FEYNMAN'S OPERATIONAL CALCULUS AND THE STOCHASTIC FUNCTIONAL CALCULUS IN HILBERT SPACE

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ABSTRACT. Let A_1, A_2 be bounded linear operators acting on a Banach space E. A pair (μ_1, μ_2) of continuous probability measures on $[0,1]$ determines a functional calculus $f \longmapsto f_{\mu_1,\mu_2}(A_1, A_2)$ for analytic functions f by weighting all possible orderings of operator products of A_1 and A_2 via the probability measures μ_1 and μ_2. For example, $f \longmapsto f_{\mu,\mu}(A_1, A_2)$ is the Weyl functional calculus with equally weighted operator products. Replacing μ_1 by Lebesgue measure λ on $[0,t]$ and μ_2 by stochastic integration with respect to a Wiener process W, we show that there exists a functional calculus $f \longmapsto f_{\lambda,W;t}(A + B)$ for bounded holomorphic functions f if A is a densely defined Hilbert space operator with a bounded holomorphic functional calculus and B is small compared to A relative to a square function norm. By this means, the solution of the stochastic evolution equation $dX_t = AX_t dt + BX_t dW_t$, $X_0 = x$, is represented as $t \longmapsto e^{A+B}_{\lambda,W;t}x$, $t \geq 0$.

1. Introduction

In a series of papers [10, 11, 12], the author and G.W. Johnson studied a family of functional calculi for bounded linear operators $A_1, \dots, A_n$ acting on a Banach space E. Each functional calculus is determined by n continuous Borel probability measures $\mu_1, \dots, \mu_n$ on $[0,1]$. The *time-ordering* measures $\mu_1, \dots, \mu_n$ determine an operational calculus or *disentangling map* $\mathcal{T}_{\mu_1,\dots,\mu_n}$ from a commutative Banach algebra $\mathbb{D}(A_1, \dots, A_n)$ of analytic functions into the noncommutative Banach algebra $\mathcal{L}(E)$, see [10]. The idea originated from a paper by the physicist R. Feynman [5] and its mathematical implementation by E. Nelson [21].

For $f \in \mathbb{D}(A_1, \dots, A_n)$, the bounded linear operator $f_{\mu_1,\dots,\mu_n}(A_1, \dots, A_n) := \mathcal{T}_{\mu_1,\dots,\mu_n} f$ represents the function f of the (constant) operator valued functions $A_j(t) := A_j$, $0 \leq t \leq 1$, after disentangling with respect to the time-ordering measures $\mu_1, \dots, \mu_n$. A similar construction works if $\mu_1, \dots, \mu_n$ are any continuous Borel measures on $[0,1]$, not necessarily probability measures. We refer to these functional calculi loosely as *Feynman's operational calculus*.

A major application for developing an operational calculus is for representing solutions of evolution equations in a fashion similar to the way Feynman path integrals are used to represent solutions of 'quantum equations'. For example, if λ denotes Lebesgue measure on $[0,1]$, then

$$e^{A+B}_{\lambda,\lambda} = e^{A+B} = e^A + \sum_{n=1}^{\infty} \int_0^1 \int_0^{s_n} \cdots \int_0^{s_2} e^{(1-s_n)A} B e^{(s_n - s_{n-1})A} \cdots \cdots e^{(s_2-s_1)A} B e^{s_1 A} \, ds_1 \dots ds_n \quad (1.1)$$

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is the well known perturbation series expansion for the exponential of the sum of bounded linear operators A and B, see [11, Corollary 5.3]. Feynman's idea seems to have been to describe a general procedure for deriving formulae of this sort.

Suppose that A is a nonnegative selfadjoint operator on a Hilbert space H. If B is a "small" perturbation of A, then $A + B$ is selfadjoint and the solution $u_t = e^{-t(A+B)}x$ of the deterministic linear equation

$$du_t = Au_t dt + Bu_t dt, \quad u_0 = x,$$

is defined for all $t \geq 0$ and all sufficiently regular $x \in H$: "small" in this sense could mean a small *form* perturbation of A. Moreover, $A + B$ has an L^∞-functional calculus $f \longmapsto f(A + B)$, so that we can form a much larger class of functions of the operator $A + B$ than just those defined by $z \longmapsto e^{-tz}$ for $t \geq 0$. Similarly, we find that if B is a small perturbation of A and W is a Brownian motion process, then the solution $t \longmapsto e^{A+B}_{dt,dW_t;t}x$ of the stochastic equation

$$dX_t = AX_t\, dt + BX_t\, dW_t, \quad X_0 = x$$

is defined and there is a functional calculus

$$f \longmapsto f_{dt,dW_t;t}(A + B)$$

for $A + B$. In the stochastic setting, the relevant properties are that A should have an H^∞-functional calculus and B should be small compared to A relative to a "square function norm".

In the case that A and B are *commuting* bounded linear operators acting on a Banach space E and W is a Brownian motion process, a solution of the linear operator valued stochastic differential equation

$$dX_t + AX_t dt = BX_t dW_t \tag{1.2}$$

in $\mathcal{L}(E)$ can be written as $X_t = \exp(-t(A + B^2/2) + BW_t)$, or,

$$X_t = \exp\left[-\int_0^t \left(A + \frac{1}{2}B^2\right) ds + \int_0^t B\, dW_s\right]; \tag{1.3}$$

for the proof, it suffices to apply Itô's formula scalarly. Formula (1.3) suggests that by taking $f(z_1, z_2) = e^{z_1+z_2}$, we ought to be able to write the solution of equation (1.2) as

$$X_t = e^{-A+B}_{dt,dW_t;t} := f_{dt,dW_t;t}(-A, B)$$

by using Feynman's operational calculus with time ordering "measures" (dt, dW_t) for the pair $(-A, B)$ of bounded linear operators, even if they do not commute.

G.W. Johnson and G. Kallianpur [14] have represented X by the stochastic Dyson series

$$X_t = e^{-tA} + \sum_{n=1}^{\infty} \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} e^{-(t-s_n)A} B_{s_n} \cdots e^{-(s_2-s_1)A} B_{s_1} e^{-s_1 A}\, dW_{s_1} \cdots dW_{s_n} \tag{1.4}$$

with respect to time-ordered operator valued multiple Wiener integrals in the case that $-A$ is the generator of a C_0-contraction semigroup on Hilbert space and $\int_0^T \|B_s\|^2\, ds < \infty$. Wiener chaos expansions like equation (1.4) have been used for some time to represent solutions of linear stochastic PDE, see for example [8], [17], [20], but the comparison of (1.4) with the perturbation expansion (1.1) above reveals the connection with the expression $e^{-A+B}_{dt,dW_t;t}$ suggested by Feynman's operational calculus.

So, we are seeking a *stochastic functional calculus* $f \longmapsto f_{dt,dW_t}(-A, B)$ based on time-ordering with respect to white noise dW_t, which will enable us to make sense of $e^{-A+B}_{dt,dW_t;t}$ even if A and B are both unbounded linear operators. The present work is a first step in that direction. A systematic study of the existence, uniqueness and regularity of solutions of parabolic stochastic evolution equations in UMD Banach spaces that includes equation (1.2) as a special case is given in [27]. The emphasis here is on the joint functional calculus properties of A and B in the stochastic setting related to Feynman's operational calculus.

The paper is organised as follows. In Section 2 we start with a brief discussion of Feynman's operational calculus for a pair (A, B) bounded linear operators on a Banach space and two time-ordering measures (μ, ν) associated with the pair (A, B). This motivates the later treatment of possibly unbounded operators (A, B) for which one measure ν is replaced by Brownian motion. Replacing integration with respect to a measure by stochastic integration requires a discussion of multiple stochastic integrals, which we outline in Section 3 and use for stochastic disentangling in Banach spaces in Section 4. For our purpose, we need just a few simple estimates involving projective tensor products. A comprehensive treatment of multiple stochastic integration for Banach space valued deterministic functions has recently been developed by J. Maas [18].

The key idea to our approach to the stochastic functional calculus is the stochastic Dyson series which we obtain in Section 5 for Hilbert space operators from simple square function estimates. For bounded linear operators, the stochastic Dyson series is derived directly by stochastic disentangling just as in the deterministic setting. In Section 6, similar square function estimates are enough to establish the existence of the stochastic H^∞-functional calculus $f \longmapsto f_{dt,dW_t;t}(A + B)$ mentioned above. Example 6.10 shows that the assumptions are satisfied in the familiar case of a stochastic parabolic evolution equation with nonsymmetric boundary conditions.

2. Feynman's operational calculus

Let E be a Banach space and let $A_1, \ldots, A_n$ be nonzero bounded linear operators E. We first introduce a commutative Banach algebra consisting of 'analytic functions' $f(\tilde{A}_1, \ldots, \tilde{A}_n)$, where $\tilde{A}_1, \ldots, \tilde{A}_n$ are treated as purely formal commuting objects. The collection $\mathbb{D} = \mathbb{D}(A_1, \ldots, A_n)$ consists of all expressions of the form

$$f(\tilde{A}_1, \ldots, \tilde{A}_n) = \sum_{m_1,\ldots,m_n=0}^{\infty} c_{m_1,\ldots,m_n} \tilde{A}_1^{m_1} \cdots \tilde{A}_n^{m_n} \tag{2.1}$$

where $c_{m_1,\ldots,m_n} \in \mathbb{C}$ for all $m_1, \ldots, m_n = 0, 1, \ldots,$ and

$$\begin{aligned} \|f(\tilde{A}_1, \ldots, \tilde{A}_n)\| &= \|f(\tilde{A}_1, \ldots, \tilde{A}_n)\|_{\mathbb{D}(A_1,\ldots,A_n)} \\ &:= \sum_{m_1,\ldots,m_n=0}^{\infty} |c_{m_1,\ldots,m_n}| \|A_1\|^{m_1} \cdots \|A_n\|^{m_n} < \infty. \end{aligned} \tag{2.2}$$

The norm on $\mathbb{D}(A_1, \ldots, A_n)$ defined by (2.2) makes $\mathbb{D}(A_1, \ldots, A_n)$ into a commutative Banach algebra under pointwise operations. We refer to $\mathbb{D}(A_1, \ldots, A_n)$ as the *disentangling algebra* associated with the n-tuple $(A_1, \ldots, A_n)$ of bounded linear operators acting on E.

Fix $t > s \geq 0$. Let $A_1, \ldots, A_n$ be nonzero operators from $\mathcal{L}(E)$ and let $\mu_1, \ldots, \mu_n$ be continuous measures defined at least on $\mathcal{B}([s, t])$, the Borel σ-algebra of $[s, t]$. The total mass of a measure μ is written as $\|\mu\|_{[s,t]}$.

The idea is to replace the operators $A_1, \ldots, A_n$ with the elements $\tilde{A}_1, \ldots, \tilde{A}_n$ from $\mathbb{D} = \mathbb{D}(\|\mu_1\|A_1, \ldots, \|\mu_n\|A_n)$ and then form the desired function of $\tilde{A}_1, \ldots, \tilde{A}_n$. Still working in $\mathbb{D}$, we time order the expression for the function and then pass back to $\mathcal{L}(E)$ simply by removing the tildes.

Given nonnegative integers $m_1, \ldots, m_n$, we let $m = m_1 + \cdots + m_n$ and

$$P^{m_1,\ldots,m_n}(z_1, \ldots, z_n) = z_1^{m_1} \cdots z_n^{m_n}. \tag{2.3}$$

We are now ready to define the *disentangling map* $\mathcal{T}_{\mu_1,\dots,\mu_n}$ which will return us from our commutative framework $\mathbb{D}(A_1,\dots,A_n)$ to the noncommutative setting of $\mathcal{L}(E)$. For $i = 1,\dots,m$, we define

$$\mathcal{A}_i := \begin{cases} A_1 & \text{if } i \in \{1,\dots,m_1\},\\ A_2 & \text{if } i \in \{m_1+1,\dots,m_1+m_2\},\\ \vdots & \qquad\qquad \vdots \\ A_n & \text{if } i \in \{m_1+\cdots+m_{n-1}+1,\dots,m\}. \end{cases} \tag{2.4}$$

For each $m = 0,1,\dots,$ let S_m denote the set of all permutations of the integers $\{1,\dots,m\}$, and given $\pi \in S_m$, we let

$$\Delta_m(\pi;s,t) = \{(s_1,\dots,s_m) \in [s,t]^m : s < s_{\pi(1)} < \cdots < s_{\pi(m)} < t\}.$$

If π is the identity, then we write $\Delta_m(s,t)$ instead. We write $\Delta_m(\pi;t)$ and $\Delta_m(t)$ if $s = 0$.

Definition 2.1. $\mathcal{T}_{\mu_1,\dots,\mu_n;s,t}\left(P^{m_1,\dots,m_n}(\tilde{A}_1,\dots,\tilde{A}_n)\right) :=$

$$\sum_{\pi\in S_m}\int_{\Delta_m(\pi;s,t)} \mathcal{A}_{\pi(m)}\cdots\mathcal{A}_{\pi(1)}(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n})(ds_1,\dots,ds_m). \tag{2.5}$$

The notation μ_j^k denotes the k-fold product measure $\mu_j \times\cdots\times \mu_j$ of μ_j with itself for $j = 1,\dots,n$ and μ_j^0 means that the integral with respect to the s_j-variable is simply omitted. We adopt this convention even if μ_j is the zero measure.

Then, for $f(\tilde{A}_1,\dots,\tilde{A}_n) \in \mathbb{D}(A_1,\dots,A_n)$ given by

$$f(\tilde{A}_1,\dots,\tilde{A}_n) = \sum_{m_1,\dots,m_n=0}^{\infty} c_{m_1,\dots,m_n}\tilde{A}_1^{m_1}\cdots\tilde{A}_n^{m_n}, \tag{2.6}$$

we set $\mathcal{T}_{\mu_1,\dots,\mu_n;s,t}\big(f(\tilde{A}_1,\dots,\tilde{A}_n)\big)$ equal to

$$\sum_{m_1,\dots,m_n=0}^{\infty} c_{m_1,\dots,m_n}\mathcal{T}_{\mu_1,\dots,\mu_n;s,t}\left(P^{m_1,\dots,m_n}(\tilde{A}_1,\dots,\tilde{A}_n)\right). \tag{2.7}$$

In the commutative setting and with probability measures, the right-hand side of (2.5) gives us what we would expect [10, Proposition 2.2], namely

$$P^{m_1,\dots,m_n}(\tilde{A}_1,\dots,\tilde{A}_n).$$

We shall sometimes write the bounded linear operator

$$\mathcal{T}_{\mu_1,\dots,\mu_n;s,t}\big(f(\tilde{A}_1,\dots,\tilde{A}_n)\big)$$

as $f_{\mu_1,\dots,\mu_n;s,t}(A_1,\dots,A_n)$. In particular,

$$P^{m_1\dots m_n}_{\mu_1,\dots,\mu_n;s,t}(A_1,\dots,A_n) = \mathcal{T}_{\mu_1,\dots,\mu_n;s,t}\left(P^{m_1,\dots,m_n}(\tilde{A}_1,\dots,\tilde{A}_n)\right). \tag{2.8}$$

The following result appeared in [11, Corollary 5.3] in the case that $t = 1$ and $s = 0$. A similar proof works for the case below.

Theorem 2.2. *Let E be a Banach space and let μ and ν be continuous measures on the Borel σ-algebra of $[0,\infty)$. Let A, B be elements of $\mathcal{L}(E)$. Then for all $t > s \geq 0$,*

$$\begin{aligned} e^{A+B}_{\mu,\nu;s,t} &:= \mathcal{T}_{\mu\restriction_{[s,t]},\nu\restriction_{[s,t]};s,t}\left(e^{\tilde{A}+\tilde{B}}\right)\\ &= e^{A\mu([s,t])} + \sum_{n=1}^{\infty}\Bigg[\int_s^t\int_s^{s_n}\cdots\int_s^{s_2} e^{A\mu([s_n,t])}Be^{A\mu([s_{n-1},s_n])}\\ &\qquad \cdots e^{A\mu([s_1,s_2])}Be^{At\mu([s,s_1])}\,\nu^n(ds_1,\dots,ds_n)\Bigg]. \end{aligned} \tag{2.9}$$

It follows that $e^{A+B}_{\mu,\nu;s,t}$ satisfies the integral equation

$$e^{A+B}_{\mu,\nu;s,t} = e^{A\mu([s,t])} + \int_s^t e^{A\mu([r,t])} B e^{A+B}_{\mu,\nu;s,r}\, d\nu(r) \tag{2.10}$$

by substituting equation (2.9) into the right-hand side of equation (2.10). Feynman's disentangling ideas suggest that for every $0 \le r < s \le t$, the equation

$$e^{A+B}_{\mu,\nu;s,t} e^{A+B}_{\mu,\nu;r,s} = e^{A+B}_{\mu,\nu;r,t} \tag{2.11}$$

ought to be valid, that is, $e^{A+B}_{\mu,\nu;s,t}$, $0 \le s \le t$, is an *evolution system*, see [22, Theorem 5.3.1]. A proof of equation (2.11) and a more general disentangling formula appears in [13].

3. Stochastic disentangling in Banach spaces

Suppose that in the situation of the preceding section, $\mu(dt)$ is Lebesgue measure dt and $\nu(dt)$ is integration with respect to "white noise" dW_t. Then the multiple integrals in the perturbation series expansion (2.9) need to be replaced by multiple stochastic integrals with respect to the Brownian motion process.

More precisely, let W denote Brownian motion in $\mathbb{R}$ with respect to the probability measure space $(\Omega, \mathcal{S}, \mathbb{P})$ such that $W_0 = 0$ almost surely. In the case that Ω is taken to be the set of all continuous functions $\omega : [0, \infty) \to \mathbb{R}$, the σ-algebra $\mathcal{S}$ is the Borel σ-algebra of Ω for the compact-open topology and $W_t(\omega) = \omega(t)$ for every $\omega \in \Omega$ and $t \ge 0$. There exists a unique Borel probability measure $\mathbb{P}$ on Ω — *Wiener measure*, such that for every $0 < t_1 < \cdots < t_k$, Borel subsets $B_1, \dots, B_k$ of $\mathbb{R}$ and $k = 1, 2, \dots$, the measure of the elementary event

$$E = \{\omega \in \Omega : \omega(t_1) \in B_1, \dots, \omega(t_k) \in B_k\ \}$$

is given by

$$\mathbb{P}(E) = \int_{B_k} \dots \int_{B_1} p_{t_k - t_{k-1}}(x_k - x_{k-1}) \cdots p_{t_2 - t_1}(x_2 - x_1) p_{t_1}(x_1)\, dx_1 \dots dx_k,$$

where $p_t(x) = (2\pi t)^{-\frac{1}{2}} e^{-x^2/(2t)}$, $t > 0$, $x \in \mathbb{R}$, is the associated transition function. Then Wiener measure $\mathbb{P}$ has the property that W_t, $t \ge 0$, is a process with stationary and independent increments such that W_t is a gaussian random variable with mean zero and variance t for $t > 0$, properties which define a Brownian motion W_t, $t \ge 0$, with $W_0 = 0$ $\mathbb{P}$-a.e. over a general probability measure space $(\Omega, \mathcal{S}, \mathbb{P})$.

For a Banach space E and $1 \le p < \infty$, the space of E-valued pth-Bochner integrable functions with respect to $\mathbb{P}$ is denoted by $L^p(\mathbb{P}, E) = L^p(\Omega, \mathcal{S}, \mathbb{P}, E)$. The linear space $L^0(\mathbb{P}, E) = L^0(\Omega, \mathcal{S}, \mathbb{P}, E)$ of strongly measurable E-valued functions has the (metrisable) topology of convergence in probability.

3.1. **Multiple stochastic integrals.** For the purpose of expanding solutions of linear stochastic equations like (1.2) as a stochastic Dyson series (1.4), we need to consider multiple Wiener-Itô integrals of deterministic functions. We follow the account in [16, Section 10.3] with suitable modifications for vector valued functions. Wiener-Itô chaos in Banach spaces is treated in [18, Section 3].

Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$. Let $T > 0$ and $k = 1, 2 \dots$. The case $k = 1$ corresponds to the Wiener integral. Let $D_1 = (0, T]$ and let

$$D_k = \{(t_1, \dots, t_k) \in (0, T]^k : \exists i, j = 1, \dots, k, i \ne j, \text{ such that } \ t_i = t_j\}$$

whenever $k = 2, 3, \dots$. Let $A_1, \dots, A_n$ be a partition of $(0, T]$ into disjoint intervals of the form $(s, t]$ for $0 \le s < t \le T$ and suppose that

$$f = \sum_{1 \le j_1, \dots, j_k \le n} \alpha_{j_1, \dots, j_k} \chi_{A_{j_1} \times \cdots \times A_{j_k}} \tag{3.1}$$

is a H-valued function such that $\alpha_{j_1,\dots,j_k} = 0$ whenever two indices $j_1,\dots,j_k$ are equal and f vanishes on D_k. Then

$$I_k(f) = \int_{[0,T]^k} f(t_1,\dots,t_k)\,dW_{t_1}\dots dW_{t_k}$$

is defined by

$$I_k(f) = \sum_{1\le j_1,\dots,j_k\le n} \alpha_{j_1,\dots,j_k} W(A_{j_1})\cdots W(A_{j_k}).$$

Here $W((s,t])$ denotes the random variable $W_t - W_s$ for $0 \le s < t \le T$. Let $\mathcal{D}((0,T]^k, H)$ denote the linear space of H-valued step functions f of the above form. Then I_k is well defined and $I_k : \mathcal{D}((0,T]^k,H) \to L^0(\Omega,\mathcal{S},\mathbb{P},H)$ is a linear map. Moreover, the maps I_k, $k = 1,2,\dots$, enjoy the following properties.

1) The integral $I_k(f)$ is invariant under the symmetrisation of the function f, that is, if $\tilde f \in \mathcal{D}((0,T]^k,H)$ is the symmetrisation

$$\tilde f(t_1,\dots,t_k) = \frac{1}{k!}\sum_{\sigma\in S_k} f(t_{\sigma(1)},\dots,t_{\sigma(k)}),\quad t_1,\dots,t_k \in (0,T]$$

of $f \in \mathcal{D}((0,T]^k,H)$, then $I_k(f) = I_k(\tilde f)$.

2) If k and k' are positive integers such that $k \ne k'$ and $f \in \mathcal{D}((0,T]^k,H)$, $g \in \mathcal{D}((0,T]^{k'},H)$, then $\mathbb{E}(\langle I_k(f), I_{k'}(g)\rangle_H) = 0$.

3) If $f \in \mathcal{D}((0,T]^k,H)$ and $g \in \mathcal{D}((0,T]^k,H)$, then

$$\mathbb{E}(\langle I_k(f), I_k(g)\rangle_H) = k!\langle \tilde f, \tilde g\rangle_{L^2((0,T]^k,H)}.$$

The inner product on the right hand side is taken in the Hilbert space $L^2((0,T]^k,H)$.

By property 3), we have a version of the Itô isometry

$$\mathbb{E}(\|I_k(f)\|_H^2) = \mathbb{E}(\|I_k^2(\tilde f)\|_H^2) = k!\|\tilde f\|^2_{L^2((0,T]^k,H)} \le k!\|f\|^2_{L^2((0,T]^k,H)}, \tag{3.2}$$

so that the mapping I_k can be extended to a bounded linear operator

$$I_k : L^2((0,T]^k,H) \to L^2(\Omega,\mathcal{S},\mathbb{P},H).$$

We also write $I_k(f)$ as $\int_{[0,T]^k} f(s)\,W^k(ds)$. In the case that $0 \le s < t \le T$ and $f \in L^2((0,T]^k,H)$ is zero off $\Delta_k(s,t)$, then

$$I_k(f) = \int_s^t\int_s^{t_k}\cdots\int_s^{t_2} f(t_1,\dots,t_k)\,dW_{t_1}\cdots dW_{t_k}, \tag{3.3}$$

where the right-hand side is interpreted as an iterated stochastic integral. The equality is easily seen to be valid for all $f \in \mathcal{D}((0,T]^k,H)$ vanishing off $\Delta_k(s,t)$ and the linear subspace of all such functions is dense in the closed subspace of $L^2((0,T]^k,H)$ consisting of all H-valued functions belonging to $L^2((0,T]^k,H)$ which are zero almost everywhere outside $\Delta_k(s,t) \subset (0,T]^k$. The Itô isometry (3.2) for the integral (3.3) takes the form

$$\mathbb{E}(\|I_k(f)\|_H^2) = \int_s^t\int_s^{t_k}\cdots\int_s^{t_2} |f(t_1,\dots,t_k)|^2\,dt_1\cdots dt_k. \tag{3.4}$$

Let m and n be nonnegative integers. We note the following obvious estimate.

Lemma 3.1. *Let μ be a finite Borel measure on $\mathbb{R}$, $A \subset [0,t]^{m+n}$ a Borel set and*

$$A(\xi) = \{(s_1,\dots,s_m,\xi_1,\dots,\xi_n) \in A\},\ \xi \in \mathbb{R}^n.$$

Then $\int_{\mathbb{R}^n} \mu^m(A(\xi))^2\,d\xi \le \|\mu\|^{2m}_{[0,t]} t^n$.

Let $A \subset [0,t]^{m+n}$ be a measurable set. The random variable $(\mu^m \times W^n)(A)$ is defined by

$$(\mu^m \times W^n)(A) = \int_A (\mu^m \times W^n)(ds_1, \dots, ds_{m+n}) = \int_{[0,t]^n} \mu^m(A(s))\, W^n(ds),$$

where the integral with respect to W^n is the multiple Wiener-Itô integral of order n defined above. Appealing to Lemma 3.1 and the bound (3.2), we note that

$$\|(\mu^m \times W^n)(A)\|_2 \le \sqrt{n!} \left(\int_{\mathbb{R}^n} \mu^m(A(\xi))^2 \, d\xi \right)^{\frac{1}{2}} \le \sqrt{n!} \|\mu\|_{[0,t]}^m t^{n/2}. \tag{3.5}$$

3.2. **Stochastic disentangling.** Let E be a Banach space and $A_1, A_2 \in \mathcal{L}(E)$. As in equation (2.4), we define

$$\mathcal{A}_i := \begin{cases} A_1 & \text{if } i \in \{1, \dots, m\}, \\ A_2 & \text{if } i \in \{m+1, \dots, m+n\}. \end{cases}$$

for $m, n = 1, 2, \dots$.

Definition 3.2. Let μ be a continuous Borel measure on $[0, \infty)$ and let E be a Banach space and $A_1, A_2 \in \mathcal{L}(E)$. The $\mathcal{L}(E)$-valued random variable

$$\mathcal{T}_{\mu,W;t}\left(P^{m,n}(\tilde{A}_1, \tilde{A}_2)\right)$$

is defined for each $t > 0$ and for nonnegative integers m and n by

$$\mathcal{T}_{\mu,W;t}\left(P^{m,n}(\tilde{A}_1, \tilde{A}_2)\right) := \sum_{\pi \in S_{m+n}} \int_{\Delta_{m+n}(\pi;0,t)} \mathcal{A}_{\pi(m+n)} \cdots \mathcal{A}_{\pi(1)} (\mu^m \times W^n)(ds_1, \dots, ds_{m+n}). \tag{3.6}$$

The notation μ^0 or W^0 means that the corresponding integral is simply omitted, with the understanding that $\mathcal{T}_{\mu,W;t}\left(P^{0,0}(\tilde{A}_1, \tilde{A}_2)\right) = I$ $\mathbb{P}$-a.e.. We refer to $\mathcal{T}_{\mu,W;t}$ as the *stochastic disentangling map.*

Note that this expression is just a finite sum of operators times real valued random variables. For each $x \in E$, we take $\mathcal{T}_{\mu,W;t}\left(P^{m,n}(\tilde{A}_1, \tilde{A}_2)\right) x$ to be the E-valued random variable

$$\sum_{\pi \in S_{m+n}} \int_{\Delta_{m+n}(\pi;0,t)} \left(\mathcal{A}_{\pi(m+n)} \cdots \mathcal{A}_{\pi(1)} x\right) (\mu^m \times W^n)(ds_1, \dots, ds_{m+n}).$$

A simple calculation disentangles a polynomial in two commutative variables [9, Lemma 3.4].

Lemma 3.3. *Let m and n be nonnegative integers. Let E be a Banach space and $A_1, A_2 \in \mathcal{L}(E)$. Then*

$$\begin{aligned} \mathcal{T}_{\mu,W;t} P^{m,n}(\tilde{A}_1, \tilde{A}_2) &= \sum_{j_0 + \cdots + j_n = m} \frac{m!n!}{j_0! \dots j_n!} A_1^{j_n} A_2 A_1^{j_{n-1}} \cdots A_1^{j_1} A_2 A_1^{j_0} \\ &\quad \times \int_{\Delta_n(t)} \mu([0, s_1])^{j_0} \mu([s_1, s_2])^{j_1} \cdots \mu([s_n, t])^{j_n} W^n(ds_1, \dots, ds_n). \end{aligned} \tag{3.7}$$

The following result [9, Theorem 3.5] follows from the Itô isometry (3.4).

Theorem 3.4. *Let H be a Hilbert space and $A_1, A_2 \in \mathcal{L}(H)$. Then for each $x \in H$ and $m, n = 0, 1, \dots,$ we have*

$$\|\mathcal{T}_{\mu,W;t}\left(P^{m,n}(\tilde{A}_1,\tilde{A}_2)\right)x\|_{L^2(\mathbb{P},H))} \leq \sqrt{n!}(\|\mu\|_{[0,t]}\|A_1\|)^m(t^{\frac{1}{2}}\|A_2\|)^n\|x\|. \tag{3.8}$$

The collection $\mathbb{D}_W(A_1, A_2)$ consists of all expressions of the form

$$f(\tilde{A}_1,\tilde{A}_2) = \sum_{m_1,m_2=0}^{\infty} c_{m_1,m_2}\tilde{A}_1^{m_1}\tilde{A}_2^{m_2} \tag{3.9}$$

where $c_{m_1,m_2} \in \mathbb{C}$ for all $m_1, m_2 = 0, 1, \dots,$ and

$$\begin{aligned}\|f(\tilde{A}_1,\tilde{A}_2)\| &= \|f(\tilde{A}_1,\tilde{A}_2)\|_{\mathbb{D}_W(A_1,A_2)} \\ &:= \sum_{m_1,m_2=0}^{\infty} \sqrt{m_2!}|c_{m_1,m_2}|\|A_1\|^{m_1}\|A_2\|^{m_2} < \infty.\end{aligned} \tag{3.10}$$

Then, for $f(\tilde{A}_1,\tilde{A}_2) \in \mathbb{D}_W(\|\mu\|_{[0,t]}A_1, t^{\frac{1}{2}}A_2)$ given by

$$f(\tilde{A}_1,\tilde{A}_2) = \sum_{m_1,m_2=0}^{\infty} c_{m_1,m_2}\tilde{A}_1^{m_1}\tilde{A}_2^{m_2}, \tag{3.11}$$

we set $f_{\mu,W;t}(A_1, A_2) := \mathcal{T}_{\mu,W;t}\big(f(\tilde{A}_1,\tilde{A}_2)\big)$ equal to

$$\sum_{m_1,m_2=0}^{\infty} c_{m_1,m_2}\mathcal{T}_{\mu,W;t}\left(P^{m_1,m_2}(\tilde{A}_1,\tilde{A}_2)\right). \tag{3.12}$$

According to Theorem 3.4, the series converges absolutely in the strong operator topology of the space $\mathcal{L}(H, L^2(\mathbb{P}, H))$ of *random linear operators* [25].

The following result was proved in [9, Proposition 3.6].

Proposition 3.5. *Let H be a Hilbert space and $A_1, A_2 \in \mathcal{L}(H)$. Suppose that $T > 0$ and μ is a continuous measure on the Borel σ-algebra of $[0, T]$. Let $f(\tilde{A}_1,\tilde{A}_2) \in \mathbb{D}_W(\|\mu\|_{[0,T]}A_1, T^{\frac{1}{2}}A_2)$.*

Then $t \longmapsto f_{\mu,W;t}(A_1, A_2)x$, $0 \leq t \leq T$, is a continuous function with values in $L^2(\mathbb{P}, H)$ for each $x \in H$.

Furthermore, for each $x \in H$, the vector valued process $\langle f_{\mu,W;t}(A_1, A_2)x\rangle_{0\leq t\leq T}$ has a pathwise continuous modification — there exists a strongly progressively measurable function $\Phi : [0, T] \times \Omega \to H$, such that $\Phi(t, \cdot) = f_{\mu,W;t}(A_1, A_2)x$ ($\mathbb{P}$ a.e.) for each $0 \leq t \leq T$ and for each $\omega \in \Omega$, the function $t \longmapsto \Phi(t, \omega)$, $0 \leq t \leq T$, is norm continuous in H.

4. Stochastic equations in Banach spaces

A comprehensive treatment of stochastic integration of Banach space valued deterministic functions appears in [26]. Muliple Wiener-Itô integrals for Banach space valued functions are treated in [18, Section 3]. A full treatment requires a discussion of γ-radonifying operators and their tensor products. In some situations it is possible to get by with simpler arguments which we now describe.

4.1. **Stochastic integration of vector valued functions.** We first mention some terminology related to stochastic integration. Let $\mathbb{R}_+ = [0, \infty)$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability measure space. A *filtration* is a family $\{\mathcal{F}_t : t \in \mathbb{R}_+\}$ of sub σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_s \subseteq \mathcal{F}_t$, $\forall s < t$. A filtration $\{\mathcal{F}_t : t \in \mathbb{R}_+\}$ is called a *standard filtration* if

(1) $\mathcal{F}_t = \mathcal{F}_{t+} := \cap_{s>t}\mathcal{F}_s \quad \forall t$ (*right continuity*)
(2) $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets (*completeness)*

Given an increasing family $\{\mathcal{F}_t : t \in \mathbb{R}_+\}$ of σ-algebras, a process $X : \mathbb{R}\times\Omega \to \mathbb{C}$ is *adapted* to $\mathcal{F}_t$ or *progressively measurable* if X_t is $\mathcal{F}_t$ measurable for all $t \in \mathbb{R}_+$.

Definition 4.1. Consider the subsets of $\mathbb{R}_+ \times \Omega$ consisting of all sets of the form

$$\begin{aligned}&\{0\}\times F_0, \qquad F_0 \in \mathcal{F}_0 \text{ and}\\ &(s,t]\times F, \qquad F \in \mathcal{F}_s \quad \text{for } s<t \text{ in } \mathbb{R}_+\end{aligned}$$

These are called *predictable rectangles.* Let $\mathcal{R}$ denote the family of all predictable rectangles. The σ-algebra $\mathcal{P}$ generated by $\mathcal{R}$ is called the *predictable σ-algebra.* A function $X : \mathbb{R}_+ \times \Omega \longrightarrow \mathbb{R}$ is called *predictable* if it is $\mathcal{P}$-measurable.

If $A \in \mathcal{R}$, then $\chi_A(t,\cdot)$ is $\mathcal{F}_t$-measurable $\forall t \geq 0$, so χ_A is an *adapted process.* By the monotone class theorem any real-valued $\mathcal{P}$-measurable function is adapted. A $\mathcal{P}$-measurable function is called a *predictable process.*

Let W_t, $t \geq 0$, be a Brownian motion process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 4.2. Let E be a Banach space. An E-valued random process Φ_t, $t \geq 0$, is said to be *stochastically integrable* in E, if for each $\xi \in E'$, the scalar valued process $\langle \Phi_t, \xi\rangle$, $t \geq 0$ is stochastically integrable with respect to W_t, $t \geq 0$, and there exists an E-valued random process Ψ_t, $t \geq 0$, such that

$$\langle \Psi_t, \xi\rangle = \int_0^t \langle \Phi_s, \xi\rangle \, dW_s \quad \text{a.e.} \tag{4.1}$$

for every $\xi \in E'$ and $t \geq 0$.

The definition only requires that the process $(t,\omega) \longmapsto \langle \Phi_t(\omega), \xi\rangle$, $(t,\omega) \in \mathbb{R}_+\times\Omega$ be measurable with respect to the *predictable* σ-algebra $\mathcal{P}$ for each $\xi \in E'$.

Let $T > 0$ and $k = 1, 2, \dots$. An E-valued function $s \longmapsto \Phi_s$, $s \in [0,T]^k$, is said to be *k-stochastically integrable* or *W^k-integrable* in E if for each $\xi \in E'$, the scalar valued function $t \longmapsto \langle \Phi_s, \xi\rangle$, $s \in [0,T]^k$ belongs to $L^2([0,T]^k)$, and there exists an E-valued random process Ψ_t, $t \in [0,T]$, such that

$$\langle \Psi_t, \xi\rangle = \int_{[0,t]^k} \langle \Phi_s, \xi\rangle \, W^k(ds_1, \dots, ds_k) \quad \text{a.e.}$$

for every $\xi \in E'$ and $t \geq 0$. We shall mainly be concerned with E-valued functions of the form $\Phi_s = \chi_{\Delta_k(T)}(s) f(s)$ for $s \in [0,T]^k$.

If a function $\phi : (0,T) \to E$ is *stochastically integrable* in E and it is weakly L^2, it follows that for every Borel subset A of $(0,T)$, there exists an E-valued Gaussian random variable X_A such that

$$\langle X_A, \xi\rangle = \int_0^T \chi_A(t)\langle \phi(t), \xi\rangle \, dW_t$$

for every $\xi \in E'$ [26]: it suffices that an E-valued random variable $X_{(0,T)}$ exists.

Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be Banach spaces. The *projective tensor product topology* π on the algebraic tensor product $E \otimes F$ of the linear spaces E and F is the topology [23, III.6.3] defined by the norm

$$\|x\|_\pi = \inf\left\{\sum_{j=1}^n \|e_j\|_E \|f_j\|_F : x = \sum_{j=1}^n e_j \otimes f_j, \ n = 1, 2, \dots\right\}.$$

The completion $E\hat{\otimes}_\pi F$ of the normed vector space $(E \otimes F, \|\cdot\|_\pi)$ is called the *projective tensor product* of E and F. Every element x of $E\hat{\otimes}_\pi F$ has a representation

$$x = \sum_{j=1}^\infty \lambda_j (e_j \otimes f_j)$$

with $\lambda_j > 0$, $\|e_j\|_E \le 1$ $\|f_j\|_F \le 1$ for $j = 1, 2, \dots$ and $\sum_{j=1}^{\infty} \lambda_j < \infty$ and

$$\|x\|_\pi = \inf \left\{ \sum_{j=1}^{\infty} \lambda_j \|e_j\|_E \|f_j\|_F \right\}, \tag{4.2}$$

see [23, III.7.1].

In the following result, we obtain a simple sufficient condition for stochastic integrability in a general Banach space.

Proposition 4.3. *Let E be a Banach space. Any function $\Phi : [0,T] \to E$ belonging to the projective tensor product $L^2([0,T])\hat{\otimes}_\pi E$ is stochastically integrable in E and*

$$\left\| \int_0^T \chi_A(s) \Phi_s \, dW_s \right\|_{L^2(\mathbb{P},E)} \le \|\Phi\|_{L^2([0,T])\hat{\otimes}_\pi E}, \quad A \in \mathcal{B}([0,T]).$$

Similarly, any function $\Phi : [0,T]^k \to E$ belonging to the projective tensor product $L^2([0,T]^k)\hat{\otimes}_\pi E$ is W^k-integrable in E on $[0,T]^k$ and

$$\left\| \int_{[0,T]^k} \chi_A(s) \Phi_s \, W^k(ds) \right\|_{L^2(\mathbb{P},E)} \le \sqrt{k!} \|\Phi\|_{L^2([0,T]^k)\hat{\otimes}_\pi E}, \quad A \in \mathcal{B}([0,T]^k).$$

Proof. Any $\Phi \in L^2([0,T])\hat{\otimes}_\pi E$ can be written as $\Phi = \sum_{j=1}^{\infty} \phi_j . x_j$ with $\phi_j \in L^2([0,T])$ and $x_j \in E$ and $\sum_{j=1}^{\infty} \|\phi_j\|_{L^2([0,T])} . \|x_j\|_E < \infty$. For each $0 \le t \le T$ and finite subset A of $\mathbb{N}$, we have

$$\begin{aligned}
&\left\| \sum_{j \in A} \left(\int_0^t \phi_j(s) \, dW_s \right) x_j \right\|_{L^2(\mathbb{P},E)}^2 \\
&\quad = \mathbb{E} \left\| \sum_{j \in A} \left(\int_0^t \phi_j(s) \, dW_s \right) x_j \right\|_E^2 \\
&\quad \le \mathbb{E} \left(\sum_{j \in A} \left| \int_0^t \phi_j(s) \, dW_s \right| \|x_j\| \right)^2 \\
&\quad \le \left(\sum_{j \in A} \left\| \int_0^t \phi_j(s) \, dW_s \right\|_{L^2(\mathbb{P})} \|x_j\| \right)^2, \quad \text{[Minkowski]} \\
&\quad = \left(\sum_{j \in A} \|\phi_j\|_{L^2([0,t])} \|x_j\| \right)^2, \quad \text{[Itô]}.
\end{aligned}$$

Therefore, $\sum_{j=1}^{\infty} \left(\int_0^t \phi_j(s) \, dW_s \right) x_j$ converges absolutely in $L^2(\mathbb{P}, E)$. For each $t > 0$, let $\Psi_t = \sum_{j=1}^{\infty} \left(\int_0^t \phi_j(s) \, dW_s \right) x_j$. Then the E-valued random process Ψ_t, $t \ge 0$, has a continuous version and the equalities

$$\begin{aligned}
\langle \Psi_t, \xi \rangle &= \sum_{j=1}^{\infty} \left(\int_0^t \phi_j(s) \, dW_s \right) \langle x_j, \xi \rangle \\
&= \sum_{j=1}^{\infty} \left(\int_0^t \langle x_j, \xi \rangle \phi_j(s) \, dW_s \right) \\
&= \int_0^t \langle \Phi_s, \xi \rangle \, dW_s \quad \text{a.e.}
\end{aligned}$$

hold for each $\xi \in E'$, because $\sum_{j=1}^{\infty}\langle x_j, \xi\rangle\phi_j$ converges absolutely in $L^2([0,T])$ to $\langle\Phi, \xi\rangle$.

Similarly, any $\Phi \in L^2([0,T]^k)\hat{\otimes}_\pi E$ can be written as $\Phi = \sum_{j=1}^{\infty}\phi_j.x_j$ with $\phi_j \in L^2([0,T]^k)$ and $x_j \in E$ and $\sum_{j=1}^{\infty}\|\phi_j\|_{L^2([0,T]^k)}.\|x_j\|_E < \infty$. The sum

$$\sum_{j=1}^{\infty}\left(\int_{[0,t]^k}\phi_j(s)\,W^k(ds)\right)x_j$$

converges absolutely in $L^2(\mathbb{P}, E)$. For each $t > 0$, let

$$\Psi_t = \sum_{j=1}^{\infty}\left(\int_{[0,t]^k}\phi_j(s)\,W^k(ds)\right)x_j.$$

Then the E-valued random process Ψ_t, $t \geq 0$, has a continuous version and equation (4.1) holds for every $\xi \in E'$. □

Remark 4.4. a) An element Φ of the projective tensor product $L^2([0,T])\hat{\otimes}_\pi E$ is associated with a *nuclear map* $T_\Phi : L^2([0,T]) \to E$ [23, p.98] and in the language of [26], nuclear maps are γ-radonifying. Indeed, a nuclear map radonifies *any* cylindrical measure on $L^2([0,T])$ with continuous weak moments [24].

b) A similar result holds if the Brownian motion process W is replaced by a square-integrable martingale M.

In the following result, we obtain a norm estimate for the disentanglement of a polynomial in elements of the disentangling algebra $\mathbb{D}(A_1, A_2)$.

Theorem 4.5. *Let E be a Banach space and $A_1, A_2 \in \mathcal{L}(E)$. Then for each $m, n = 0, 1, \ldots$ and $0 \leq t \leq T$, we have*

$$\|\mathcal{T}_{\mu,W;t}\left(P^{m,n}(\tilde{A}_1, \tilde{A}_2)\right)\|_{L^2(\mathbb{P},\mathcal{L}(E))} \leq n!(n\|A_1\|\mu([0,T]))^m(t^{\frac{1}{2}}\|A_2\|)^n. \tag{4.3}$$

Proof. First we appeal to Lemma 3.3 and note that

$$\begin{aligned}
&\left(\mathcal{T}_{\mu,W;t}P^{m,n}(\tilde{A}_1, \tilde{A}_2)\right)\\
&\quad= \sum_{\pi\in S_{m+n}}\int_{\Delta_{m+n}(\pi;0,t)}\mathcal{A}_{\pi(m+n)}\cdots\mathcal{A}_{\pi(1)}(\mu^m \times W^n)(ds_1, \ldots, ds_{m+n})\\
&\quad= \sum_{j_0+\cdots+j_n=m}\frac{m!n!}{j_0!\ldots j_n!}A_1^{j_n}A_2A_1^{j_{n-1}}\cdots A_1^{j_1}A_2A_1^{j_0}\\
&\qquad\times\int_{\Delta_n(t)}\mu([0,s_1])^{j_0}\mu([s_1,s_2])^{j_1}\cdots\mu([s_n,t])^{j_n}W^n(ds_1,\ldots,ds_n).
\end{aligned}$$

Applying Proposition 4.3, we have

$$\begin{aligned}
&\left\|\left(\mathcal{T}_{\mu,W;t}P^{m,n}(\tilde{A}_1, \tilde{A}_2)\right)\right\|_{L^2(\mathbb{P},\mathcal{L}(E))}\\
&\quad\leq (n!)^{\frac{3}{2}}\sum_{j_0+\cdots+j_n=m}\frac{m!}{j_0!\ldots j_n!}\left\|A_1^{j_n}A_2A_1^{j_{n-1}}\cdots A_1^{j_1}A_2A_1^{j_0}\right\|_{\mathcal{L}(E)}\\
&\qquad\times\left(\int_{\Delta_n(t)}\mu([0,s_1])^{2j_0}\mu([s_1,s_2])^{2j_1}\cdots\mu([s_n,t])^{2j_n}ds_1\ldots ds_n\right)^{\frac{1}{2}}.
\end{aligned}$$

But the sum

$$\sum_{j_0+\cdots+j_n=m} \frac{m!}{j_0!\dots j_n!}$$
$$\times \left(n! \int_{\Delta_n(t)} \nu([0,s_1])^{2j_0}\nu([s_1,s_2])^{2j_1}\cdots\nu([s_n,t])^{2j_n}\,ds_1\dots ds_n \right)^{\frac{1}{2}}$$

is bounded by $n^m t^{n/2}$ for $\nu = \mu/\|\mu\|_{[0,t]}$ for $m = 1, 2, \dots$, so that the estimate (4.3) is valid. □

Note that in the Hilbert space case $E = H$ with the $\mathcal{L}(H, L^2(\mathbb{P}, H))$-topology, we used the Itô isometry and calculated

$$\sqrt{n!}\left(\int_{\Delta_n(t)} \left(\sum_{j_0+\cdots+j_n=m} \frac{m!}{j_0!\dots j_n!}\nu([0,s_1])^{j_0}\cdots\nu([s_n,t])^{j_n} \right)^2 ds_1\dots ds_n \right)^{\frac{1}{2}}$$
$$= \sqrt{n!}\left(\int_{\Delta_n(t)} (\nu([0,s_1]) + \nu([s_1,s_2]) + \cdots + \nu([s_n,t]))^{2m} ds_1\dots ds_n\right)^{\frac{1}{2}}$$
$$= \sqrt{n!}\left(\int_{\Delta_n(t)} \nu([0,t])^{2m} ds_1\dots ds_n\right)^{\frac{1}{2}}$$
$$= t^{n/2},$$

which leads to a better estimate (3.8).

4.2. **Bilinear stochastic equations in Banach spaces.** The case of stochastic linear equations with bounded operators is rather trivial, but its solution does lead to considerations important for the treatment of unbounded linear operators.

Definition 4.6. Let A and B be bounded linear operators acting on a Banach space E. We say that a continuous progressively measurable E-valued process X satisfies the stochastic equation

$$dX_t = AX_t\,dt + BX_t\,dW_t$$

if BX_t, $t > 0$ is stochastically integrable in E and

$$X_t = X_0 + \int_0^t AX_s\,ds + \int_0^t BX_s\,dW_s.$$

In the first integral we are integrating a continuous E-valued function $s \longmapsto AX_s(\omega)$, $s \in [0,t]$, over the bounded interval $[0,t]$ for each $\omega \in \Omega$.

In the case of the Banach algebra $\mathcal{L}(E)$ of bounded linear operators acting on a Banach space E, in Definition 4.6 we interpret bounded linear operators to be acting on $\mathcal{L}(E)$ by left multiplication.

To motivate the following result, suppose that A is the generator of a C_0-semigroup on a Banach space E and B is a bounded linear operator on E. The simplest way to solve

$$du_t = Au_t\,dt + Bu_t\,dt, \quad u_0 = x \in E,$$

in general, is to use the "Dyson series" expansion (2.9).

Theorem 4.7. *Let A and B be bounded linear operators acting on a Banach space E. For each $k = 1, 2, \dots$ and $t > 0$, the $\mathcal{L}(E)$-valued function*

$$(s_1,\dots,s_k) \longmapsto e^{(t-s_k)A}B\dots e^{(s_2-s_1)A}Be^{s_1A}, \quad (s_1,\dots,s_k) \in \Delta_k(t),$$

is W^k-integrable in $\mathcal{L}(E)$ for the uniform operator norm and the series defined by

$$X_t = e^{tA} + \sum_{k=1}^{\infty} \int_{\Delta_k(t)} e^{(t-s_k)A} B \dots e^{(s_2-s_1)A} B e^{s_1 A} \, W^k(ds_1, \dots, ds_k) \qquad (4.4)$$

converges absolutely in $L^2(\mathbb{P}, \mathcal{L}(E))$ for all times t satisfying $0 \le te^{2t\|A\|} < 1/\|B\|^2$, the map $t \longmapsto X_t$, $0 \le te^{2t\|A\|} < 1/\|B\|^2$ is continuous in $L^2(\mathbb{P}, \mathcal{L}(E))$ and the $\mathcal{L}(E)$-valued process X_t, $0 \le te^{2t\|A\|} < 1/\|B\|^2$, represents the solution of the stochastic equation

$$dX_t = AX_t \, dt + BX_t \, dW_t, \quad X_0 = I. \qquad (4.5)$$

Moreover, $e^{(t-s)A} BX_s$, $0 \le s \le t$ is stochastically integrable in $\mathcal{L}(E)$ on the interval $[0, t]$ and X is the unique solution of the stochastic integral equation

$$X_t = e^{tA} + \int_0^t e^{(t-s)A} BX_s \, dW_s \qquad (4.6)$$

and satisfies the bound

$$\|X_t\|_{L^2(\mathbb{P}, \mathcal{L}(E))} \le \sum_{k=0}^{\infty} e^{kt\|A\|} (t^{\frac{1}{2}} \|B\|)^k \qquad (4.7)$$

for all $0 \le te^{2t\|A\|} < 1/\|B\|^2$.

Proof. Although the Itô isometry (3.2) is no longer available, we can appeal to Proposition 4.3 by noting that

$$e^{(t-s_k)A} B \cdots e^{(s_2-s_1)A} B e^{s_1 A}$$
$$= \sum_{m=0}^{\infty} \sum_{j_0+\cdots+j_k=m} \frac{1}{j_0! \dots j_k!} (A^{j_k} B A^{j_{k-1}} \cdots A^{j_1} B A^{j_0}) s_1^{j_0} (s_2 - s_1)^{j_1} \cdots (t - s_k)^{j_k}.$$

Let $a_{m,k}$ denote the sum

$$\sum_{j_0+\cdots+j_k=m} \frac{m!}{j_0! \dots j_k!} \left(k! \int_{\Delta_k(1)} s_1^{2j_0} (s_2 - s_1)^{2j_1} \cdots (1 - s_k)^{2j_k} \, ds_1 \dots ds_k \right)^{\frac{1}{2}}$$

for each $m = 0, 1, \dots$ and $k = 0, 1, \dots$. The integers $j_0, \dots, j_k$ are asumed to be nonnegative. Then $a_{m,k} \le k^m$. It follows that the function $\Phi : [0, t]^k \to \mathcal{L}(E)$ defined by

$$\Phi_k(s_1, \dots, s_k) = e^{(t-s_k)A} B \cdots e^{(s_2-s_1)A} B e^{s_1 A} \quad \text{for } (s_1, \dots, s_k) \in \Delta_k(t)$$

and $\Phi_k(s_1, \dots, s_k) = 0$ for $(s_1, \dots, s_k) \in [0, t]^k \setminus \Delta_k(t)$ belongs to $L^2([0, t]^k) \hat{\otimes}_\pi \mathcal{L}(E)$ and an appeal to equation (4.2) shows that the bound

$$\|\Phi_k\|_{L^2([0,t]^k) \hat{\otimes}_\pi \mathcal{L}(E)} \quad \le \quad \frac{t^{k/2} \|B\|^k}{\sqrt{k!}} \sum_{m=0}^{\infty} a_{m,k} t^m \|A\|^m / m!$$

holds. According to Proposition 4.3, the $\mathcal{L}(E)$-valued function Φ_k is W^k-integrable in $\mathcal{L}(E)$ on $[0, t]^k$ for the uniform operator norm and

$$\left\| \int_{\Delta_k(t)} e^{(t-s_k)A} B \cdots e^{(s_2-s_1)A} B e^{s_1 A} \, W^k(ds_1, \dots, ds_k) \right\|_{L^2(\mathbb{P}, \mathcal{L}(E))}$$
$$\le \sqrt{k!} \|\Phi_k\|_{L^2([0,t]^k) \hat{\otimes}_\pi \mathcal{L}(E)}$$
$$\le t^{k/2} \|B\|^k \sum_{m=0}^{\infty} a_{m,k} t^m \|A\|^m / m!$$
$$\le t^{k/2} \|B\|^k e^{kt\|A\|}.$$

By the ratio test, the sum (4.4) converges absolutely in $L^2(\mathbb{P}, \mathcal{L}(E))$ for $0 \leq te^{2t\|A\|} < 1/\|B\|^2$, the bound (4.7) holds and the map $t \longmapsto X_t$, $0 \leq te^{2t\|A\|} < 1/\|B\|^2$ is continuous in $L^2(\mathbb{P}, \mathcal{L}(E))$.

By substituting the series expansion (4.4) into the right hand side of equation (4.6), we see that X_t, $0 \leq te^{2t\|A\|} < 1/\|B\|^2$ satisfies (4.6) and by an argument analogous to [2, Chapter 6, pp. 150–156], the $\mathcal{L}(E)$-valued process X_t, $0 \leq e^{2t\|A\|} < 1/\|B\|^2$, also satisfies (4.5). □

Corollary 4.8. *Let A and B be bounded linear operators acting on a Banach space E. Let $(\Omega, \mathcal{S}, \mathbb{P})$ be Wiener measure and suppose that $W_t(\omega) = \omega(t)$ for all $t \geq 0$ and $\omega \in \Omega$.*

There exists a progressively measurable $\mathcal{L}(E)$-valued process X_t, $t \geq 0$, continuous for the uniform operator topology, such that

$$dX_t = AX_t\, dt + BX_t\, dW_t, \quad X_0 = I. \tag{4.8}$$

Moreover, if $(\theta_t(\omega))(s) = \omega(t+s)$ for each $\omega \in \Omega$ and $s, t \geq 0$, then X can be chosen such that for every $s, t \geq 0$, the equality

$$X_{t+s} = (X_t \circ \theta_s)X_s \tag{4.9}$$

holds everywhere.

Proof. Let $\delta e^{2\delta\|A\|} < 1/\|B\|^2$. By a standard stopping time argument, we can find a continuous progressively measurable solution of (4.8) for $0 \leq t \leq \delta$ and extend it to $[\delta, 2\delta]$ with the formula $X_{\delta+s} = (X_s \circ \theta_\delta)X_\delta$ for $0 < s < \delta$. Repeating the process, we obtain an operator-norm continuous solution of the stochastic operator equation (4.8) for all $t \geq 0$. By discarding a null set, if necessary, equation (4.9) holds for all rational $s, t \geq 0$, because this is a feature of the representation (4.4). By continuity, equation (4.9) must hold for all $s, t \geq 0$. □

Equation (4.9) shows that the operator solution of the stochastic equation (4.8) is a *random evolution*. We hasten to add that these results can be achieved by many other *ad hoc* methods. For example, a similar result holds if A is the generator of a C_0-semigroup on the Banach space E and B is the generator of a continuous *group* of operators on E [1].

Remark 4.9. For an arbitrary Banach space E, can we ensure that

$$X_t = e^{tA} + \sum_{k=1}^{\infty} \int_{\Delta_k(t)} \left(e^{(t-s_k)A} B \dots e^{(s_2-s_1)A} B e^{s_1 A} \right) x\, W^k(ds_1, \dots, ds_k) \tag{4.10}$$

converges absolutely in $L^2(\mathbb{P}, E)$ for all $x \in E$ and $t > 0$? For example, a series of E-valued martingales converging almost everywhere. We have seen that absolute convergence holds at least if $0 \leq te^{2t\|A\|} < 1/\|B\|^2$.

If E is a Hilbert space, then Theorem 5.1 in the next section is applicable and in this case, the expansion (4.10) is an absolutely convergent orthogonal series in $L^2(\mathbb{P}, E)$ for every $t > 0$, even if A is the generator of a C_0-semigroup. The theory of multiple stochastic integration for Banach spaces developed by J.Maas [18] may prove useful in this context.

5. The stochastic Dyson series in Hilbert space

If we apply stochastic disentangling to the exponential function, we obtain the following result [9, Theorem 4.1] in the Hilbert space case where the Itô isometry is available.

Theorem 5.1. *Let H be a Hilbert space and $x \in H$. Let A and B be bounded linear operators on H. Suppose that $\mu : \mathcal{B}([0,T]) \to [0,\infty)$ is a continuous Borel measure, $f(z_1,z_2) = e^{z_1+z_2}$ for all $z_1, z_2 \in \mathbb{C}$ and $X_t = f_{\mu,W;t}(A,B)x$, $t \in [0,T]$.*

The H-valued random variable X_t is given by the stochastic Dyson series

$$X_t = e^{\mu([0,t])A}x+\sum_{n=1}^{\infty}\int_{\Delta_n(t)} e^{\mu([s_n,t])A}B\dots e^{\mu([s_1,s_2])A}Be^{\mu([0,s_1])A}x\,W^n(ds_1,\dots,ds_n), \tag{5.1}$$

which converges absolutely in $L^2(\mathbb{P},H)$ for all $t>0$. Furthermore, the bounds

$$\left\|\int_{\Delta_n(t)} e^{\mu([s_n,t])A}B\dots e^{\mu([s_1,s_2])A}Be^{\mu([0,s_1])A}x\,W^n(ds_1,\dots,ds_n)\right\|_{L^2(\mathbb{P},H)}$$
$$\le \|x\|e^{\|\mu\|_{[0,t]}\|A\|}\frac{(t^{\frac12}\|B\|)^n}{\sqrt{n!}}, \quad n=1,2,\dots, \tag{5.2}$$

and

$$\|X_t\|_{L^2(\mathbb{P},H)} \le \|x\|e^{\|\mu\|_{[0,t]}\|A\|}\sum_{n=0}^{\infty}\frac{(t^{\frac12}\|B\|)^n}{\sqrt{n!}} \tag{5.3}$$

hold for all $t \ge 0$.

Remark 5.2. a) For $0<\alpha<1$, let $G_\alpha(z)$ be defined by

$$G_\alpha(z) = \sum_{n=0}^{\infty}\frac{z^n}{(n!)^\alpha}, \quad \text{for every } z \in \mathbb{C}.$$

Note that by squaring both sides of the inequality below, the bound

$$e^{t\|B\|^2/2} \le \sum_{n=0}^{\infty}\frac{(t^{\frac12}\|B\|)^n}{\sqrt{n!}} = G_{\frac12}\left(t^{\frac12}\|B\|\right)$$

holds for all $t \ge 0$ and $B \in \mathcal{L}(H)$. The entire function $G_{\frac12} : \mathbb{C} \to \mathbb{C}$ arises in the kernel of the intertwining unitary operator between the Bargmann-Segal representation and the Hardy space representation of the canonical commutation relations of quantum mechanics, see [4, Equation (5.7.32a)].

b) A similar result holds if A is the generator of a C_0-semigroup on H and $B \in \mathcal{L}(H)$. The proof proceeds by approximating A by suitable bounded linear operators in the strong resolvent sense.

We now see that the expansion (5.1) is valid under quite general conditions for unbounded operators A and B acting in Hilbert space H. Such expansions are frequently called *Wiener chaos expansions* in the probability literature. If stochastic integration with respect to Brownian motion W is replaced by integration with respect to Lebesgue measure, then we obtain what is known in the physics literature as a *Dyson series expansion* associated with F. Dyson's fundamental work on quantum field theory. I prefer to emphasise the connection with quantum physics rather than probability theory, so (5.1) is referred to as a *stochastic Dyson series* in this paper.

It is well-known in the probability literature that

$$\mathbb{C} \oplus \bigoplus_{n=1}^{\infty}\left\{\int_{\Delta_n(t)} f(s_1,\dots,s_n)\,W^n(ds_1,\dots,ds_n) : f \in L^2(\Delta_n(t))\right\}$$

is a complete orthogonal decomposition of the space $L^2(\mathbb{P})$ with respect to Wiener measure $\mathbb{P}$, for each $t>0$ [8]. If $P_{t,n}$, $n=0,1,\dots,$ denote the corresponding projection operators, then $f = \sum_{n=0}^{\infty} P_{t,n}f$ is called the *Wiener chaos expansion* of $f \in L^2(\mathbb{P})$ for $t > 0$. There exist unitary operators $U_t : L^2(\mathbb{P}) \to L^2(\mathbb{P})$ (time scales) such that $P_{t,n} = U_t^* P_{1,n} U_t$ for all $t > 0$ and all $n = 1, 2, \dots$.

In the physics literature, a Wiener chaos expansion defines a natural isomorphism with *Fock space*. This isomorphism is fundamental to the study of *Euclidean quantum field theory* [7].

5.1. **Sectorial operators.** Let $0 < \omega < \pi/2$. The sectors $S_{\omega\pm}$ are defined by

$$S_{\omega-} = \{-z : z \in \mathbb{C},\ |\arg z| \le \omega\ \} \cup \{0\}, \quad S_{\omega+} = \{z : z \in \mathbb{C},\ |\arg z| \le \omega\ \} \cup \{0\}.$$

Suppose that $A : \mathcal{D}(A) \longrightarrow H$ is a closed densely defined linear operator acting in the Hilbert space H. The spectrum of A is denoted by $\sigma(A)$. If $0 \le \omega < \pi/2$, then A is said to be of *type* $\omega-$, if $\sigma(A) \subset S_{\omega-}$ and for each $\nu > \omega$, there exists $C_\nu > 0$ such that

$$\|(zI - A)^{-1}\| \le C_\nu |z|^{-1}, \quad z \notin S_{\nu-}. \tag{5.4}$$

An operator A is of type $\omega-$ if and only if it is the generator of an analytic semigroup e^{zA} in the region $|\arg z| < \pi/2 - \omega$ so that for each $\nu > \omega$, there exists $C_\nu > 0$ such that $\|e^{zA}\| \le C_\nu$ for all $z \in \mathbb{C}$ with $|\arg z| < \pi/2 - \nu$. [22, §2.5]. An operator A is of type $\omega+$ if and only if $-A$ is the generator of an analytic semigroup in the region $|\arg z| < \pi/2 - \omega$.

Let $T > 0$. Let H be a real Hilbert space, A an operator of type $\omega-$, $\omega < \pi/2$ and let V be a real separable Banach space with norm $\|\cdot\|_V$ such that $\mathcal{D}(A) \subset V \subset H$ with continuous inclusions such that $B : V \to H$ is bounded. Suppose that there exists $c_1 > 0$ such that

$$\int_0^T \|e^{tA}x\|_V^2\, dt \le c_1^2 \|x\|_H^2 \tag{5.5}$$

for all $x \in \mathcal{D}(A)$.

Lemma 5.3. *Let $c_1 > 0$. The inequality (5.5) holds if and only if*

$$\int_0^T \int_0^t \|e^{(t-s)A} g(s)\|_V^2\, dsdt \le c_1^2 \int_0^T \|g(t)\|_H^2\, dt \tag{5.6}$$

for all H-valued simple functions g.

Proof. The inequality (5.5) holds for all $x \in H$ because there exists $c > 0$ such that $\|Ae^{tA}x\| \le c\|x\|/t$ for all $t > 0$. Moreover, if the bound (5.5) holds, then

$$\begin{aligned}
\int_0^T \int_0^t \|e^{(t-s)A} g(s)\|_V^2\, dsdt &= \int_0^T \int_s^T \|e^{(t-s)A} g(s)\|_V^2\, dtds \\
&= \int_0^T \int_0^{T-s} \|e^{tA} g(s)\|_V^2\, dtds \\
&\le \int_0^T \int_0^T \|e^{tA} g(s)\|_V^2\, dtds \\
&\le c_1^2 \int_0^T \|g(s)\|_H^2\, ds, \qquad \text{by (5.5).}
\end{aligned}$$

Now suppose that (5.6) holds. By taking $g = \chi_R.x$, $x \in H$, we obtain

$$\frac{\int_R \int_0^{T-s} \|e^{tA}x\|_V^2\, dtds}{|R|} \le c_1^2 \|x\|_H^2$$

for all finite unions R of intervals. Because $s \mapsto \int_0^{T-s} \|e^{tA}x\|_V^2\, dt$ is continuous, this is only possible if equation (5.5) holds. ∎

Theorem 5.4. *Suppose that the estimate (5.5) holds for all $x \in H$ and $\|Bx\|_H \le c_2\|x\|_V$ for all $x \in V$. If $c_1c_2 < 1$, then the stochastic Dyson series*

$$e^{tA}u_0 + \sum_{k=1}^{\infty} \int_0^t \int_0^{s_k} \cdots \int_0^{s_2} \left[e^{(t-s_k)A} B e^{(s_k - s_{k-1})A} \cdots B e^{s_1 A} u_0 \right] dW_{s_1} \ldots dW_{s_k} \quad (5.7)$$

converges in $L^2(\mathbb{P}; H)$ for every $0 < t \le T$ and every $u_0 \in H$.

Proof. Suppose that the estimate (5.5) holds for all $x \in H$ and $\|Bx\|_H \le c_2\|x\|_V$ for all $x \in V$. The estimate (5.6) in Lemma 5.3 is also valid for all square integrable H-valued functions g by continuity. Then by the Itô isometry (3.2), we have

$$\begin{aligned}
&\mathbb{E}\left\| \int_0^t \int_0^{s_k} \cdots \int_0^{s_2} \left[e^{(t-s_k)A} B e^{(s_k - s_{k-1})A} \cdots B e^{s_1 A} u_0 \right] dW_{s_1} \ldots dW_{s_k} \right\|_H^2 \\
&= \int_0^t \int_0^{s_k} \cdots \int_0^{s_2} \left\| e^{(t-s_k)A} B e^{(s_k - s_{k-1})A} \cdots B e^{s_1 A} u_0 \right\|_H^2 ds_1 \ldots ds_k \\
&\le C^2 \int_0^t \int_0^{s_k} \cdots \int_0^{s_2} \left\| B e^{(s_k - s_{k-1})A} \cdots B e^{s_1 A} u_0 \right\|_H^2 ds_1 \ldots ds_k \\
&\le C^2 c_2^2 \int_0^t \int_0^{s_k} \cdots \int_0^{s_2} \left\| e^{(s_k - s_{k-1})A} B e^{(s_{k-1} - s_{k-2})A} \cdots B e^{s_1 A} u_0 \right\|_V^2 ds_1 \ldots ds_k \\
&\le C^2 (c_1c_2)^2 \int_0^t \int_0^{s_{k-1}} \cdots \int_0^{s_2} \left\| B e^{(s_{k-1} - s_{k-2})A} \cdots B e^{s_1 A} u_0 \right\|_H^2 ds_1 \ldots ds_{k-1} \\
&\vdots \\
&\le C^2 (c_1c_2)^{2(k-1)} \int_0^t \left\| B e^{s_1 A} u_0 \right\|_H^2 ds_1 \\
&\le C^2 (c_1c_2)^{2k} \|u_0\|_H^2.
\end{aligned}$$

Here we have used the bound $\|e^{sA}\| \le C$ for all $s \ge 0$. If $c_1c_2 < 1$, then the sum (5.7) converges in $L^2(\mathbb{P}; H)$ for every $0 < t \le T$ and every $u_0 \in H$. □

Suppose that the conditions of Theorem 5.4 hold. For each $u_0 \in H$ and $0 < t \le T$, the H-valued random variable defined by the series (5.7) is denoted by $e^{A+B}_{\lambda,W;t}u_0$. We define $e^{A+B}_{\lambda,W;0}u_0 = u_0$. The notation is suggested by comparison with equation (5.1) which is valid for *bounded* linear operators A and B in Hilbert space. Lebesgue measure λ is associated with the operator A and stochastic integration with respect to W is associated with the operator B in disentangling over the interval $[0, t]$.

The mapping $u_0 \longmapsto e^{A+B}_{\lambda,W;t}u_0$ is an element of the space $\mathcal{L}(H, L^2(\mathbb{P}, H))$ of *random linear operators* [25] which we denote by $e^{A+B}_{\lambda,W;t}$. It is easy to see that

$$t \longmapsto e^{A+B}_{\lambda,W;t}, \quad 0 \le t \le T,$$

is a continuous map from the closed interval $[0, T]$ into $\mathcal{L}(H, L^2(\mathbb{P}, H))$.

The following corollary follows from the observation that the stochastic Dyson series (5.7) is the solution obtained from the contraction mapping principle for the stochastic equation (5.8) below, see [6, Lemma 2.2]. By a *mild solution*, we mean an H-valued solution X_t, $t \ge 0$, of the stochastic equation

$$X_t = e^{tA}x + \int_0^t e^{(t-s)A} B X_t \, dW_s.$$

A general treatment of stochastic equations in Hilbert space is given in [2].

Corollary 5.5. *Suppose that the conditions of Theorem 5.4 hold. Then for each* $x \in H$*, the* H*-valued process*

$$t \longmapsto e^{A+B}_{\lambda,W;t}x, \quad 0 \le t \le T,$$

is the unique mild solution of the stochastic equation

$$dX_t = AX_t\,dt + BX_t\,dW_t, \quad X_0 = x. \tag{5.8}$$

For the definition of fractional powers of operators, see [22], [15, Appendix]. The possibility of different choices of the space V are studied in [6, §3.1].

Corollary 5.6. *Suppose that there exists* $c_1 > 0$ *such that*

$$\int_0^\infty \|e^{tA}x\|_V^2\,dt \le c_1^2\|x\|_H^2 \tag{5.9}$$

for all $x \in H$ *and* $\|Bx\|_H \le c_2\|x\|_V$ *for all* $x \in V$*. If* $c_1c_2 < 1$*, then there exists* $K > 0$ *such that* $\|e^{A+B}_{\lambda,W;t}x\|_{L^2(\mathbb{P},H)} \le K\|x\|_H$ *for all* $t \ge 0$*.*

Furthermore, suppose that A *is a one-to-one operator of type* $\omega-$ *and the norm* $\|\cdot\|_V$ *is defined by* $\|x\|_V = \|(-A)^{\frac{1}{2}}x\|_H$*. Then for every* $t > 0$*, there exists* $K_t > 0$ *such that* $\|(-A)^{\frac{1}{2}}e^{A+B}_{\lambda,W;t}x\|_{L^2(\mathbb{P},H)} \le K_t\|x\|_H$ *for all* $x \in H$ *and* $t \longmapsto e^{A+B}_{\lambda,W;t}x$ *is a predictable continuous process with values in* $L^2(\mathbb{P},\mathcal{D}((-A)^{\frac{1}{2}}))$ *for* $t > 0$*.*

Proof. Under condition (5.9), the bound giving the convergence of (5.7) is uniform in $T > 0$, from which the uniform bound for $t \longmapsto e^{A+B}_{\lambda,W;t}x$, $t > 0$, is obtained.

For the last statement, it suffices to apply Lemma 5.3 to note that

$$\begin{aligned}
&\mathbb{E}\left\|\int_0^t\int_0^{s_k}\cdots\int_0^{s_2}\left[(-A)^{\frac{1}{2}}e^{(t-s_k)A}Be^{(s_k-s_{k-1})A}\cdots Be^{s_1A}u_0\right]dW_{s_1}\dots dW_{s_k}\right\|_H^2\\
&= \int_0^t\int_0^{s_k}\cdots\int_0^{s_2}\left\|(-A)^{\frac{1}{2}}e^{(t-s_k)A}Be^{(s_k-s_{k-1})A}\cdots Be^{s_1A}u_0\right\|_H^2 ds_1\dots ds_k\\
&= \int_0^t\int_0^{s_k}\cdots\int_0^{s_2}\left\|e^{(t-s_k)A}Be^{(s_k-s_{k-1})A}\cdots Be^{s_1A}u_0\right\|_V^2 ds_1\dots ds_k\\
&\le c_1^2\int_0^t\int_0^{s_k}\cdots\int_0^{s_2}\left\|Be^{(s_k-s_{k-1})A}\cdots Be^{s_1A}u_0\right\|_H^2 ds_1\dots ds_k,
\end{aligned}$$

and then continue as in the proof of Theorem 5.4. The first term of (5.7) is treated by noting that $e^{tA}x \in \mathcal{D}(A)$ for every $x \in H$ and $t > 0$ [22, §2.5]. □

The condition $c_1c_2 < 1$ can be relaxed if we only require the sum (5.7) to converge absolutely for small times [6]. The solution of (5.8) is then obtained by piecing together the solutions obtained from the stochastic Dyson series (5.7).

6. Stochastic Functional Calculus

The significance of Corollary 5.6 above is that the bound (5.9) required for the existence of the solution $t \longmapsto e^{A+B}_{\lambda,W;t}x$, $t \ge 0$, of the stochastic equation (5.8) is a type of *square function estimate* for the operator A. It has been known since the work of A. McIntosh [19] that such estimates are associated with the existence of an H^∞-functional calculus for A. Furthermore, it has been shown in [3, Theorem 6.5] that the regularity of solutions of simple stochastic equations involving the operator A implies that A has an H^∞-functional calculus.

A good reference for many of the results we need for an operator acting in Hilbert space is [15, Chap. 2]. We now set down the basic definitions.

6.1. H^∞ **functional calculus.** Let $0 < \omega < \pi/2$ and suppose that T is an operator of type $\omega-$ as defined at the beginning of Section 4.

Then the bounded linear operator $f(T)$ is defined by the Riesz-Dunford formula

$$f(T) = \frac{1}{2\pi i}\int_C (zI - T)^{-1} f(z)\, dz. \tag{6.1}$$

for any function f satisfying the bounds

$$|f(z)| \le K_\nu \frac{|z|^s}{1+|z|^{2s}}, \quad z \in S_\nu^\circ.$$

The contour C can be taken to be $\{z \in \mathbb{C} : \Re(z) \le 0,\ |\Im(z)| = -\tan\theta.\Re(z)\ \}$, with $\omega < \theta < \nu$. The integral (6.1) converges as a Bochner integral in the uniform norm due to the estimate (5.4) for the resolvent $z \longmapsto (zI - T)^{-1}$ of T.

The operator T of type $\omega-$ is said to have a *bounded H^∞-functional calculus* if for each $\omega < \nu < \pi/2$, there exists an algebra homomorphism $f \longmapsto f(T)$ from $H^\infty(S_{\nu-}^\circ)$ to $\mathcal{L}(H)$ agreeing with (6.1) and a positive number C_ν such that $\|f(T)\| \le C_\nu \|f\|_\infty$ for all $f \in H^\infty(S_\nu^\circ)$. The following result is from [19], see also [15, Theorem 11.9].

Theorem 6.1. *Suppose that T is a one-to-one operator of type $\omega-$ in H. Then T has a bounded H^∞-functional calculus if and only if for every $\omega < \nu < \pi/2$, there exists $c_\nu > 0$ such that T and its adjoint T^* satisfy the square function estimates*

$$\int_0^\infty \|\psi_t(T)u\|^2 \frac{dt}{t} \le c_\nu \|u\|^2, \quad u \in H, \tag{6.2}$$

$$\int_0^\infty \|\psi_t(T^*)u\|^2 \frac{dt}{t} \le c_\nu \|u\|^2, \quad u \in H, \tag{6.3}$$

for some function (every function) $\psi \in H^\infty(S_{\nu-}^\circ)$, which satisfies

$$\int_0^\infty \psi^2(-t)\frac{dt}{t} = 1, \text{ and} \tag{6.4}$$

$$|\psi(z)| \le K_\nu \frac{|z|^s}{1+|z|^{2s}}, \quad z \in S_{\nu-}^\circ, \tag{6.5}$$

for some $s > 0$. Here $\psi_t(z) = \psi(tz)$ for $z \in S_{\nu-}^\circ$.

For the function $\psi(z) = Cz^{\frac{1}{2}}e^z$ with $C > 0$ chosen such that (6.4) holds,

$$\int_0^\infty \|\psi_t(T)u\|^2 \frac{dt}{t} = C^2 \int_0^\infty \|(-tT)^{\frac{1}{2}} e^{tT}u\|^2 \frac{dt}{t}$$
$$= C^2 \int_0^\infty \|(-T)^{\frac{1}{2}} e^{tT}u\|^2 dt.$$

With this choice for ψ, the bound (6.2) is equivalent to the bound (5.9) with $\|x\|_V = \|(-T)^{\frac{1}{2}}x\|$ for $x \in \mathcal{D}((-T)^{\frac{1}{2}})$.

6.2. **Random Resolvents.** Suppose that $T : \mathcal{D}(T) \to H$ is a closed linear map defined in the Hilbert space H. Then the *resolvent* $R(\zeta)$, $\zeta \in \rho(T)$, of T is the bounded linear map defined by $R(\zeta) = (\zeta I - T)^{-1}$ for all $\zeta \in \mathbb{C}$ belonging to the set $\rho(T)$ for which the inverse is defined. If T is the generator of a C_0-semigroup e^{tT}, $t \ge 0$, then we also have

$$(\zeta I - T)^{-1} = \int_0^\infty e^{-\zeta t} e^{tT}\, dt \tag{6.6}$$

for all $\zeta \in \mathbb{C}$ in some right half-plane. We adopt the right-hand side of equation (6.6) as the definition of a resolvent in the setting of *stochastic disentangling*. In the case that A and B are bounded linear operators, $\beta > 0$ and $T = A + B$, the disentangling $e^{A+B}_{\beta dt,\beta dt}$ of the exponential $e^{\tilde{A}+\tilde{B}}$ with respect to the pair of measures $(\beta dt, \beta dt)$ on the interval $[0,1]$ is $e^{\beta(A+B)}$ [11, Proposition 5.5], so that equation (6.6) becomes

$$(\zeta I - (A+B))^{-1} = \int_0^\infty e^{-\zeta\beta} e^{A+B}_{\beta dt,\beta dt}\, d\beta. \tag{6.7}$$

For the stochastic disentangling, we replace $(\beta dt, \beta dt)$ by the pair $(\beta dt, dW_{\beta t})$ with $t \longmapsto W_{\beta t}$, $t \geq 0$, the Wiener process such that $W_{\beta t}$ is a Gaussian random variable with mean zero and variance βt for each $t > 0$. Because $\beta^{-\frac{1}{2}} W_{\beta t}$ has mean zero and variance t, a change of variables in the expansion (5.7) shows that for each $x \in H$, we have

$$e^{A+B}_{\lambda,W;\beta t} x = e^{A+B}_{\beta dt, dW_{\beta t};t} x = e^{\beta A+\sqrt{\beta}B}_{\lambda,W;t} x \qquad \mathbb{P}\text{-a.e.}.$$

To see that these equalities hold, we look at one term

$$\int_0^{\beta t}\int_0^{s_k}\cdots\int_0^{s_2}\left[e^{(\beta t - s_k)A} B e^{(s_k - s_{k-1})A}\cdots B e^{s_1 A} u_0\right] dW_{s_1}\ldots dW_{s_k}$$

from the stochastic Dyson series (5.7). Using the substitution $s_j = \beta t_j$, $j = 1,\ldots,k$, we obtain

$$\begin{aligned}
&\int_0^{\beta t}\int_0^{s_k}\cdots\int_0^{s_2}\left[e^{(\beta t - s_k)A} B e^{(s_k - s_{k-1})A}\cdots B e^{s_1 A} u_0\right] dW_{s_1}\ldots dW_{s_k}\\
&= \int_0^{t}\int_0^{t_k}\cdots\int_0^{t_2}\left[e^{(t - t_k)\beta A} B e^{(t_k - t_{k-1})\beta A}\cdots B e^{t_1 \beta A} u_0\right] dW_{\beta t_1}\ldots dW_{\beta t_k}\\
&= \int_0^{t}\int_0^{t_k}\cdots\int_0^{t_2}\left[e^{(t - t_k)\beta A} (\sqrt{\beta}B) e^{(t_k - t_{k-1})\beta A}\cdots (\sqrt{\beta}B) e^{t_1 \beta A} u_0\right] dW_{t_1}\ldots dW_{t_k}
\end{aligned}$$

$\mathbb{P}$-almost everywhere. The validity of the substitution can be checked on simple functions from the definition of multiple stochastic integrals.

In the case of unbounded linear operators defined in Hilbert space H, we adopt the following assumptions.

1) A is an operator of type $\omega-$ for $0 < \omega < \pi/2$.
2) There exists a real separable Banach space V with norm $\|\cdot\|_V$ such that $\mathcal{D}(A) \subset V \subset H$ and $B : V \to H$ is a bounded linear operator with $\|Bx\|_H \leq c_B\|x\|_V$ for all $x \in V$.
3) Let $A_\theta = e^{i\theta}A$ for $0 \leq |\theta| < \pi/2 - \omega$. For each $0 \leq |\theta| < \pi/2 - \omega$, there exists $m_\theta > 0$ such that
$$\int_0^\infty \|e^{tA_\theta}x\|_V^2\, dt \leq m_\theta^2 \|x\|_H^2 \tag{6.8}$$
for all $x \in H$.
4) There exists $0 < \delta < \pi/2 - \omega$ such that $\sup_{|\theta|\leq\delta} m_\theta c_B < 1$.

According to Corollary 5.6, the random process $t \longmapsto e^{A+B}_{\lambda,W;t}$, $t \geq 0$, is uniformly bounded in $\mathcal{L}(H, L^2(\mathbb{P}, H))$ by a constant K. If the pair (A, B) of linear operators satisfies conditions 1) – 4) above, then so does the pair $(\beta A, \sqrt{\beta}B)$ for any $\beta > 0$, so the mapping $(\beta, t) \longmapsto e^{\beta A+\sqrt{\beta}B}_{\lambda,W;t}$, $\beta, t \geq 0$, is also uniformly bounded in $\mathcal{L}(H, L^2(\mathbb{P}, H))$ by K. Consequently, the following definition makes sense.

Definition 6.2. Let H be a Hilbert space and suppose that the conditions 1) – 4) above hold. The *stochastic resolvent* $R_{\lambda,W;t}(z; A+B)$, $t \geq 0$, of the process $t \longmapsto e^{A+B}_{\lambda,W;t}$, $t \geq 0$, is the $\mathcal{L}(H, L^2(\mathbb{P}, H))$-valued mapping $t \longmapsto R_{\lambda,W;t}(z; A+B)$, $t \geq 0$, given by

$$R_{\lambda,W;t}(z; A+B)x \quad = \quad \int_0^\infty e^{-z\beta} e^{A+B}_{\lambda,W;\beta t} x\, d\beta \tag{6.9}$$

for all $x \in H$, $t \geq 0$ and $\Re z > 0$.

We denote by the same symbol $R_{\lambda,W;t}(z; A + B)$ the analytic continuation of (6.9) as an element of $\mathcal{L}(H, L^2(\mathbb{P}, H))$ to the left half-plane. We obtain an $\mathcal{L}(H, L^2(\mathbb{P}, H))$-valued function of time t because we are considering disentangling over an interval $[0, t]$ as in Section 2.

Appealing to the orthogonality property 2) of multiple stochastic integrals, we see that (5.7) is an orthogonal expansion in H-valued random variables. According to formula (6.9), the stochastic resolvent $R_{\lambda,W;t}(z; A+B)x$ also has an orthogonal expansion in H-valued random variables. We use this expansion in order to establish the following bound.

Lemma 6.3. *Suppose that conditions 1)-4) above hold. Then the $L^2(\mathbb{P}, H)$-valued function $z \longmapsto R_{\lambda,W;t}(z; A+B)x$ is holomorphic in $\mathbb{C} \setminus S_{\delta-}$ for all $t > 0$ and $x \in H$ and for each $\pi/2 - \delta < \mu < \pi/2$ there exists $C_\mu > 0$ such that*

$$\|R_{\lambda,W;t}(z; A + B)x\|_{L^2(\mathbb{P},\mathcal{H})} \leq \frac{C_\mu}{|z|}\|x\|, \quad z \in \mathbb{C} \setminus S_{\mu-} \tag{6.10}$$

for all $x \in \mathcal{H}$ and $t > 0$.

Proof. Let $\sqrt{z}$ denote the square root of z with positive real part. Under conditions 1) - 4), replacing A by zA and B by $\sqrt{z}B$ in the expansion (5.7), we obtain a uniformly bounded $L^2(\mathbb{P}, H)$-valued holomorphic function $z \longmapsto e^{zA+\sqrt{z}B}_{\lambda,W;t}x$ in $S^\circ_{\delta+}$ for each $t > 0$ and $x \in H$.

For each $0 < \mu < \pi/2$, let $\Xi_{\pm\mu} = \{se^{\pm i\mu} : s \geq 0\}$. Then for $0 < \nu < \delta$, by the vector version of Cauchy's Theorem we have

$$R_{\lambda,W;t}(z, A + B) = \int_{\Xi_{-\nu}} e^{-z\zeta} e^{\zeta A+\sqrt{\zeta}B}_{\lambda,W;t} x \, d\zeta \tag{6.11}$$

if $\Re(ze^{-i\nu}) > 0$ and

$$R_{\lambda,W;t}(z, A + B) = \int_{\Xi_{\nu}} e^{-z\zeta} e^{\zeta A+\sqrt{\zeta}B}_{\lambda,W;t} x \, d\zeta \tag{6.12}$$

if $\Re(ze^{i\nu}) > 0$. Because $\pi/2 - \delta < \mu < \pi/2$, we can choose $0 < \nu < \delta$ such that $\pi/2 - \nu < \mu < \pi/2$. Then the bound (6.10) follows for all $z \in \mathbb{C} \setminus S_{\mu-}$ with $\Im z \geq 0$ from the representation (6.11) and the uniform boundedness of $z \longmapsto e^{zA+\sqrt{z}B}_{\lambda,W;t}x$ in $S^\circ_{\delta+}$. For $\Im z < 0$, the representation (6.12) is used. □

For any holomorphic function φ in a sector $S^\circ_{\nu-}$ with $\pi/2 - \delta < \nu < \pi/2$ and satisfying the bound

$$|\varphi(z)| \quad \leq \quad M_\nu \frac{|z|^s}{1 + |z|^{2s}}, \quad z \in S^\circ_{\nu-}, \tag{6.13}$$

for some $M_\nu, s > 0$ we may define the integral

$$\varphi_{\lambda,W;t}(A + B)x = \frac{1}{2\pi i}\int_C \varphi(z)R_{\lambda,W;t}(z; A + B)x \, dz, \quad x \in H, \tag{6.14}$$

in $L^2(\mathbb{P}, H)$ for the contour $C = \{z \in \mathbb{C} : |\Im(z)| = -\tan\mu.\Re(z),\ \Re(z) \leq 0\}$ taken anticlockwise around $S_{\delta-}$ for $\pi/2 - \delta < \mu < \nu$. By Lemma 6.3 and the estimate (6.13), the contour integral converges as a Bochner integral in $L^2(\mathbb{P}, H)$ and $\varphi_{\lambda,W;t}(A+B)x$ admits an orthogonal expansion in H-valued random variables. In the case that $B = 0$, we obtain the Riesz-Dunford formula (6.1).

The following result says that the random part $\varphi_{\lambda,W;t}(A+B)-\varphi(A)$ of $\varphi_{\lambda,W;t}(A+B)$ has an H^∞-bound under the assumptions 1)-4) above.

Theorem 6.4. *Suppose that conditions 1)-4) above hold. Then for every $\pi/2-\delta < \nu < \pi/2$, there exists $C_\nu > 0$ such that*

$$(\mathbb{E}\|\varphi_{\lambda,W;t}(A+B)x - \varphi(A)x\|^2)^{\frac{1}{2}} \leq C_\nu \|\varphi\|_\infty \|x\|$$

for every holomorphic function φ on $S^\circ_{\nu-}$ satisfying the bound (6.13) and every $t > 0$.

Proof. For each $0 < \mu < \pi/2$, let $\Xi_{\pm\mu} = \{se^{\pm i\mu} : s \geq 0\}$ and

$$\Gamma_{\mu,1} = \{se^{i\mu} : -\infty \leq s \leq 0\}, \quad \Gamma_{\mu,2} = \{-se^{-i\mu} : 0 \leq s < \infty\}.$$

Then for $0 < \nu < \delta$, by the vector version of Cauchy's Theorem $R_{\lambda,W;t}(z, A+B)$ is given by equation (6.11) if $\Re(ze^{-i\nu}) > 0$ and equation (6.12) if $\Re(ze^{-i\nu}) > 0$. Let φ be a uniformly bounded holomorphic function in a sector $S^\circ_{\nu-}$ with $\pi/2-\delta < \nu < \pi/2$. Let $\pi/2-\delta < \mu < \nu$. Then

$$2\pi i\varphi_{\lambda,W;t}(A+B)x = \int_{\Gamma_{\mu,1}} \varphi(z)R_{\lambda,W;t}(z, A+B)x\,dz + \int_{\Gamma_{\mu,2}} \varphi(z)R_{\lambda,W;t}(z, A+B)x\,dz,$$

if the integrals converge. The Laplace transform

$$\mathcal{L}\varphi(\zeta) = \begin{cases} -\int_{\Gamma_{\mu,1}} e^{-z\zeta}\varphi(z)\,dz, & \Re(\zeta e^{i\mu}) < 0 \\ \int_{\Gamma_{\mu,2}} e^{-z\zeta}\varphi(z)\,dz, & \Re(\zeta e^{-i\mu}) < 0 \end{cases}$$

of φ is defined for $\pi/2-\nu < |\arg\zeta| < \pi$.

From equation (5.7), the random part of $R_{\lambda,W;t}(\zeta, A+B)x$ is given by

$$\tilde{R}_{\lambda,W;t}(\zeta, A+B)x = R_{\lambda,W;t}(\zeta, A+B)x - (\zeta I - A)^{-1}x$$

In order to estimate

$$\mathbb{E}\left\|\int_{\Gamma_{\mu,2}} \varphi(\zeta)\tilde{R}_{\lambda,W;t}(\zeta, A+B)x\,d\zeta\right\|^2, \tag{6.15}$$

we apply the Itô isometry and consider the sum

$$\sum_{n=1}^{\infty}\int_0^t\int_0^{t_n}\cdots\int_0^{t_1}\left\|\int_{\Xi_{-\theta}} \mathcal{L}\varphi(\zeta)e^{\zeta A(t-t_n)}(\zeta^{\frac{1}{2}}B)e^{\zeta A(t_n-t_{n-1})} \cdots \zeta^{\frac{1}{2}}B)e^{\zeta At_1}x d\zeta\right\|^2 dt_1\ldots dt_n \tag{6.16}$$

for $\pi/2-\mu < \theta < \pi/2-\omega$. For each such θ, there exists $K_\theta > 0$ such that

$$|\mathcal{L}\varphi(\zeta)| \leq \frac{K_\theta}{|\zeta|}\|\varphi\|_\infty, \quad \zeta \in \Xi_{-\theta},$$

for every uniformly bounded holomorphic function in a sector $S^\circ_{\nu-}$. It suffices to show that the sum

$$\left(\sum_{n=1}^{\infty}\int_0^t\int_0^{t_n}\cdots\int_0^{t_1}\left(\int_{\Xi_{-\theta}} \frac{\|\varphi\|_\infty}{|\zeta|}\|e^{\zeta A(t-t_n)}(\zeta^{\frac{1}{2}}B)e^{\zeta A(t_n-t_{n-1})} \cdots \zeta^{\frac{1}{2}}B)e^{\zeta At_1}x\||d\zeta|\right)^2 dt_1\ldots dt_n\right)^{\frac{1}{2}} \tag{6.17}$$

converges. The notation $|d\zeta|$ means arclength measure. Then an application of the Fubini-Tonelli Theorem shows that (6.15) is equal to (6.16) and is estimated by the expression (6.17). Here we don't actually appeal to the bound (6.13) which is only needed to make sense of $\varphi(A)$.

Applying Minkowski's inequality, (6.17) is estimated by

$$\|\varphi\|_\infty \int_{\Xi_{-\theta}} \Big(\sum_{n=1}^{\infty} \int_0^t \int_0^{t_n} \cdots \int_0^{t_1} \Big(\|e^{\zeta A(t-t_n)}(\zeta^{\frac12}B)e^{\zeta A(t_n-t_{n-1})} \cdots \zeta^{\frac12}B)e^{\zeta A t_1}x\|^2 dt_1 \dots dt_n \Big)^{\frac12} \frac{|d\zeta|}{|\zeta|}$$

$$= \|\varphi\|_\infty \int_0^{\infty} \Big(\sum_{n=1}^{\infty} \int_0^t \int_0^{t_n} \cdots \int_0^{t_1} \Big(\|e^{sA_{-\theta}(t-t_n)}(s^{\frac12}B)e^{sA_{-\theta}(t_n-t_{n-1})} \cdots s^{\frac12}B)e^{sA_{-\theta}t_1}x\|^2 dt_1 \dots dt_n \Big)^{\frac12} \frac{ds}{s}$$

$$= \|\varphi\|_\infty \int_0^{\infty} \Big(\sum_{n=1}^{\infty} \int_0^{st} \int_0^{s_n} \cdots \int_0^{s_1} \Big(\|e^{A_{-\theta}(st-s_n)}Be^{A_{-\theta}(s_n-s_{n-1})} \cdots Be^{A_{-\theta}s_1}x\|^2 ds_1 \dots ds_n \Big)^{\frac12} \frac{ds}{s}, \qquad [s_j = st_j \text{ for } j = 1, \dots, n]$$

$$= \|\varphi\|_\infty \int_0^{\infty} \Big(\sum_{n=1}^{\infty} \int_0^{r} \int_0^{s_n} \cdots \int_0^{s_1} \Big(\|e^{A_{-\theta}(r-s_n)}Be^{A_{-\theta}(s_n-s_{n-1})} \cdots Be^{A_{-\theta}s_1}x\|^2 ds_1 \dots ds_n \Big)^{\frac12} \frac{dr}{r}, \qquad [r = st]$$

We would like to know that this integral is finite. Split it into $r \geq 1$ and $r < 1$. Applying the Cauchy-Schwarz inequality for $r \geq 1$, we obtain

$$\|\varphi\|_\infty \Big(\int_1^{\infty} \sum_{n=1}^{\infty} \int_0^{r} \int_0^{s_n} \cdots \int_0^{s_1} \Big(\|e^{A_{-\theta}(r-s_n)}Be^{A_{-\theta}(s_n-s_{n-1})} \cdots Be^{A_{-\theta}s_1}x\|^2 ds_1 \dots ds_n dr \Big)^{\frac12}.$$

Each term

$$\int_1^{\infty} \int_0^{r} \int_0^{s_n} \cdots \int_0^{s_1} \|e^{A_{-\theta}(r-s_n)}Be^{A_{-\theta}(s_n-s_{n-1})} \cdots Be^{A_{-\theta}s_1}x\|^2 ds_1 \dots ds_n dr$$

in the sum is bounded by

$$\int_0^{\infty} \int_0^{r} \int_0^{s_n} \cdots \int_0^{s_1} \|e^{A_{-\theta}(r-s_n)}Be^{A_{-\theta}(s_n-s_{n-1})} \cdots Be^{A_{-\theta}s_1}x\|^2 ds_1 \dots ds_n dr. \tag{6.18}$$

For every $t > 0$ and $y \in H$, the vector $e^{tA_{-\theta}}y$ is an element of $\mathcal{D}(A)$. But $\mathcal{D}(A) \subset V \subset H$ with continuous embeddings, so there exists $C > 0$ such that (6.18) is bounded by

$$C^2 \int_0^{\infty} \int_0^{r} \int_0^{s_n} \cdots \int_0^{s_1} \|e^{A_{-\theta}(r-s_n)}Be^{A_{-\theta}(s_n-s_{n-1})} \cdots Be^{A_{-\theta}s_1}x\|_V^2 ds_1 \dots ds_n dr. \tag{6.19}$$

Applying the inequality (6.8) and Lemma 5.3, the integral (6.18) is bounded by

$$C^2 m_{-\theta}^2 \int_0^{\infty} \int_0^{s_n} \cdots \int_0^{s_1} \|Be^{A_{-\theta}(s_n-s_{n-1})} \cdots Be^{A_{-\theta}s_1}x\|^2 ds_1 \dots ds_n$$

$$\leq C^2 m_{-\theta}^2 c_B^2 \int_0^{\infty} \int_0^{s_n} \cdots \int_0^{s_1} \|e^{A_{-\theta}(s_n-s_{n-1})}B \cdots Be^{A_{-\theta}s_1}x\|_V^2 ds_1 \dots ds_n.$$

Repeating the process, we obtain the bound

$$C^2(m_{-\theta}c_B)^{2n}\int_0^\infty \|e^{A_{-\theta}s_1}x\|_V^2\,ds_1 \le C^2(m_{-\theta}c_B)^{2n}m_{-\theta}\|x\|^2.$$

By condition 4), $m_{-\theta}c_B < 1$ and so the integral over $r \ge 1$ converges.

For $r < 1$, we can similarly estimate

$$\int_0^r\int_0^{s_n}\cdots\int_0^{s_1}\|e^{A_{-\theta}(r-s_n)}Be^{A_{-\theta}(s_n-s_{n-1})}\cdots Be^{A_{-\theta}s_1}x\|^2ds_1\ldots ds_n$$

to get a bound

$$C'\|\varphi\|_\infty\int_0^1\left(\sum_{n=1}^\infty(m_{-\theta}c_B)^{2n-2}\int_0^r\|x\|^2ds_n\right)^{\frac12}\frac{dr}{r}$$

which is finite. Combining the estimates for $r \ge 1$ and $r < 1$, we obtain the required bound for (6.17) and together with a similar argument for the integral over $\Gamma_{\mu,1}$, this finishes the proof of the theorem. □

Remark 6.5. The above result also holds if we replace 4) by the condition

4′) $$\sup_{\|x\|\le1,|\theta|\le\delta}\int_0^\infty\|Be^{tA_\theta}x\|^2\,dt<1.$$

Combined with the characterisation of Hilbert space operators with an H^∞-functional calculus [19] we have the following result establishing the existence of a *stochastic functional calculus* for "$A+B$".

Theorem 6.6. *Suppose that A is a one-to-one operator of type $\omega-$ in H such that A has an H^∞-functional calculus on $S_{\omega-}$. Let $V=\mathcal{D}((-A)^{\frac12})$ with $\|x\|_V=\|(-A)^{\frac12}x\|$ for $x\in V$.*

Then for every $\omega<\nu<\pi/2$, there exists $b_\nu>0$ such that for every bounded linear map $B:V\to H$ with operator norm $\|B\|_{\mathcal{L}(V,H)}<b_\nu$, there exists a linear map

$$\varphi\longmapsto\varphi_{\lambda,W;t}(A+B)$$

from $H^\infty(S_{\nu-})$ with values in the linear space $\mathcal{L}(H,L^2(\mathbb{P},H))$ such that

$$(\mathbb{E}\|\varphi_{\lambda,W;t}(A+B)x\|^2)^{\frac12}\le C_\nu\|\varphi\|_\infty\|x\|,\ t>0,$$

for every uniformly bounded holomorphic function φ on $S^\circ_{\nu-}$.

The element $\varphi_{\lambda,W;t}(A+B)$ of $\mathcal{L}(H,L^2(\mathbb{P},H))$ is given by equation (6.14) for every uniformly bounded holomorphic function φ on $S^\circ_{\nu-}$ satisfying the bound (6.13). Furthermore, the number b_ν is given by

$$b_\nu=\left(\sup_{\|x\|\le1,|\theta|\le\frac\pi2-\nu}\int_0^\infty\|(-A)^{\frac12}e^{te^{i\theta}A}x\|^2\,dt\right)^{-\frac12}. \tag{6.20}$$

Proof. Let $\omega<\nu<\pi/2$ and $\psi(z)=(-z)^{\frac12}e^z$, for all $z\in\mathbb{C}\setminus[0,\infty)$. Then for each $0\le\theta<\pi/2-\nu$, the function $z\longmapsto\psi(e^{i\theta}z)$, $z\in S_{\nu-}$, satisfies the bound (6.5). Because A has an H^∞-functional calculus on $S_{\omega-}$, the square function estimate (6.2) holds and there exists $c_{\nu,\theta}>0$ such that

$$\begin{aligned}\int_0^\infty\|\psi_t(A)u\|^2\frac{dt}{t} &= \int_0^\infty\|(-A)^{\frac12}e^{te^{i\theta}A}x\|^2\,dt\\ &\le c_{\nu,\theta}\|x\|^2\end{aligned}$$

for all $x\in H$. Because A has an H^∞-functional calculus, the square function norms (6.2) and (6.3) are equivalent to the Hilbert space norm [19], [15, Theorem 11.9] and depend continuously on functions ψ uniformly satisfying the bound (6.5). It follows that

$$(x,\theta) \longmapsto \int_0^\infty \|(-A)^{\frac{1}{2}} e^{te^{i\theta}A}x\|^2\,dt, \quad 0 \le \theta < \pi/2-\omega,\ x \in H$$

is a continuous function. By the uniform boundedness principle,

$$\sup_{\|x\|\le 1, |\theta| \le \frac{\pi}{2}-\nu} \int_0^\infty \|(-A)^{\frac{1}{2}} e^{te^{i\theta}A}x\|^2$$

is finite for each $\omega < \nu < \pi/2$ and conditions 1)-4) above are satisfied with $\delta = \nu$ and the given value b_ν.

The random linear operator $\varphi_{\lambda,W;t}(A+B) \in \mathcal{L}(H, L^2(\mathbb{P}, H))$ is defined by continuous extension from functions satisfying the bound (6.5). The nonrandom part of $\varphi_{\lambda,W;t}(A+B)$ has a limit by the convergence lemma of [19] and for the random part of $\varphi_{\lambda,W;t}(A+B)$, from the proof of Theorem 6.4 it is clear that we can appeal to dominated convergence. □

Remark 6.7. If the operator A satisfies the conditions above and $B : V \to H$ is bounded, then $-\nu I + A + B$ has an H^∞-functional calculus for ν sufficiently large [15, Proposition 13.1].

We cannot expect the linear map

$$\varphi \longmapsto \varphi_{\lambda,W;t}(A+B), \quad \varphi \in H^\infty(S_{\nu-})$$

to be a homomorphism of the algebra $H^\infty(S_{\nu-})$ unless $B = 0$. However, we can calculate $\varphi_{\lambda,W;t}(A+B)$ in some simple cases with the appropriate estimates of the norm of B.

Example 6.8. (a) Let $c \in \mathbb{C}$ and $\varphi(z) = c$ for all $z \in \mathbb{C}$. Then

$$\varphi_{\lambda,W;t}(A+B) = cI \quad \mathbb{P}\text{-a.e.}.$$

The nonrandom part of $\varphi_{\lambda,W;t}(A+B)$ is cI because it is given by an algebra homomorphism of $H^\infty(S_{\nu-})$. The estimate for $r \le 1$ in the proof of Theorem 6.4 shows that the random part is zero $\mathbb{P}$-a.e..

(b) Let $\sigma \in \mathbb{C}\setminus\{0\}$ with $|\arg\sigma| < \pi/2-\omega$ and $\varphi(z) = e^{\sigma z}$ for all $z \in \mathbb{C}$. Then

$$\varphi_{\lambda,W;t}(A+B) = e^{\sigma A+\sqrt{\sigma}B}_{\lambda,W;t} \quad \mathbb{P}\text{-a.e.}.$$

The nonrandom part of $\varphi_{\lambda,W;t}(A+B)$ is $e^{\sigma tA}$ because A is the generator of a holomorphic semigroup and the proof of Theorem 6.4 shows that we can apply Cauchy's integral formula for each term of the orthogonal expansion of $\varphi_{\lambda,W;t}(A+B)$ in multiple stochastic integrals to obtain the expansion for $e^{\sigma A+\sqrt{\sigma}B}_{\lambda,W;t}$.

(c) Let $\sigma \in \mathbb{C}\setminus\{0\}$ with $|\arg\sigma| < \pi/2-\omega$, $n = 1, 2, \dots$ and $\varphi(z) = z^n e^{\sigma z}$ for all $z \in \mathbb{C}$. Then

$$\varphi_{\lambda,W;t}(A+B) = \frac{d^n}{d\sigma^n} e^{\sigma A+\sqrt{\sigma}B}_{\lambda,W;t} \quad \mathbb{P}\text{-a.e.}$$

because $e^{\sigma A+\sqrt{\sigma}B}_{\lambda,W;t}$ is holomorphic in σ.

(d) Let $\Xi_{\pm\mu} = \{se^{\pm i\mu} : s \ge 0\}$ for $0 \le \mu < \pi/2-\omega$. Suppose that $\mu < \nu < \pi/2-\omega$, ψ is a bounded holomorphic function in $S_{\nu+}$ and

$$\varphi(z) = \begin{cases} \int_{\Xi_\mu} e^{\sigma z}\psi(\sigma)\,d\sigma, & \Re(ze^{i\mu}) < 0, \\ \int_{\Xi_{-\mu}} e^{\sigma z}\psi(\sigma)\,d\sigma, & \Re(ze^{-i\mu}) < 0. \end{cases}$$

Then φ is a holomorphic function on $S_{\alpha-}$ for every $\omega < \alpha < \pi/2 - \mu$. If, in addition, $\psi \in L^1([0,\infty))$, then φ is a uniformly bounded holomorphic function in the left half-plane and

$$\varphi_{\lambda,W;t}(A+B) = \int_0^\infty e^{\sigma A + \sqrt{\sigma} B}_{\lambda,W;t}\psi(\sigma)\, d\sigma \quad \mathbb{P}\text{-a.e.}.$$

In particular, if $\zeta \in \mathbb{C}$, $\Re\zeta > 0$ and $\varphi(z) = (\zeta - z)^{-1}$ for $z \in \mathbb{C}$, $z \neq \zeta$, then

$$\varphi_{\lambda,W;t}(A+B) = R_{\lambda,W;t}(\zeta; A+B).$$

Let $0 \leq \omega \leq \pi/2$. A closed injective operator A defined in a Hilbert space H is said to be *ω-accretive* if

$$\sigma(A) \subset S_{\omega+} \text{ and } \langle Ax, x\rangle_H \subset S_{\omega+} \text{ for all } x \in H.$$

In the following result, we formulate conditions for the existence of a stochastic functional calculus in terms of bilinear forms.

Theorem 6.9. *Let $0 < \nu < \pi/2$. Suppose that $-A$ is a one-to-one $(\pi/2 - \nu)$-accretive operator defined in H. Let $V = \mathcal{D}((-A)^{\frac{1}{2}})$ with $\|x\|_V = \|(-A)^{\frac{1}{2}}x\|$ for $x \in V$.*

Let $B : V \to H$ be a bounded linear map such that for some $\eta \in (0,1)$, the bound

$$\frac{1}{2}\|Bx\|^2 \leq \eta\Re(-e^{i\theta}\langle Ax, x\rangle_H)$$

holds for all $x \in \mathcal{D}(A)$ and $|\theta| \leq \nu$.

Then for every $\pi/2 - \nu < \mu < \pi/2$, there exists a linear map

$$\varphi \longmapsto \varphi_{\lambda,W;t}(A+B)$$

from $H^\infty(S_{\mu-})$ with values in the linear space $\mathcal{L}(H, L^2(\mathbb{P}, H))$ such that

$$(\mathbb{E}\|\varphi_{\lambda,W;t}(A+B)x\|^2)^{\frac{1}{2}} \leq C_\mu\|\varphi\|_\infty\|x\|, \ t > 0,$$

for every uniformly bounded holomorphic function φ on $S^\circ_{\mu-}$.

The element $\varphi_{\lambda,W;t}(A+B)$ of $\mathcal{L}(H, L^2(\mathbb{P}, H))$ is given by equation (6.14) for every uniformly bounded holomorphic function φ on $S^\circ_{\mu-}$ satisfying the bound (6.13).

Proof. Under the assumption that $-A$ is a one-to-one $(\pi/2-\nu)$-accretive operator, $-e^{i\theta}A$ is a $(\pi/2 - \nu + \theta)$-accretive operator, so $e^{i\theta}A$ has a H^∞-functional calculus on $S_{\alpha-}$ for every $|\theta| < \nu$ and $\pi/2 - \nu + \theta < \alpha < \pi/2$ [15, Theorem 11.13]. Hence condition 3) holds by an appeal to Theorem 6.1.

Because

$$\begin{aligned}\int_0^T \|Be^{tA_\theta}x\|^2\, dt &\leq -\eta\int_0^T \frac{d}{dt}\|e^{tA_\theta}x\|^2\, dt \\ &\leq \eta\|x\|^2\end{aligned}$$

for all $|\theta| \leq \nu$, $x \in H$ and $T > 0$, condition $4')$ of Remark 5.5 holds. □

By [15, Theorem 11.13], A has a H^∞-functional calculus on a sector if and only if $-A$ is accretive in an equivalent Hilbert space norm. The following example is adapted from [6, Example 4.1].

Example 6.10. Let $D \subset \mathbb{R}^d$ be a bounded open domain with regular boundary ∂D. Let A be the operator

$$\sum_{i,j=1}^d \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)$$

with the boundary condition

$$\frac{\partial u}{\partial \eta_A} = \sum_{i,j=1}^{d} \left(a_{ij}(x) \frac{\partial u}{\partial x_i} \eta_j(x) \right) = 0,$$

where $\eta(x) = (\eta_1(x), \dots, \eta_d(x))$ is the outward unit normal at $x \in \partial D$. Then

$$\mathcal{D}(A) = \{ u \in H^2(D) : \frac{\partial u}{\partial \eta_A} = 0 \text{ on } \partial D \ \}.$$

The operator B is given by

$$Bu(x) = \sum_{i=1}^{d} b_i(x) \frac{\partial u}{\partial x_i}(x), \quad u \in H^1(D).$$

If the coefficients $a_{ij}(x)$ and $b_i(x)$ are real valued, regular and satisfy the joint ellipticity condition

$$\sum_{i,j=1}^{d} \left(\cos \nu . a_{ij} - \frac{1}{2} b_i(x) b_j(x) \right) \xi_i \xi_j \geq \rho |\xi|^2, \quad \xi \in \mathbb{R}^d, \ x \in D,$$

for some $0 < \nu < \pi/2$, then the operators A and B satisfy the conditions of Theorem 6.9 so that (A, B) has a stochastic $H^\infty(S_{\mu-})$-functional calculus $\varphi \longmapsto \varphi_{\lambda,W;t}(A+B)$ on the sector $S_{\mu-}$ for every $\pi/2 - \nu < \mu < \pi/2$. If the matrix $(a_{ij}(x))$ is not symmetric, then $\mathcal{D}(A) \neq \mathcal{D}(A^*)$, so A is not selfadjoint.

Further examples of operators A and B satisfying conditions 1)-4) can be deduced from the examples given in [6, Section 4].

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