\Lambda^1$ to $\mathbb{C}\Lambda^2$ by setting

$$\partial(\omega_{1,0} + \omega_{0,1}) := d\omega_{0,1}, \quad \bar{\partial}(\omega_{1,0} + \omega_{0,1}) := d\omega_{1,0}$$

if $\omega_{0,1} \in T^*_{0,1}M$ and $\omega_{1,0} \in T^*_{1,0}M$. In coordinates this is simply

$$\partial(udz + vd\bar{z}) = \partial v \wedge d\bar{z}, \quad \bar{\partial}(udz + vd\bar{z}) = \bar{\partial} u \wedge dz.$$

There is a natural operator, the Laplacian acting on functions and defined by

$$\Delta f := -2i \star \bar{\partial}\partial f = d^* d$$

where d^* is the adjoint of d through the metric g and $\star$ is the Hodge star operator mapping Λ^2 to Λ^0 and induced by g as well.

To construct Carleman weights, we will use strongly the Riemann-Roch theorem, so for the convenience of the reader we recall it (see Farkas-Kra [9] for more details). A divisor D on M is an element

$$D = \big((p_1, n_1), \dots, (p_k, n_k)\big) \in (M \times \mathbb{Z})^k, \text{ where } k \in \mathbb{N}$$

which will also be denoted $D = \prod_{i=1}^k p_i^{n_i}$ or $D = \prod_{p \in M} p^{\alpha(p)}$ where $\alpha(p) = 0$ for all p except $\alpha(p_i) = n_i$. The inverse divisor of D is defined to be $D^{-1} := \prod_{p \in M} p^{-\alpha(p)}$ and the degree of the divisor D is defined by $\deg(D) := \sum_{i=1}^k n_i = \sum_{p \in M} \alpha(p)$. A meromorphic function on M is said to have divisor D if $(f) := \prod_{p \in M} p^{\mathrm{ord}(p)}$ is equal to D, where $\mathrm{ord}(p)$ denotes the order of p as a pole or zero of f (with positive sign convention for zeros). Notice that in this case we have $\deg(f) = 0$. For divisors $D' = \prod_{p \in M} p^{\alpha'(p)}$ and $D = \prod_{p \in M} p^{\alpha(p)}$, we say that $D' \geq D$ if $\alpha'(p) \geq \alpha(p)$ for all

$p \in M$. The same exact notions apply for meromorphic 1-forms on M. Then we define for a divisor D

$$r(D) := \dim\{f \text{ meromorphic functions on } M; (f) \geq D\},$$
$$i(D) := \dim\{u \text{ meromorphic 1 forms on } M; (u) \geq D\}.$$

The Riemann-Roch theorem states the following identity: for all divisor D on the closed Riemann surface M of genus g,

$$r(D^{-1}) = i(D) + \deg(D) - g + 1. \tag{1}$$

Notice also that for any divisor D with $\deg(D) > 0$, one has $r(D) = 0$ since $\deg(f) = 0$ for all f meromorphic. By [9, Th. p70], let D be a divisor, then for any non-zero meromorphic 1-form ω on M, one has

$$i(D) = r(D(\omega)^{-1}) \tag{2}$$

which is thus independent of ω.

2.2. **Morse holomorphic functions with prescribed critical points.** The main result of this section is the following

Proposition 2.1. *Let q be a point in $M \setminus M_0$ and let $\mathcal{O} \subset M\backslash\{q\}$ be an open subset with smooth boundary of the punctured Riemann surface $M \setminus \{q\}$ such that $M_0 \subset \mathcal{O}$. Then there exists a dense set of points p in $\mathcal{O}$ such that there exists a Morse holomorphic function f on $\mathcal{O}$ which has a critical point at p.*

We first prove an auxiliary result which states that for any point $p \in M\backslash\{q\}$ one can find a holomorphic function on $M\backslash\{p, q\}$, meromorphic on M and with a pole or zero of any desired order at p.

Lemma 2.1. *Let $p \in M\backslash\{q\}$ and $n \in \mathbb{N}$. Then there exist meromorphic functions h_n, k_n on M such that k_n is holomorphic on $M\backslash\{q, p\}$ with a pole of order n at p and h_n is holomorphic on $M \setminus \{q\}$ with a zero of order n at p.*

Proof. First we claim that there exists $N_0 \in \mathbb{N}$ so that for all $N \geq N_0$, there is meromorphic function on M, holomorphic on $M \setminus \{p\}$ with a pole of order N at p. Indeed, fix a meromorphic 1-form ω, then by (1), we know that for $D := p^N$ with $N > g-1$, then $r(D^{-1}) > 0$. Moreover, if $\deg(D(\omega^{-1})) > 0$, one has $r(D(\omega)^{-1}) = 0$ so we conclude by (2) and (1) that if $N_0 > g - 1$ is taken large enough and $N \geq N_0$ then $r(D^{-1}) = N - g + 1$, which implies that there is a meromorphic function f_N with a pole of order N at p and no other poles. By Riemann-Roch again (1), one has $r(D^{-1}) > 0$ if $D = p^{-\ell}q^k$ with $k, l \in \mathbb{N}$ and $k - \ell > g - 1$. Thus there exists a meromorphic function h on M, holomorphic on $M \setminus \{q\}$, with a pole of order, say $k' \leq k$, at q and a zero of order, say $\ell' \geq \ell$ at p. By possibly taking powers of h, we may assume that p is a zero of h of order say N with $N > N_0$. Then the function $h_n := (f_{N-1}h)^n$ is meromorphic on M, holomorphic on $M \setminus \{q\}$, and with a zero of order n at p. Similarly, the function $k_n := (h_{N-1}f_N)^n$ is meromorphic on M, holomorphic on $M \setminus \{p, q\}$ and with a pole of order n at p. $\square$

Fix $k > 2$ a large integer, we denote by $C^k(\bar{\mathcal{O}})$ the Banach space of C^k real valued functions on $\bar{\mathcal{O}}$. Then the set of harmonic functions on $\mathcal{O}$ which are in the Banach space $C^k(\bar{\mathcal{O}})$ (and smooth in $\mathcal{O}$ by elliptic regularity) is the kernel of the continuous map $\Delta : C^k(\bar{\mathcal{O}}) \to C^{k-2}(\bar{\mathcal{O}})$, and so it is a Banach subspace of $C^k(\bar{\mathcal{O}})$. The set $\mathcal{H} \subset C^k(\bar{\mathcal{O}})$ of harmonic function u in $C^k(\bar{\mathcal{O}})$ such there exists $v \in C^k(\bar{\mathcal{O}})$ harmonic with $u + iv$ holomorphic on $\mathcal{O}$ is a Banach subspace of $\ker \Delta \cap C^k(\bar{\mathcal{O}})$ of finite codimension. Indeed, let $\{\gamma_1, .., \gamma_N\}$ be a homology basis for $\mathcal{O}$, then

$$\mathcal{H} = \ker L, \text{ with } L : \ker \Delta \cap C^k(\bar{\mathcal{O}}) \to \mathbb{C}^N \text{ defined by } L(u) := \Big(\frac{1}{\pi i}\int_{\gamma_j} \partial u\Big)_{j=1,\dots,N}.$$

We now show

Lemma 2.2. *The set of functions $u \in \mathcal{H}$ which are Morse in $\mathcal{O}$ is dense in $\mathcal{H}$ with respect to the $C^k(\bar{\mathcal{O}})$ topology.*

Proof. We use an argument very similar to those used by Uhlenbeck [20]. We start by defining $m : \bar{\mathcal{O}} \times \mathcal{H} \to T^*\mathcal{O}$ by $(p,u) \mapsto (p, du(p)) \in T^*_p\mathcal{O}$. This is clearly a smooth map, linear in the second variable, moreover $m_u := m(.,u) = (\cdot, du(\cdot))$ is Fredholm since $\mathcal{O}$ is finite dimensional. The map u is a Morse functions if and only if m_u is transverse to the zero section, denoted $T^*_0\mathcal{O}$, of $T^*\mathcal{O}$, ie. if

$$\text{Image}(D_p m_u) + T_{m_u(p)}(T^*_0\mathcal{O}) = T_{m_u(p)}(T^*\mathcal{O}), \quad \forall p \in \mathcal{O} \text{ such that } m_u(p) = (p,0).$$

which is equivalent to the fact that the Hessian of u at critical points is nondegenerate (see for instance Lemma 2.8 of [20]). We recall the following transversality theorem ([20, Th.2] or [1, 18]):

Theorem 2.3. *Let $m : X \times \mathcal{H} \to W$ be a C^k map, where X, $\mathcal{H}$, and W are separable Banach manifolds with W and X of finite dimension. Let $W' \subset W$ be a submanifold such that $k > \max(1, \dim X - \dim V + \dim V')$. If m is transverse to W' then the set*

$$\{u \in \mathcal{H}; m_u \text{ is transverse to } W'\}$$

is dense in $\mathcal{H}$, more precisely it is a set of second category.

We want to apply it with $X := \mathcal{O}$, $W := T^*\mathcal{O}$ and $W' := T^*_0\mathcal{O}$, and the map m is defined above. We have thus proved our Lemma if one can show that m is transverse to W'. Let (p,u) such that $m(p,u) = (p,0) \in V'$. Then identifying $T_{(p,0)}(T^*\mathcal{O})$ with $T_p\mathcal{O} \oplus T^*_p\mathcal{O}$, one has

$$Dm_{(p,u)}(z,v) = (z, dv(p) + \text{Hess}_p(u)z)$$

where $\text{Hess}_p u$ is the Hessian of u at the point p, viewed as a linear map from $T_p\mathcal{O}$ to $T^*_p\mathcal{O}$. To prove that m is transverse to W' we need to show that $(z,v) \to (z, dv(p) + \text{Hess}_p(u)z)$ is onto from $T_p\mathcal{O} \oplus \mathcal{H}$ to $T_p\mathcal{O} \oplus T^*_p\mathcal{O}$, which is realized for instance if the map $v \to dv(p)$ from $\mathcal{H}$ to $T^*_p\mathcal{O}$ is onto. But from Lemma 2.1, we know that there exists a holomorphic function v on $M \setminus \{q\}$ (thus on $\bar{\mathcal{O}}$) such that $v(p) = 0$ and $dv(p) \neq 0$ as a linear map $T_p\mathcal{O} \to \mathbb{C}$, we can then take its real and imaginary parts v_1 and v_2, both are real valued harmonic smooth function on $\bar{\mathcal{O}}$ thus in $\mathcal{H}$, and $dv_1(p)$ and $dv_2(p)$ are linearly independent in $T^*_p\mathcal{O}$ by the Cauchy-Riemann equation $\bar{\partial} v = 0$. This shows our claim and ends the proof by using Theorem 2.3. □

Proof of Proposition 2.1 Let p be a point of $\mathcal{O}$ and let u be a holomorphic function with a nondegenerate critical point at p, the existence is insured by Lemma 2.1. By Lemma 2.2, there exist Morse holomorphic functions $(u_j)_{j\in\mathbb{N}}$ such that $u_j \to u$ in $C^k(\bar{\mathcal{O}}, \mathbb{C})$ for any fixed k large. Let $\epsilon > 0$ small and let $U \subset \mathcal{O}$ be a neighbourhood containing p and no other critical points of u, and with boundary a smooth circle of radius ϵ. In complex local coordinates near p, we can consider ∂u and ∂u_j as holomorphic functions on an open set of $\mathbb{C}$. Then by Rouche's theorem, it is clear that ∂u_j has precisely one zero in U if j is large enough. This completes the proof.□

3. Carleman Estimate for Harmonic Weights with Critical Points

In this section, we prove a Carleman estimate using harmonic weight with nondegenerate critical points, in way similar to [13]:

Proposition 3.1. *Let (M, g) be a Riemann surface with boundary, with $\bar{M} := M \cup \partial\bar{M}$, and let $\varphi : \bar{M} \to \mathbb{R}$ be a harmonic function with non-degenerate critical points. Then for all $V \in L^\infty$ there exists an $h_0 > 0$ such that for all $h < h_0$ and $u \in C_0^\infty(M)$, we have*

$$\frac{1}{h}\|u\|^2 + \frac{1}{h^2}\|u|d\varphi|\|^2 + \|du\|^2 \leq C\|e^{-\varphi/h}(\Delta_g + V)e^{\varphi/h}u\|^2 \tag{3}$$

Proof. We start by modifying the weight as follows: if $\varphi_0 = \varphi : \bar{M} \to \mathbb{R}$ is a real valued harmonic Morse function with critical points $\{p_1, \dots, p_N\}$ in the interior M, we let $\varphi_j : \bar{M} \to \mathbb{R}$ be harmonic functions such that p_j is not a critical point of φ_j for $j = 1, \dots, N$, their existence is insured by Lemma 2.1. For all $\epsilon > 0$ we define the convexified weight by $\varphi_\epsilon := \varphi - \frac{h}{2\epsilon}(\sum_{j=0}^N |\varphi_j|^2)$.

Lemma 3.1. *Let Ω be an open chart of M and $\varphi_\epsilon : \Omega \to \mathbb{R}$ be as above. Then for all $u \in C_0^\infty(\Omega)$ and $h > 0$ small enough, the following estimate holds:*

$$\frac{C}{\epsilon}\|u\|^2 \leq \|e^{-\varphi_\epsilon/h}\bar{\partial}e^{\varphi_\epsilon/h}u\|^2 \tag{4}$$

Proof We use complex coordinates $z = x + iy$ in the chart where u is supported and then integrate by parts so that we have

$$\begin{aligned}\|e^{-\varphi_\epsilon/h}\bar{\partial}e^{\varphi_\epsilon/h}u\|^2 &= \frac{1}{4}\Big(\|(\partial_x + \frac{i\partial_y\varphi_\epsilon}{h})u + (i\partial_y + \frac{\partial_x\varphi_\epsilon}{h})u\|^2\Big)\\ &= \frac{1}{4}\Big(\|(\partial_x + \frac{i\partial_y\varphi_\epsilon}{h})u\|^2 + \|(i\partial_y + \frac{\partial_x\varphi_\epsilon}{h})u\|^2 + \frac{1}{h}\langle u\Delta\varphi_\epsilon, u\rangle\Big)\end{aligned}$$

where $\Delta := -(\partial_x^2 + \partial_y^2)$. Then $\langle u\Delta\varphi_\epsilon, u\rangle = \frac{h}{\epsilon}(|d\varphi_0|^2 + |d\varphi_1|^2 + .. + |d\varphi_N|^2)|u|^2$, since φ_j are harmonic, so the proof follows from the fact that $|d\varphi_0|^2 + |d\varphi_1|^2 + .. + |d\varphi_N|^2$ is uniformly bounded away from zero. □

The main step to go from (4) to (3) is the following lemma which is a slight modification of the proof in [13]:

Lemma 3.2. *With the same assumption as Proposition 3.1 and if Ω is a chart of (M, g) chosen sufficiently small and containing at most one critical point of φ, then we have*

$$\frac{c}{\epsilon}(\frac{1}{h}\|u\|^2 + \frac{1}{h^2}\|u|d\varphi|\|^2 + \frac{1}{h^2}\|u|d\varphi_\epsilon|\|^2 + \|du\|^2) \leq C\|e^{-\varphi_\epsilon/h}\Delta_g e^{\varphi_\epsilon/h}u\|^2 \tag{5}$$

or equivalently,

$$\frac{c}{\epsilon}\Big(\frac{1}{h}\|e^{-\varphi_\epsilon/h}u\|^2 + \frac{1}{h^2}\|e^{-\varphi_\epsilon/h}u|d\varphi|\|^2 + \frac{1}{h^2}\|e^{-\varphi_\epsilon/h}u|d\varphi_\epsilon|\|^2 + \|e^{-\varphi_\epsilon/h}du\|^2\Big) \leq C\|e^{-\varphi_\epsilon/h}\Delta_g u\|^2$$

for all $0 < h \ll \epsilon \ll 1$ and $u \in C_0^\infty(M)$.

Proof. Since, in suitable coordinates (x, y), the metric g is conformal to the Euclidean metric, one has in these coordinates $\Delta_g = -e^{2f}(\partial_x^2 + \partial_y^2) = e^{2f}\Delta$ in the complex coordinate chart $z = x + iy$ for some smooth function f, it suffices to get the estimate (5) for Euclidean norms and Laplacian. Clearly, we can assume $u \in C_0^\infty(M)$ to be real valued without loss of generality. By (4) we have

$$\begin{aligned}\|e^{-\varphi_\epsilon/h}\Delta e^{\varphi_\epsilon/h}u\|^2 &= 4\|e^{-\varphi_\epsilon/h}\bar{\partial}e^{\varphi_\epsilon/h}e^{-\varphi_\epsilon/h}\partial e^{\varphi_\epsilon/h}u\|^2\\ &\geq \frac{c}{\epsilon}\|e^{-\varphi_\epsilon/h}\partial e^{\varphi_\epsilon/h}u\|^2\\ &= \frac{c}{\epsilon}\|\partial u + \frac{\partial\varphi_\epsilon}{h}u\|^2.\end{aligned}$$

Using the fact that u is real valued, we get that

$$\|e^{-\varphi_\epsilon/h}\Delta e^{\varphi_\epsilon/h}u\|^2 \geq \frac{c}{\epsilon}\Big(\|du\|^2 + \frac{1}{h^2}\|u|d\varphi_\epsilon|\|^2 + \frac{2}{h}\langle\partial_x u, u\partial_x\varphi_\epsilon\rangle + \frac{2}{h}\langle\partial_y u, u\partial_y\varphi_\epsilon\rangle\Big)$$

Using the fact that u is real valued, that φ is harmonic and that $\sum_{j=0}^N |d\varphi_j|^2$ is uniformly bounded below, we see that

$$\frac{2}{h}\langle\partial_x u, u\partial_x\varphi_\epsilon\rangle + \frac{2}{h}\langle\partial_y u, u\partial_y\varphi_\epsilon\rangle = \frac{1}{h}\langle u, u\Delta\varphi_\epsilon\rangle \geq \frac{C}{\epsilon}\|u\|^2 \tag{6}$$

for some $C > 0$ and therefore,

$$\|e^{-\varphi_\epsilon/h}\Delta_e e^{\varphi_\epsilon/h}u\|^2 \geq \frac{c}{\epsilon}(\|du\|^2 + \frac{1}{h^2}\|u|d\varphi_\epsilon|\|^2 + \frac{C}{\epsilon}\|u\|^2).$$

If the diameter of the chart Ω is chosen small (its size depending only on $|\mathrm{Hess}\varphi_0|(p)$) with a unique critical point p of φ_0 inside, one can use integration by part and the fact that the critical point is non-degenerate to obtain

$$\|\bar\partial u\|^2 + \frac{1}{h^2}\|u|\partial\varphi_0|\|^2 \geq \frac{1}{h}\left|\int \partial_{\bar z}(u^2)\overline{\partial_z\varphi_0}dxdy\right| \geq \frac{1}{h}\left|\int u^2\,\overline{\partial_z^2\varphi_0}\,dxdy\right| \geq \frac{C'}{h}\|u\|^2 \tag{7}$$

for some $C' > 0$. Clearly the same estimate holds trivially if Ω does not contain critical point of φ_0. Thus, combining with (6), there are positive constants c, c', C'' such that for h small enough (for instance $h \ll \epsilon^2$)

$$\begin{aligned}\|e^{-\varphi_\epsilon/h}\Delta e^{\varphi_\epsilon/h}u\|^2 &\geq \frac{c}{\epsilon}(\|du\|^2 + \frac{1}{h^2}\|u|d\varphi_0|\|^2 - \frac{C''}{\epsilon^2}\|u\|^2)\\ &\geq \frac{c'}{\epsilon}(\|du\|^2 + \frac{1}{h^2}\|u|d\varphi_0|\|^2 + \frac{1}{h}\|u\|^2).\end{aligned}$$

Combining the two above inequalities gives the desired estimate. □

Proof of Proposition 3.1. Using triangular inequality and absorbing the term $||Vu||^2$ into the left hand side of (3), it suffices to prove (3) with Δ_g instead of $\Delta_g + V$. Let $v \in C_0^\infty(M)$, we have by Lemma 3.2 that there exist constants $c, c', C, C' > 0$ such that

$$\begin{aligned}&\frac{c}{\epsilon}(\frac{1}{h}\|e^{-\varphi_\epsilon/h}v\|^2 + \frac{1}{h^2}\|e^{-\varphi_\epsilon/h}v|d\varphi|\|^2 + \frac{1}{h^2}\|e^{-\varphi_\epsilon/h}v|d\varphi_\epsilon|\|^2 + \|e^{-\varphi_\epsilon/h}dv\|^2)\\ &\quad\leq \sum_j \frac{c'}{\epsilon}\Big(\frac{1}{h}\|e^{-\varphi_\epsilon/h}\chi_j v\|^2 + \frac{1}{h^2}\|e^{-\varphi_\epsilon/h}\chi_j v|d\varphi|\|^2\\ &\qquad\qquad + \frac{1}{h^2}\|e^{-\varphi_\epsilon/h}\chi_j v|d\varphi_\epsilon|\|^2 + \|e^{-\varphi_\epsilon/h}d(\chi_j v)\|^2\Big)\\ &\quad\leq \sum_j C\|e^{-\varphi_\epsilon/h}\Delta_g(\chi_j v)\|^2\\ &\quad\leq C'\|e^{-\varphi_\epsilon/h}\Delta_g v\|^2 + C'\|e^{-\varphi_\epsilon/h}v\|^2 + C'\|e^{-\varphi_\epsilon/h}dv\|^2\end{aligned}$$

where $(\chi_j)_j$ is a partition of unity associated to the complex charts on M. Since constants on both sides are independent of ϵ and h, we can take ϵ small enough so that $C'\|e^{-\varphi_\epsilon/h}v\|^2 + C'\|e^{-\varphi_\epsilon/h}dv\|^2$ can be absorbed to the left side. Now set $v = e^{\varphi_\epsilon/h}w$, we have

$$\frac{1}{h}\|w\|^2 + \frac{1}{h^2}\|w|d\varphi|\|^2 + \frac{1}{h^2}\|w|d\varphi_\epsilon|\|^2 + \|dw\|^2 \leq C\|e^{-\varphi_\epsilon/h}\Delta_g e^{\varphi_\epsilon/h}w\|^2$$

Finally, fix $\epsilon > 0$ and set $u = e^{\frac{1}{\epsilon}\sum_{j=0}^N|\varphi_j|^2}w$ and use the fact that $e^{\frac{1}{\epsilon}\sum_{j=0}^N|\varphi_j|^2}$ is independent of h and bounded uniformly away from zero and above, we then obtain the desired estimate for $h \ll \epsilon$. □

4. Complex Geometric Optics on a Riemann Surface

As explained in the Introduction, the method for recovering the potential at a point p is to construct complex geometric optic solutions depending on a small parameter $h > 0$, with phase a Carleman weight (here a Morse holomorphic function), and such that the phase has a non-degenerate critical point at p, in order to apply the stationary phase method.

First consider any continuous extension of V to M, still denoted V for simplicity. Choose $p \in M_0$ such that there exists a Morse holomorphic function $\Phi = \varphi + i\psi$ on $\mathcal{O}$, C^k in $\bar{\mathcal{O}}$, with a critical point at p and where $\mathcal{O}$ is chosen like in first section, ie. such that $M_0 \subset \mathcal{O} \subset M$. Obviously Φ has isolated critical points in $\mathcal{O}$ and thus by reducing slightly $\mathcal{O}$ if necessary, we can assume that Φ has no critical point on its boundary $\partial\bar{\mathcal{O}}$. The purpose of this section is to construct solutions u on $\mathcal{O}$ of $(\Delta + V)u = 0$ of the form

$$u = e^{\Phi/h}(a + r_1 + r_2) \tag{8}$$

for $h > 0$ small, where a is a holomorphic function on $\mathcal{O}$ such that $a(p) \neq 0$ and r_1, r_2 will be reminder terms which are small as $h \to 0$ and have particular properties near the critical points of Φ. More precisely, r_2 will be a $O_{L^2}(h^{3/2-\epsilon})$ for all $\epsilon > 0$ and r_1 will be a $O_{L^2}(h^{1-\epsilon})$ but with an explicit expression, which can be used to obtain sufficient informations from the stationary phase method.

4.1. Construction of r_1.

For all $\epsilon > 0$ we want to construct r_1 which satisfies

$$e^{-\Phi/h}(\Delta_g + V)e^{\Phi/h}(a + r_1) = O(h^{1-\epsilon})$$

in L^2 and $\|r_1\|_{L^2} = O(h^{1-\epsilon})$. We let G be the Green operator of the Laplacian on the smooth surface with boundary $\bar{\mathcal{O}}$ with Dirichlet condition, so that $\Delta_g G = \mathrm{Id}$ on $L^2(\mathcal{O})$. In particular this implies that $\bar{\partial}\partial G = \frac{i}{2}\star^{-1}$ where $\star^{-1}$ is the inverse of $\star$ mapping functions to 2-forms. First, we will search for r_1 satisfying

$$e^{-2i\psi/h}\partial e^{2i\psi/h} r_1 = -\partial G(aV) + \omega + O_{H^1}(h^{1-\epsilon}) \tag{9}$$

with ω a holomorphic 1-form on $\mathcal{O}$ and $\|r_1\|_{L^2} = O(h^{1-\epsilon})$. Indeed, using the fact that Φ is holomorphic we have

$$\begin{aligned} e^{-\Phi/h}\Delta_g e^{\Phi/h} &= -2i \star \bar{\partial} e^{-\Phi/h}\partial e^{\Phi/h} \\ &= -2i \star \bar{\partial} e^{-\frac{1}{h}(\Phi - \bar{\Phi})}\partial e^{\frac{1}{h}(\Phi - \bar{\Phi})} \\ &= -2i \star \bar{\partial} e^{-2i\psi/h}\partial e^{2i\psi/h} \end{aligned}$$

and applying $-2i \star \bar{\partial}$ to (9), this gives

$$e^{-\Phi/h}(\Delta_g + V)e^{\Phi/h} r_1 = -aV + O_{L^2}(h^{1-\epsilon}).$$

The form ω above, will be chosen as a correction term to optimize the use of the stationary phase later, this is why we need the following

Lemma 4.1. *Let $\{p_0, ..., p_N\}$ be finitely many points on $\mathcal{O}$ and let g be a continuous section of $T^*_{1,0}\mathcal{O}$. Then there exists a holomorphic 1-form ω on $\mathcal{O}$ such that $(g - \omega)(p_i) = 0$ for all $i = 0, \ldots, N$.*

Proof. First by Riemann-Roch formula (1), there exists a meromorphic 1-form v on M, holomorphic on $\mathcal{O}$, which has a zero of order greater or equal to 1 at all $p_1, \ldots, p_N$, so using Lemma 2.1, we can multiply it by a meromorphic function f_j on $\mathcal{O}$, holomorphic on $\mathcal{O} \setminus \{p_j\}$, with a pole of order exactly n_j at p_j if n_j is the order of p_j as a zero of v, so that $v_j := f_j v$ is a holomorphic 1-form on $\mathcal{O}$ with no zero at p_j and zeros of order larger or equal to 1 at all other p_k for $k \neq j$. Now since $T^*_{1,0}\mathcal{O}$ is a complex line bundle, there is a complex number $c_j \in \mathbb{C}$ such that $g(p_j) = c_j v_j(p_j)$. Thus it is clear that $\omega = \sum_{j=1}^{N} c_j v_j$ satisfies the claim.$\Box$

With this lemma, we will choose ω to be a smooth holomorphic 1-form on M_0 such that at all critical point p' of Φ in M_0, the form $b := -\partial G(aV) + \omega$ with value in $T^*_{1,0}M_0$ vanishes at p'.

We will construct $r_1 = r_{1,1} + r_{1,2}$ in two steps. First, we will construct $r_{1,1}$ to solve equation (9) locally near critical points of Φ by using coordinate charts. Then we will construct the global correction term $r_{1,2}$ away from critical points.

Let p' be a critical point of Φ and $\mathcal{U}(p')$ be a complex coordinate chart z containing p' but no other critical points of Φ. In local coordinates one has $-\partial G(aV)+\omega = b(z)dz$ for some C^∞ function b vanishing at p'. Let $\chi_1 \in C_0^\infty(\mathcal{U}(p'))$ such that $\chi_1 = 1$ in a neighbourhood of p' and let $\chi \in C_0^\infty(\mathcal{U}(p'))$ with $\chi = 1$ on an open set containing the support of χ_1. Define for $z \in \mathcal{U}(p')$

$$r_{1,1}(z) := \chi(z)e^{-2i\psi/h}R(e^{2i\psi/h}\chi_1 b)(z) \tag{10}$$

where $Rf(z) := (2\pi i)^{-1}\int_{\mathbb{R}^2} \frac{1}{\bar\xi - \bar z} f(\xi,\bar\xi)d\xi \wedge d\bar\xi$ for $f \in L^\infty$ compactly supported is the classical Cauchy-Riemann operator. Extend $r_{1,1}$ trivially outside of $\mathcal{U}(p')$. Then

$$\begin{gathered} e^{-2i\psi/h}\partial(e^{2i\psi/h}r_{1,1}) = \chi_1(-\partial G(aV) + \omega) + \eta \\ \text{with } \eta := e^{-2i\psi/h}R(e^{2i\psi/h}\chi_1 b) \wedge \partial\chi \end{gathered} \tag{11}$$

where the form η makes sense globally on $\mathcal{O}$ since $\partial\chi$ is supported in $\mathcal{U}(p')$. Note that η is a C^∞ form with value in $T^*_{1,0}\mathcal{O}$. Now the support of $\partial\chi$, thus of η, is contained in the complement of the support of χ_1. By stationary phase and the fact that $b = 0$ at all critical points of Φ, one has

$$\|\eta\|_\infty \leq Ch^2 \quad \text{and} \quad \|\Delta\eta\|_\infty \leq C. \tag{12}$$

The term $r_{1,1}$ is supported in $\mathcal{U}(p')$ for a fixed critical point p' and depends on p', let us write it $r_{1,1}^{p'}$ instead, but since our discussion did not depend on the choice of p', we can sum the $r_{1,1}^{p'}$ over the critical points p' to define a term $r_{1,1}$.

Next we define $r_{1,2}$ by the equation

$$r_{1,2}\partial\Phi = h\Big(-\eta + (1-\chi_1)(-\partial G(aV) + \omega)\Big).$$

so that

$$e^{-\Phi/h}\partial e^{\Phi/h}r_{1,2} = \partial r_{1,2} - \eta + (1-\chi_1)(-\partial G(aV) + \omega). \tag{13}$$

There is a well defined C^∞ function $r_{1,2}$ satisfying this equation since both $\partial\Phi$ and the right hand side have values in the bundle $T^*_{1,0}\mathcal{O}$ and moreover the right hand side has support which does not intersect the critical points of Φ.

We now derive the asymptotic properties of $r_{1,1}$ and $r_{1,2}$

Lemma 4.2. *For all $\epsilon > 0$, the following estimates hold*

$$\|r_{1,1}\|_{L^2} \leq Ch^{1-\epsilon}, \quad \|r_{1,2}\|_\infty \leq Ch \quad \text{and} \quad \|\Delta r_{1,2}\|_\infty \leq Ch.$$

Proof. The first estimate is a local result and comes from classical properties of R as proved in Proposition 2.6 of [13]. The ones involving $r_{1,2}$ follow directly from (12). Moreover $r_{1,2} = hv + O_{L^\infty}(h^2)$ for some smooth $v = (1-\chi_1)(-\partial G(aV) + \omega)$ independent of h. □

Lemma 4.3. *With $r_1 := r_{1,1} + r_{1,2}$ constructed above, then for all $\epsilon > 0$*

$$e^{-\Phi/h}(\Delta + V)e^{\Phi/h}(a + r_1) = O_{L^2}(h^{1-\epsilon}).$$

Proof. First, we write

$$e^{-\Phi/h}\partial e^{\Phi/h}r_1 = e^{-2i\psi/h}\partial(e^{2i\psi/h}r_{1,1}) + (\partial + \frac{1}{h}\partial\Phi)r_{1,2}.$$

and by (11) and (13) this implies

$$e^{-\Phi/h}\partial e^{\Phi/h}r_1 = -\partial G(aV) + \omega + \partial r_{1,2}$$

and applying $-2i \star \bar{\partial}$

$$e^{-\Phi/h}\Delta e^{\Phi/h}r_1 = -aV + \Delta r_{1,2}.$$

The proof is complete since we know from Lemma 4.2 that $\|\Delta r_{1,2}\|_\infty \leq Ch$. □

4.2. **Construction of r_2.**

The goal of this section is to complete the construction of the complex geometric optic solutions by the following proposition:

Proposition 4.1. *For all $\epsilon > 0$ there exist solutions to $(\Delta + V)u = 0$ of the form (8) with $r_1 = r_{1,1} + r_{1,2}$ constructed in the previous section and r_2 satisfying $\|r_2\|_{L^2} \leq Ch^{3/2-\epsilon}$*

This is a consequence of the following Lemma (which follows from the Carleman estimate obtained above)

Lemma 4.4. *Let $V \in L^\infty(\mathcal{O})$ and $f \in L^2(\mathcal{O})$. For all $h > 0$ small enough, there exists a solution $v \in L^2$ to the equation*

$$e^{\varphi/h}(\Delta_g + V)e^{-\varphi/h}v = f$$

satisfying

$$\|v\|_{L^2} \leq Ch^{\frac{1}{2}}\|f\|_{L^2}$$

Proof. The proof is the same than Proposition 2.2 of [13], we repeat the argument for the convenience of the reader. Define for all $h > 0$ the real vector space $\mathcal{A} := \{u \in H^1_0(\mathcal{O}); (\Delta_g + V)u \in L^2(\mathcal{O})\}$ equipped with the real scalar product

$$(u, w)_{\mathcal{A}} := \int_{\mathcal{O}} e^{-2\varphi/h}(\Delta_g u + Vu)(\Delta_g w + Vw)dg.$$

By the Carleman estimate of Proposition 3.1, the space $\mathcal{A}$ is a Hilbert space equipped with the scalar product above and so the linear functional $L : w \to \int_{\mathcal{O}} e^{-\varphi/h}fw\,dg$ on $\mathcal{A}$ is continuous and norm bounded by $h^{\frac{1}{2}}||f||_{L^2}$ by Proposition 3.1, and by Riesz theorem there is an element $u \in \mathcal{A}$ such that $(., u)_{\mathcal{A}} = L$ and with norm bounded by the norm of L. It remains to take $v := e^{-\varphi/h}(\Delta_g u + Vu)$ which solves $e^{\varphi/h}(\Delta_g + V)e^{-\varphi/h}v = f$ and which in addition satisfies the desired norm estimate. □

Proof of Proposition 4.1. We note that $(\Delta + V)e^{\Phi/h}(a + r_1 + r_2) = 0$ if and only if

$$e^{-\Phi/h}(\Delta + V)e^{\Phi/h}r_2 = -e^{-\Phi/h}(\Delta + V)e^{\Phi/h}(a + r_1)$$

By Lemma 4.4 one can find such an r_2 which satisfies

$$\|r_2\|_{L^2} \leq Ch^{\frac{1}{2}}\|e^{-\Phi/h}(\Delta + V)e^{\Phi/h}(a + r_1)\|_{L^2} \leq Ch^{3/2-\epsilon}$$

where the last inequality comes from Lemma 4.3. □

5. Identifying the potential

We now assume that $V_1, V_2 \in C^\infty(\bar{M_0})$ are two real valued potentials such that the respective Cauchy data spaces $\mathcal{C}_1, \mathcal{C}_2$ for the operators $\Delta_g + V_1$ and $\Delta_g + V_2$ are equal. Let $p \in M_0$ and $\mathcal{O}$ with $M_0 \subset \mathcal{O} \subset M \setminus \{q\}$ such that, using Proposition 2.1, we can choose a holomorphic Morse function $\Phi = \varphi + i\psi$ on $\mathcal{O}$, C^k in $\bar{\mathcal{O}}$ for some large $k \in \mathbb{N}$, with a critical point at p. By reducing slightly $\mathcal{O}$ if necessary, we can assume that Φ has no critical points on $\partial\bar{\mathcal{O}}$ and finitely many critical points in $\mathcal{O}$.

Proposition 5.1. *If the Cauchy data spaces agree, i.e. if $\mathcal{C}_1 = \mathcal{C}_2$, then $V_1(p) = V_2(p)$.*

Proof. By boundary identifiability (see for example [8]), one has $V_1 = V_2$ on ∂M_0 to second order and therefore we can extend V_1, V_2 to be C^∞ to $\bar{\mathcal{O}}$ such that they agree outside of ∂M_0. Let a be a holomorphic function on $\mathcal{O}$ with $a(p) \neq 0$ and $a(p') = 0$ for all other critical point p' of Φ. The existence is insured by Lemma 2.1 as follows: by Riemann-Roch, we can find a holomorphic function on $M \setminus \{q\}$ such that $a(p') = 0$ for all $p' \neq p$. Either at p this function does not vanish and we have our function a, or there is a zero of order say N, in which case one can multiply it by a meromorphic function on M, holomorphic on $M \setminus \{q, p\}$ with a pole of order exactly N at p (the existence of which is proved in Lemma 2.1). Let u_1 and u_2 be H^2 solutions on $\bar{\mathcal{O}}$ to

$$(\Delta_g + V_j)u_j = 0$$

constructed in Section 4 with Φ for Carleman weight for u_1 and $-\Phi$ for u_2, thus of the form

$$u_1 = e^{\Phi/h}(a + r_1^1 + r_2^1), \quad u_2 = e^{-\Phi/h}(a + r_1^2 + r_2^2)$$

and with boundary value $u_j|_{\partial M_0} = f_j$. Since $\overline{u_2}$ is also a solution, we can write by Green formula

$$\begin{aligned}\int_{M_0} u_1(V_1 - V_2)\overline{u_2}\mathrm{dv}_g &= -\int_{M_0} (\Delta_g u_1.\overline{u_2} - u_1.\Delta_g\overline{u_2})\mathrm{dv}_g \\ &= -\int_{\partial M_0} (\partial_n u_1.\overline{f_2} - f_1.\partial_n\overline{u_2})\mathrm{dv}_g\end{aligned}$$

Since the Cauchy data for $\Delta_g + V_1$ agrees with that of $\Delta_g + V_2$, there exists a solution v of the boundary value problem

$$(\Delta_g + V_2)v = 0, \qquad v|_{\partial M_0} = f_1$$

satisfying $\partial_n v = \partial_n u_1$ on ∂M_0. This implies that

$$\begin{aligned}\int_{M_0} u_1(V_1 - V_2)\overline{u_2}\mathrm{dv}_g &= -\int_{M_0} (\Delta_g u_1.\overline{u_2} - u_1\Delta_g\overline{u_2})\mathrm{dv}_g \\ &= -\int_{\partial M_0} (\partial_n u_1\overline{f_2} - f_1\partial_n\overline{u_2})\mathrm{dv}_g \\ &= -\int_{\partial M_0} (\partial_n v\overline{f_2} - v\partial_n\overline{u_2})\mathrm{dv}_g \\ &= -\int_{M_0} (\Delta_g v.\overline{u_2} - v\Delta_g\overline{u_2})\mathrm{dv}_g \\ &= 0\end{aligned}$$

since $\Delta_g + V_2$ annihilates both v and u_2. Then by using the estimates in Lemma 4.2 and Proposition 4.1 that we have, as $h \to 0$,

$$\int_{M_0} e^{2i\psi/h}|a|^2(V_1 - V_2)\mathrm{dv}_g + \int_{M_0} e^{2i\psi/h}(\bar{a}r_1^1 + a\overline{r_1^2})(V_1 - V_2)dv_g + O(h^{3/2-\epsilon}) = 0$$

for all $\epsilon > 0$. By stationary phase the first term can be developed as follows

$$\int_{M_0} e^{2i\psi/h}|a|^2(V_1-V_2)\mathrm{dv}_g = \int_{\mathcal{O}} e^{2i\psi/h}|a|^2(V_1-V_2)\mathrm{dv}_g = Ch(V_1(p)-V_2(p))+O(h^2)$$

for some $C \neq 0$. Therefore,

$$Ch(V_1(p) - V_2(p)) + \int_{\mathcal{O}} e^{2i\psi/h}(\bar{a}r_1^1 + a\overline{r_1^2})(V_1 - V_2)\mathrm{dv}_g + O(h^{3/2-\epsilon}) = 0.$$

It suffices then to show that

$$\int_{M_0} e^{2i\psi/h}(\bar{a}r_1^1 + a\overline{r_1^2})(V_1 - V_2)\mathrm{dv}_g = o(h).$$

This can be accomplished by the following argument: let us deal with the $\bar{a}r_1^1$ term since the other one is exactly similar, then we localize near critical points and by stationary phase,

$$\int_{\mathcal{O}} e^{2i\psi/h}\bar{a}r_1^1(V_1 - V_2)\mathrm{dv}_g = \sum_{p'\in\mathcal{O}, d\Phi(p')=0} \int_{\mathcal{U}(p')} e^{2i\psi/h}\chi_{p'}\bar{a}r_1^1(V_1 - V_2)\mathrm{dv}_g + O(h^2)$$

where $\mathcal{U}(p')$ is a small neighbourhood of p' that we used to define $r_{1,1}$ in the Subsection 4.1 and $\chi_{p'}$ the smooth cutoff functions supported in that neighbourhood. First we observe that by stationary phase and the fact that $r_{1,2}^1 \in hC^\infty(M_0) + O_{L^\infty}(h^2)$ (proof of Lemma 4.2), we have

$$\begin{aligned}\int_{\mathcal{U}(p')} e^{2i\psi/h}\chi_{p'}\bar{a}r_1^1(V_1 - V_2)\mathrm{dv}_g &= \int_{\mathcal{U}(p')} e^{2i\psi/h}\chi_{p'}\bar{a}(r_{1,1}^1 + r_{1,2}^1)(V_1 - V_2)\mathrm{dv}_g \\ &= \int_{\mathcal{U}(p')} e^{2i\psi/h}\chi_{p'}\bar{a}r_{1,1}^1(V_1 - V_2)\mathrm{dv}_g + O(h^2)\end{aligned}$$

The complex coordinates are denoted $z = x + iy$ in these charts and the volume form is written $\sqrt{\det(g)(z)}dxdy$. For each critical points p' using the local representation in $\mathcal{U}(p')$ of r_1^1 in (10) (and the same notations for R and b), we obtain

$$\begin{aligned}&\int_{\mathcal{U}(p')} e^{i\psi/h}\chi_{p'}\bar{a}r_1^1(V_1 - V_2)\mathrm{dv}_g \\ &\quad= \int_{\mathcal{U}(p')} e^{2i\psi/h}\chi_{p'}\bar{a}r_{1,1}^1(V_1 - V_2)\mathrm{dv}_g + O(h^2) \\ &\quad= \int_{\mathbb{R}^2} \chi R(e^{2i\psi/h}\chi_1 b)\bar{a}(V_1 - V_2)\sqrt{\det g}\, dxdy + O(h^2) \\ &\quad= \int_{\mathbb{R}^2} e^{2i\psi/h}\chi_1 b R^*\Big(\chi\bar{a}(V_1 - V_2)\sqrt{\det g}\Big)\, dxdy + O(h^2).\end{aligned}$$

where R^* is the adjoint of R on L^2 compactly supported functions in $\mathbb{R}^2$, which has the same mapping properties as R on C^∞ (its integral kernel being of the same form). Since b and $R^*(\chi_{p'}^2 a(V_1 - V_2)\sqrt{\det g})$ are C^∞ in (10) and b vanishes at p', we obtain by stationary phase that the above integral is $O(h^2)$, which completes the proof. □

Proof of Theorem 1.1. By combining Proposition 5.1 and the Proposition 2.1, we show that $V_1 = V_2$ on a dense set of M_0, so by continuity of V_1, V_2 they agree everywhere. □

5.1. **Acknowledgements.** This work started during a summer evening in Pisa thanks to the hospitality of M. Mazzucchelli and A.G. Lecuona. We thank Mikko Salo, Eleny Ionel and Rafe Mazzeo for pointing out helpful references. C.G. thanks MSRI and the organizers of the 'Analysis on Singular spaces' 2008 program for support during part of this project. This works was achieved while C.G. was visiting IAS under an NSF fellowship number No. DMS-0635607. L.T is supported by NSF Grant No. DMS-0807502.

References

[1] R. Abraham, *Transversality in manifolds of mapping*, Bull. Amer. Math. Soc. **69** (1963), 470-474.

[2] K. Astala, L. Päivärinta, *Calderón's inverse conductivity problem in the plane*, Ann. of Math. (2) **163** (2006), no. 1, 265–299.

[3] K. Astala, M. Lassas, L. Päivärinta, *Calderón's inverse problem for anisotropic conductivity in the plane*, Comm. Partial Differential Equations **30** (2005), no. 1-3, 207–224.

[4] M.I. Belishev, *The Calderon problem for two-dimensional manifolds by the BC-method.* SIAM J. Math. Anal. **35** (2003), no. 1, 172–182

[5] A.L. Bukhgeim, *Recovering a potential from Cauchy data in the two-dimensional case.* J. Inverse Ill-Posed Probl. **16** (2008), no. 1, 19–33.

[6] A.L Bukhgeim, G. Ulmann, *Recovering a potential from partial Cauchy data*, Comm. PDE **27** (3,4) (2002), 653–668.

[7] A.P. Calderón, *On an inverse boundary value problem*, Seminar on Numerical analysis and its applications to continuum physics, Soc. Brasileira de Matematica, Rio de Janeiro, 1980.

[8] D. Dos Santos Ferreira, C. E. Kenig, M. Salo, G. Uhlmann, *Limiting Carleman weights and anisotropic inverse problems*, arXiv:0803.3508.

[9] H.M Farkas, I. Kra, *Riemann surfaces.* Second edition. Graduate Texts in Mathematics, 71. Springer-Verlag, New York, 1992. 363 pp.

[10] C. Guillarmou, A. Sa Barreto, *Inverse Problems for Einstein manifolds*, Inverse Problems and Imaging. **3** (2009), no 1, 1-15.

[11] C. Guillarmou, L. Tzou, *Calderón problem with partial data on Riemann surfaces*, preprint arXiv 0908.1417.

[12] G. Henkin, V. Michel, *Inverse conductivity problem on Riemann surfaces.* J. Geom. Anal. **18** (2008), no. 4, 1033–1052.

[13] O. Imanuvilov, G. Uhlmann, M. Yamamoto, *The Calderon Problem with Partial Data in Two Dimensions.* to appear in Journ. of the A.M.S..

[14] C. Kenig, J. Sjöstrand, G. Uhlmann, *The Calderón Problem with Partial Data.* Ann. of Math. (2) **165** (2007), no. 2, 567–591.

[15] M. Lassas, M. Taylor, G. Uhlmann, *The Dirichlet-to-Neumann map for complete Riemannian manifolds with boundary*, Communications in Analysis and Geometry **11** (2003), 207-222.

[16] M. Lassas, G. Uhlmann, *On determining a Riemannian manifold from the Dirichlet-to-Neumann map.* Ann. Sci. École Norm. Sup. (4) **34** (2001), no. 5, 771-787.

[17] A. Nachman, *Reconstructions from boundary measurements.* Ann. of Math. (2) **128** (1988), no. 3, 531-576.

[18] F. Quinn, *Transversal approcimation on Banach manifolds*, AMS. Proc. Sym. in Pure Math. XV (1970), 213-222.

[19] J. Sylvester G. Uhlmann, *A global uniqueness theorem for an inverse boundary value problem.* Ann. of Math. (2) **125** (1987), no. 1, 153–169.

[20] K. Uhlenbeck, *Generic properties of eigenfunctions.* Amer. J. Math. **98** (1976), no. 4, 1059-1078.

Colin Guillarmou, DMA, U.M.R. 8553 CNRS, Ecole Normale Supérieure, 45 rue d'Ulm, F 75230 Paris cedex 05, France
E-mail address: `cguillar@dma.ens.fr`

Leo Tzou, Department of Mathematics, Stanford University, Stanford, CA 94305, USA.
E-mail address: `ltzou@math.stanford.edu.`