LOCAL QUADRATIC ESTIMATES AND HOLOMORPHIC FUNCTIONAL CALCULI

ANDREW J. MORRIS

Abstract. We construct holomorphic functional calculi and introduce local quadratic estimates for operators in a reflexive Banach space that are bisectorial except possibly in a neighbourhood of the origin. The main result is an equivalence of local quadratic estimates with bounded holomorphic functional calculi. For operators with spectrum in a neighbourhood of the origin, the results are weaker than those for bisectorial operators. For operators with a spectral gap in a neighbourhood of the origin, the results are stronger. In each case, however, local quadratic estimates are a more appropriate tool than standard quadratic estimates for establishing that our functional calculi are bounded. This shows that in certain applications it suffices to establish local quadratic estimates.

Contents

1. Introduction

Given an operator T on a Banach space X and a space of functions $\mathcal{F}$, a functional calculus is a mapping from $\mathcal{F}$ into the space of linear operators on X that is canonical in a certain sense. It is usual to denote this mapping by $f \mapsto f(T)$ for all f in $\mathcal{F}$. In applications, such as those discussed below, it is often desirable to know that $f(T)$ is bounded with operator norm controlled by some property of f in $\mathcal{F}$. This is the important notion of a bounded functional calculus.

Given a closed operator T with nonempty resolvent set and a domain Ω in $\mathbb{C}$ that contains a neighbourhood of the spectrum of T, the Dunford–Riesz–Taylor functional calculus, which is given in Definition 2.1, is defined on the space of functions that are holomorphic in Ω. The idea of McIntosh in [11] was to instead design a functional calculus suited to operators of type S_ω. These are closed operators satisfying certain resolvent bounds and having spectrum contained in the bisector S_ω centered at the origin in the complex plane of angle ω in $(0, \pi/2)$. The advantage

Date: Received 4 March 2010 / Accepted 2 April 2010.

2000 *Mathematics Subject Classification.* Primary: 47A60; Secondary: 47B44.

Key words and phrases. Holomorphic functional calculi, quadratic estimates, sectorial operators.

of the resulting functional calculus is that it is defined for functions that need not be holomorphic in a neighbourhood of the origin nor the point at infinity.

It is shown in [11] that, for an operator T of type S_ω on a Hilbert space $\mathcal{H}$, the functional calculus designed by McIntosh is bounded if and only if quadratic estimates of the form

$$\int_0^\infty \|tT(I+t^2T^2)^{-1}u\|_{\mathcal{H}}^2 \, \frac{\mathrm{d}t}{t} \lesssim \|u\|_{\mathcal{H}}^2 \quad \text{and} \quad \int_0^\infty \|tT^*(I+t^2T^{*2})^{-1}u\|_{\mathcal{H}}^2 \, \frac{\mathrm{d}t}{t} \lesssim \|u\|_{\mathcal{H}}^2$$

hold for all $u \in \mathcal{H}$, where T^* is the adjoint operator. Establishing these types of quadratic estimates in order to have a bounded holomorphic functional calculus has been used with great effect in many applications. Most notable is the proof of the Kato Conjecture in [2, 9] and it's many extensions, including [3, 10].

More generally, given an operator T of type S_ω and a domain Ω that touches the spectrum of T nontangentially at a point, the functional calculus on the space of functions that are holomorphic in Ω depends on quadratic estimates approaching the point of contact. Indeed, the lower and upper limits in the quadratic estimates above correspond to the spectral points at infinity and at the origin, respectively. The case of several points of contact has also been considered in [7].

In this paper, we replicate the construction in [11] for operators on a Banach space X that satisfy resolvent bounds and have spectrum contained in either the set $S_{\omega\cup R}$ or the set $S_{\omega\setminus R} \cup \{0\}$, as depicted in Figure 1. Operators of type $S_{\omega\cup R}$ are introduced in Section 3.1. The functional calculus that we construct is defined for functions that must be holomorphic in a neighbourhood of the origin but need not be holomorphic in a neighbourhood of the point at infinity. As a result, the functional calculus only depends on quadratic estimates near the spectral point at infinity. We refer to these as local quadratic estimates, since they are of the form

$$\int_0^1 \|tT(I+t^2T^2)^{-1}u\|_X^2 \, \frac{\mathrm{d}t}{t} + \|(I+T^2)^{-1}u\|_X^2 \lesssim \|u\|_X^2$$

for all $u \in X$. These are defined in Section 3.2 and the equivalence with bounded holomorphic functional calculi is proved in Section 3.3.

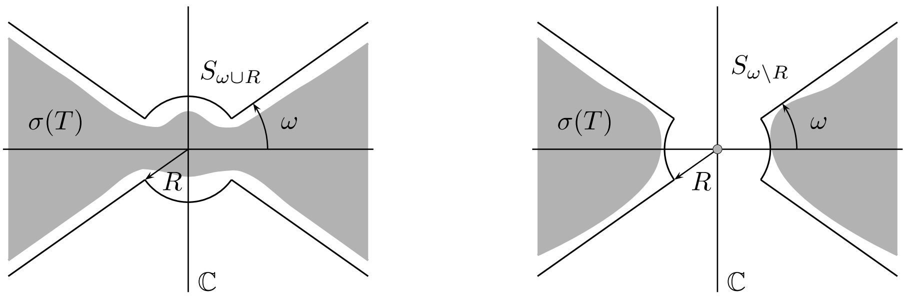

FIGURE 1. The sets $S_{\omega\cup R}$ and $S_{\omega\setminus R}$ for $\omega \in (0, \pi/2)$ and $R > 0$. The shaded areas depict the spectrums of an operator of type $S_{\omega\cup R}$ and an operator of type $S_{\omega\setminus R}$. In both cases, the origin may be in the spectrum.

The theory of type $S_{\omega\cup R}$ operators, which is a weak version of the theory in [11], is actually more suited to certain applications. For example, consider the gradient operator $D = -i\nabla$ on the Sobolev space $W^{1,2}(\mathbb{R}^n)$. The connection between singular convolution operators and the functional calculus of D is well-understood. In particular, local Riesz transforms $r = \{r_j\}_{j=1,\dots,n}$ are defined for each $a > 0$ as the multiplier $(r_j u)\check{}(\xi) = i\xi_j(|\xi|^2 + a)^{-1/2}\hat{u}(\xi)$. These then correspond to the operator $r(D)$ under our new functional calculus, where $r(z) = z(z^2 + a)^{-1/2}$ is holomorphic at the origin but not at the point at infinity. The local Riesz transforms also motivate the definition of the local Hardy spaces $h^p(\mathbb{R}^n)$ in [8]. Furthermore, the theory developed in this paper is applied in [4] to define local Hardy spaces of differential forms $h^p_{\mathcal{D}}(\wedge T^*M)$ that are adapted to a class of first-order differential operators $\mathcal{D}$ of type $S_{\omega\cup R}$ on Riemannian manifolds M with exponential volume growth.

The analogous results for operators of type $S_{\omega\setminus R}$ are in Section 4. This is a special case of the theory of type S_ω operators and the results are stronger. In the sequel, the author applies the theory to Kato-type problems for first order differential operators $\mathcal{D}$ of type S_ω that have a spectral gap. The presence of the spectral gap implies that there exists $R > 0$ such that $\mathcal{D}$ is of type $S_{\omega\setminus R}$, so the existence of a bounded functional calculus follows from local quadratic estimates. The advantage of local quadratic estimates in this context is that they allow for techniques in harmonic analysis that usually require at most polynomial volume growth to be applied on Riemannian manifolds with exponential volume growth.

2. Notation and Preliminaries

Throughout this paper, let X denote a nontrivial complex reflexive Banach space with norm $\|\cdot\|_X$. An *operator* T on X is a linear mapping $T : \mathsf{D}(T) \to X$, where the *domain* $\mathsf{D}(T)$ is a subspace of X. The *range* $\mathsf{R}(T) = \{Tu : u \in \mathsf{D}(T)\}$ and the *null-space* $\mathsf{N}(T) = \{u \in \mathsf{D}(T) : Tu = 0\}$. Let $\overline{\mathsf{D}(T)}$ and $\overline{\mathsf{R}(T)}$ denote the closure of these subspaces in X. An operator T is *closed* if the graph $\mathsf{G}(T) = \{(u, Tu) : u \in \mathsf{D}(T)\}$ is a closed subspace of $X \times X$, and *bounded* if the operator norm

$$\|T\| = \sup\{\|Tu\|_X : u \in \mathsf{D}(T) \text{ and } \|u\|_X \leq 1\}$$

is finite. To minimise notation, we also denote the norm on X by $\|\cdot\|$ when there is no danger of confusion. The unital algebra of bounded operators on X is denoted by $\mathcal{L}(X)$, where the unit is the identity operator I on X. The *resolvent set* $\rho(T)$ is the set of all $z \in \mathbb{C}$ for which the operator $zI - T$ has a bounded inverse with domain equal to X. The *resolvent* $R_T(z)$ is the operator on X defined by

$$R_T(z) = (zI - T)^{-1}$$

for all $z \in \rho(T)$. The *spectrum* $\sigma(T)$ is the complement of the resolvent set in the extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.

We adopt the convention for estimating $x, y \in \mathbb{R}$ whereby $x \lesssim y$ means that $x \leq cy$ for some constant $c \geq 1$ that may only depend on constants specified in the relevant preceding hypotheses.

Given an open set $\Omega \subseteq \mathbb{C}_\infty$, let $H(\Omega)$ denote the algebra of holomorphic functions on Ω. Note that a function f is holomorphic in a neighbourhood of the point at infinity if $f(1/z)$ is holomorphic in a neighbourhood of the origin. The following functional calculus is usually attributed to N. Dunford, F. Riesz and A. E. Taylor. The precise formulation below is from [12].

Definition 2.1 (Dunford–Riesz–Taylor $H(\Omega)$ functional calculus)**.** Let T be a closed operator on X with nonempty resolvent set. If Ω is a proper open subset of $\mathbb{C}_\infty$ that contains $\sigma(T) \cup \{\infty\}$ and $f \in H(\Omega)$, then define $f(T) \in \mathcal{L}(X)$ by

$$f(T)u = f(\infty)u + \frac{1}{2\pi i}\int_\gamma f(z)R_T(z)u \,\mathrm{d}z \tag{2.1}$$

for all $u \in X$, where $f(\infty) = \lim_{z\to\infty} f(z)$ and γ is the boundary of an unbounded Cauchy domain that is oriented clockwise and envelopes $\sigma(T)$ in Ω.

If T is a bounded operator on X, then Ω in Definition 2.1 need not contain the point at infinity, in which case $f(T)u = \frac{1}{2\pi i}\int_\gamma f(z)R_T(z)u \,\mathrm{d}z$. A comprehensive list of attributes and references to the literature on this topic can be found at the end of Chapter VII in [6]. The following theorem, which is set as an exercise in [1], is a consequence of Runge's Theorem.

Theorem 2.2. *The mapping given by* (2.1) *is the unique algebra homomorphism from $H(\Omega)$ into $\mathcal{L}(X)$ with following properties:*

(1) *If $\mathbf{1}(z) = 1$ for all $z \in \Omega$, then $\mathbf{1}(T) = I$ on X;*
(2) *If $\lambda \in \rho(T) \setminus \Omega$ and $f(z) = (\lambda - z)^{-1}$ for all $z \in \Omega$, then $f(T) = R_T(\lambda)$;*
(3) *If $(f_n)_n$ is a sequence in $H(\Omega)$ that converges uniformly on compact subsets of Ω to $f \in H(\Omega)$, then $f_n(T)$ converges to $f(T)$ in $\mathcal{L}(X)$.*

We conclude this section by introducing the following setup.

Definition 2.3. Given $0 \le \mu < \theta < \pi/2$, define the closed and open bisectors in the complex plane as follows:

$$S_\mu = \{z \in \mathbb{C} : |\arg z| \le \mu \text{ or } |\pi - \arg z| \le \mu\};$$
$$S^o_\theta = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \theta \text{ or } |\pi - \arg z| < \theta\}.$$

Given $r \ge 0$, define the closed and open discs as follows:

$$D_r = \{z \in \mathbb{C} : |z| \le r\}$$
$$D^o_r = \{z \in \mathbb{C} : |z| < r\}.$$

These are combined together as follows:

$$S_{\mu\cup r} = S_\mu \cup D_r; \qquad S_{\mu\setminus r} = S_\mu \setminus D^o_r;$$
$$S^o_{\theta\cup r} = S^o_\theta \cup D^o_r; \qquad S^o_{\theta\setminus r} = S^o_\theta \setminus D_r.$$

Note that $D_0 = \{0\}$ and $D^o_0 = \emptyset$ so that $S_{\mu\cup 0} = S_{\mu\setminus 0} = S_\mu$ and $S^o_{\theta\cup 0} = S^o_{\theta\setminus 0} = S^o_\theta$. Let $S^o_{\theta,r}$ denote either $S^o_{\theta\cup r}$ or $S^o_{\theta\setminus r}$. A function on $S^o_{\theta,r}$ is called *nondegenerate* if it is not identically zero on either component of $S^o_{\theta,r}$.

Let $H^\infty(S^o_{\theta,r})$ denote the algebra of bounded holomorphic functions on $S^o_{\theta,r}$. Given $f \in H^\infty(S^o_{\theta,r})$ and $t \in (0,1]$, define $f^* \in H^\infty(S^o_{\theta,r})$ and $f_t \in H^\infty(S^o_{\theta,r/t})$ as follows:

$$f^*(z) = \overline{f(\bar{z})} \quad \text{for all} \quad z \in S^o_{\theta,r};$$
$$f_t(z) = f(tz) \quad \text{for all} \quad z \in S^o_{\theta,r/t}.$$

Given $\alpha, \beta > 0$, define the following sets:

$$\Psi^\beta_\alpha(S^o_{\theta,r}) = \{\psi \in H^\infty(S^o_{\theta,r}) : |\psi(z)| \lesssim \min(|z|^\alpha, |z|^{-\beta})\};$$
$$\Theta^\beta(S^o_{\theta,r}) = \{\phi \in H^\infty(S^o_{\theta,r}) : |\phi(z)| \lesssim |z|^{-\beta}\}.$$

Let $\Psi(S^o_{\theta,r}) = \bigcup_{\alpha,\beta>0} \Psi^\beta_\alpha(S^o_{\theta,r})$ and $\Theta(S^o_{\theta,r}) = \bigcup_{\beta>0} \Theta^\beta(S^o_{\theta,r})$.

3. Operators of Type $S_{\omega\cup R}$

3.1. **Holomorphic Functional Calculi.** We construct holomorphic functional calculi for the following class of operators, where X denotes a nontrivial complex reflexive Banach space.

Definition 3.1. Let $\omega \in [0, \pi/2)$ and $R \ge 0$. An operator T on X is of *type* $S_{\omega\cup R}$ if $\sigma(T) \subseteq S_{\omega\cup R}$, and for each $\theta \in (\omega, \pi/2)$ and $r > R$, there exists $C_{\theta\cup r} > 0$ such that

$$\|R_T(z)\| \le \frac{C_{\theta\cup r}}{|z|}$$

for all $z \in \mathbb{C} \setminus S_{\theta\cup r}$.

The following important lemma allows us to obtain stronger results in reflexive Banach spaces. The proof below is derived from the proof of Theorem 3.8 in [5].

Lemma 3.2. Let $\omega \in [0, \pi/2)$ and $R \geq 0$. Let T be an operator of type $S_{\omega \cup R}$ on X. If $r > R$, then

$$\overline{\mathsf{D}(T)} = \{u \in X : \lim_{n\to\infty} (I + \tfrac{i}{rn}T)^{-1}u = u\} = X.$$

Proof. If $u \in X$ and $\lim_{n\to\infty}(I + \frac{i}{rn}T)^{-1}u = u$, then $u \in \overline{\mathsf{D}(T)}$ simply because $\mathsf{R}((I + \frac{i}{rn}T)^{-1}) = \mathsf{D}(T)$ for all $n \in \mathbb{N}$.

To prove the converse, first suppose that $u \in \mathsf{D}(T)$. The resolvent bounds in Definition 3.1 imply that

$$\|(I + \tfrac{i}{rn}T)^{-1}u - u\| = \|\tfrac{i}{rn}(I + \tfrac{i}{rn}T)^{-1}Tu\| = \|R_T(irn)Tu\| \lesssim (1/rn)\|Tu\|$$

for all $n \geq 1$, which implies that $\lim_{n\to\infty}(I + \frac{i}{rn}T)^{-1}u = u$. Now suppose that $u \in \overline{\mathsf{D}(T)}$. For each $\epsilon > 0$, there exists $v \in \mathsf{D}(T)$ and $N \in \mathbb{N}$ such that $\|u - v\| < \epsilon$ and

$$\begin{aligned}\|(I + \tfrac{i}{rn}T)^{-1}u - u\| &\leq \|(I + \tfrac{i}{rn}T)^{-1}(u - v)\| + \|(I + \tfrac{i}{rn}T)^{-1}v - v\| + \|v - u\| \\ &\lesssim (rn\|R_T(irn)\| + 1)\|u - v\| + (1/rn)\|Tv\| \\ &\lesssim \epsilon\end{aligned}$$

for all $n > N$, as required.

The proof that $\overline{\mathsf{D}(T)} = X$ uses the fact that X is reflexive and follows exactly as in the proof of Theorem 3.8 in [5]. □

For the remainder of this section, fix $\omega \in [0, \pi/2)$ and $R \geq 0$, and let T be an operator of type $S_{\omega\cup R}$ on X. An operator of type $S_{\omega\cup R}$ has a nonempty resolvent set, which of course implies that it is closed, so the Dunford–Riesz–Taylor $H(\Omega)$ functional calculus applies. Following the ideas in [11], however, we introduce the following preliminary functional calculus.

Definition 3.3 ($\Theta(S^o_{\theta\cup r})$ functional calculus)**.** Given $\theta \in (\omega, \pi/2)$, $r > R$ and $\phi \in \Theta(S^o_{\theta\cup r})$, define $\phi(T) \in \mathcal{L}(X)$ by

$$\phi(T)u = \frac{1}{2\pi i}\int_{+\partial S^o_{\tilde\theta\cup\tilde r}} \phi(z)R_T(z)u \,\mathrm{d}z := \lim_{\rho\to\infty}\frac{1}{2\pi i}\int_{(+\partial S^o_{\tilde\theta\cup\tilde r})\cap D_\rho} \phi(z)R_T(z)u \,\mathrm{d}z \tag{3.1}$$

for all $u \in X$, where $\tilde\theta \in (\omega, \theta)$, $\tilde r \in (R, r)$ and $+\partial S^o_{\tilde\theta\cup\tilde r}$ denotes the boundary of $S^o_{\tilde\theta\cup\tilde r}$ oriented clockwise.

The exceptional feature of (3.1) is that the contour of integration is allowed to touch the spectrum of T at infinity. This is made possible by the decay of ϕ and the resolvent bounds in Definition 3.1. A standard calculation using the resolvent equation shows that the mapping $\Theta(S^o_{\theta\cup r}) \mapsto \mathcal{L}(X)$ given by (3.1) is an algebra homomorphism. There is also no ambiguity in our notation, since if Ω is an open set in $\mathbb{C}_\infty$ that contains $S_{\theta\cup r} \cup \{\infty\}$, then the operators defined by (2.1) and (3.1) coincide for functions in $\Theta(S^o_{\theta\cup r}) \cap H(\Omega)$. This is because $\phi \in \Theta(S^o_{\theta\cup r}) \cap H(\Omega)$ is holomorphic in a neighbourhood of infinity, so the Θ-class decay implies that $\phi(\infty) = 0$. Cauchy's Theorem, the resolvent bounds and the Θ-class decay then allow us to modify the contour of integration in (3.1) to that in (2.1). In particular, if $\lambda \in \mathbb{C} \setminus S_{\theta\cup r}$ and $f(z) = (\lambda - z)^{-1}$ for all $z \in S_{\theta\cup r}$, then $f(T) = R_T(\lambda)$.

The proofs of the next two results are based on proofs for operators of type S_ω that were communicated to the author by Alan McIntosh in a graduate course. The first is a convergence lemma for the $\Theta(S^o_{\theta\cup r})$ functional calculus.

Proposition 3.4. Let $\theta \in (\omega, \pi/2)$ and $r > R$. If $(\phi_n)_n$ is a sequence in $\Theta(S^o_{\theta\cup r})$ and there exists $c, \delta > 0$ and $\phi \in \Theta(S^o_{\theta\cup r})$ such that the following hold:

(1) $\sup_n |\phi_n(z)| \leq c|z|^{-\delta}$ for all $z \in S^o_{\theta\cup r}$;
(2) ϕ_n converges to ϕ uniformly on compacts subsets of $S^o_{\theta\cup r}$,

then $\phi_n(T)$ converges to $\phi(T)$ in $\mathcal{L}(X)$.

Proof. Fix $\tilde\theta \in (\omega, \theta)$ and $\tilde r \in (R, r)$. Let γ denote the boundary of $S_{\tilde\theta\cup\tilde r}$ oriented clockwise. Given $r_0 \geq \tilde r$, divide γ into $\gamma_0 = \gamma \cap D_{r_0}$ and $\gamma_\infty = \gamma \cap (\mathbb{C} \setminus D_{r_0})$, so

$$\phi_n(T)u - \phi(T)u = \frac{1}{2\pi i}\left(\int_{\gamma_0} + \int_{\gamma_\infty}\right)(\phi_n(z) - \phi(z))R_T(z)u \,\mathrm{d}z = I_1 + I_2$$

for all $u \in X$. Given $\epsilon > 0$, choose $r_0 > \tilde r$ such that

$$\|I_2\| \lesssim \int_{r_0}^\infty (|\phi_n(z)| + |\phi(z)|)\|R_T(z)u\| \,\frac{\mathrm{d}|z|}{|z|} \lesssim \int_{r_0}^\infty |z|^{-\delta}\,\frac{\mathrm{d}|z|}{|z|}\|u\| < \epsilon\|u\|$$

for all $n \in \mathbb{N}$ and $u \in X$. Now, since ϕ_n converges to ϕ uniformly on compact subsets of $S^o_{\theta\cup r}$, there exists $N \in \mathbb{N}$ such that

$$\|I_1\| \lesssim \int_{|z|=\tilde r} |\phi_n(z) - \phi(z)|\frac{|\mathrm{d}z|}{|z|}\|u\| + \int_{\tilde r}^{r_0} |\phi_n(z) - \phi(z)|\,\frac{\mathrm{d}|z|}{|z|}\|u\| < \epsilon\|u\|$$

for all $n > N$ and $u \in X$. The result follows. $\square$

The next lemma allows us to derive an $H^\infty(S^o_{\theta\cup r})$ functional calculus from the $\Theta(S^o_{\theta\cup r})$ functional calculus.

Lemma 3.5. Let $\theta \in (\omega, \pi/2)$ and $r > R$. If $(\phi_n)_n$ is a sequence in $\Theta(S^o_{\theta\cup r})$ and there exists $f \in H^\infty(S^o_{\theta\cup r})$ such that the following hold:

(1) $\sup_n \|\phi_n\|_\infty < \infty$;
(2) $\sup_n \|\phi_n(T)\| < \infty$;
(3) ϕ_n converges to f uniformly on compacts subsets of $S^o_{\theta\cup r}$,

then $\lim_n \phi_n(T)u$ exists in X for all $u \in X$. Moreover, if $f \in \Theta(S^o_{\theta\cup r})$, then $\lim_n \phi_n(T)u = f(T)u$ for all $u \in X$.

Proof. Let $\tilde\phi_n(z) = (1 + \frac{i}{r}z)^{-1}\phi_n(z)$ and $\tilde\phi(z) = (1 + \frac{i}{r}z)^{-1}f(z)$ for all $z \in S^o_{\theta\cup r}$. There exists $c > 0$ such that the sequence $(\tilde\phi_n)_n$ in $\Theta(S^o_{\theta\cup r})$ satisfies $\sup_n |\tilde\phi_n(z)| \leq c|z|^{-1}$ for all $z \in S^o_{\theta\cup r}$, and converges to $\tilde\phi \in \Theta(S^o_{\theta\cup r})$ uniformly on compact subsets of $S^o_{\theta\cup r}$. Proposition 3.4 then implies that

$$\lim_n \|\tilde\phi_n(T)u - \tilde\phi(T)u\| = 0 \tag{3.2}$$

for all $u \in X$.

If $u \in \mathsf{D}(T)$, then $u = (I + \frac{i}{r}T)^{-1}v$ for some $v \in X$, so we have

$$\phi_n(T)u = \phi_n(T)(I + \tfrac{i}{r}T)^{-1}v = \tilde\phi_n(T)v$$

and (3.2) implies that $\lim_n \phi_n(T)u = \tilde\phi(T)v$. Note that the second equality above holds because $(1 + \frac{i}{r}z)^{-1}$ is in $\Theta(S^o_{\theta\cup r})$.

If $u \in X$, then $u \in \overline{\mathsf{D}(T)}$ by Lemma 3.2. For each $\epsilon > 0$, there exists $v \in \mathsf{D}(T)$ such that $\|u - v\| < \epsilon$, and it follows from what was just proved that $(\phi_n(T)v)_n$ is a Cauchy sequence in X. Therefore, there exists $N \in \mathbb{N}$ such that

$$\begin{aligned}\|\phi_n(T)u - \phi_m(T)u\| &\leq \|\phi_n(T)(u - v)\| + \|\phi_n(T)v - \phi_m(T)v\| + \|\phi_m(T)(v - u)\| \\ &\lesssim \sup_n \|\phi_n(T)\|\,\epsilon\end{aligned}$$

for all $n > m > N$, and $\lim_n \phi_n(T)u$ exists in X.

Finally, if $f \in \Theta(S^o_{\theta\cup r})$, then $\tilde{\phi}(T) = f(T)(I + \frac{i}{r}T)^{-1}$ and $\lim_n \phi_n(T)u = f(T)u$ for all $u \in \mathsf{D}(T)$ by the above. If $u \in X$, then for each $\epsilon > 0$, there exists $v \in \mathsf{D}(T)$ and $N \in \mathbb{N}$ such that

$$\begin{aligned}\|\phi_n(T)u - f(T)u\| &\leq \|\phi_n(T)(u-v)\| + \|\phi_n(T)v - f(T)v\| + \|f(T)(v-u)\| \\ &\lesssim (\sup_n \|\phi_n(T)\| + \|f(T)\|)\epsilon\end{aligned}$$

for all $n > N$, and $\lim_n \phi_n(T)u = f(T)u$. □

The usefulness of condition (2) in the preceding lemma suggests the following definition, which allows us to construct an $H^\infty(S^o_{\theta\cup r})$ functional calculus. This is based on the analogous construction for operators of type S_ω that was communicated to the author by Alan McIntosh in a graduate course.

Definition 3.6 ($H^\infty(S^o_{\theta\cup r})$ functional calculus)**.** Given $\theta \in (\omega, \pi/2)$ and $r > R$, the operator T has a *bounded* $H^\infty(S^o_{\theta\cup r})$ *functional calculus* if there exists $c > 0$ such that

$$\|\phi(T)\| \leq c\|\phi\|_\infty$$

for all $\phi \in \Theta(S^o_{\theta\cup r})$. If T has a bounded $H^\infty(S^o_{\theta\cup r})$ functional calculus, then given $f \in H^\infty(S^o_{\theta\cup r})$ define $f(T) \in \mathcal{L}(X)$ by

$$f(T)u = \lim_n (f\phi_n)(T)u \tag{3.3}$$

for all $u \in X$, where $(\phi_n)_n$ is a uniformly bounded sequence in $\Theta(S^o_{\theta\cup r})$ that converges to 1 uniformly on compact subsets of $S^o_{\theta\cup r}$.

The operator in (3.3) is well-defined by Lemma 3.5. In particular, the definition is independent of the choice of sequence $(\phi_n)_n$ in Definition 3.6. As an example, consider the sequence defined by $\phi_n(z) = (1 + \frac{i}{rn}z)^{-1}$ for all $z \in S^o_{\theta\cup r}$ and $n \in \mathbb{N}$, which satisfies $\sup_n \|\phi_n\|_\infty = 1$. The requirement that T has a bounded $H^\infty(S^o_{\theta\cup r})$ functional calculus then implies that

$$\|f(T)\| \leq \sup_n \|(f\phi_n)(T)\| \leq c \sup_n \|f\phi_n\|_\infty \leq c\|f\|_\infty$$

for all $f \in H^\infty(S^o_{\theta\cup r})$, where c is the constant from Definition 3.6.

Lemma 3.5 also shows that the operators defined by (3.1) and (3.3) coincide for functions in $\Theta(S^o_{\theta\cup r})$. Furthermore, if Ω is an open set in $\mathbb{C}_\infty$ that contains $S_{\theta\cup r} \cup \{\infty\}$, then the operators defined by (2.1) and (3.3) coincide for functions in $H^\infty(S^o_{\theta\cup r}) \cap H(\Omega)$ by Theorem 2.2. There is also the following analogue of Theorem 2.2.

Theorem 3.7. *The mapping given by* (3.3) *is an algebra homomorphism from* $H^\infty(S^o_{\theta\cup r})$ *into* $\mathcal{L}(X)$ *with following properties:*

(1) *If* $\mathbf{1}(z) = 1$ *for all* $z \in S^o_{\theta\cup r}$, *then* $\mathbf{1}(T) = I$ *on* X;
(2) *If* $\lambda \in \mathbb{C} \setminus S_{\omega\cup R}$ *and* $f(z) = (\lambda - z)^{-1}$ *for all* $z \in S^o_{\theta\cup r}$, *then* $f(T) = R_T(\lambda)$;
(3) *If* $(f_n)_n$ *is a sequence in* $H^\infty(S^o_{\theta\cup r})$ *and there exists* $f \in H^\infty(S^o_{\theta\cup r})$ *such that the following hold:*
    (i) $\sup_n \|f_n\|_\infty < \infty$;
    (ii) $\sup_n \|f_n(T)\| < \infty$;
    (iii) f_n *converges to* f *uniformly on compacts subsets of* $S^o_{\theta\cup r}$,
  then $\|f(T)\| \leq \sup_n \|f_n(T)\|$ *and* $\lim_n f_n(T)u = f(T)u$ *for all* $u \in X$.

Proof. Let $f, g \in H^\infty(S^o_{\theta\cup r})$. If $(\phi_n)_n$ satisfies the requirements of Definition 3.6, then so does $(\phi_n^2)_n$. Therefore, the algebra homomorphism property of the $\Theta(S^o_{\theta\cup r})$ functional calculus implies that

$$(fg)(T)u = \lim_n (fg\phi_n^2)(T)u = \lim_n f_n(T)g_n(T)u$$

for all $u \in X$, where $f_n = f\phi_n$ and $g_n = g\phi_n$. This shows that for each $\epsilon > 0$ and $u \in X$, there exists $N \in \mathbb{N}$ such that

$$\begin{aligned}\|(fg)(T)u - f_n(T)g(T)u\| \\ &\leq \|(fg)(T)u - f_n(T)g_n(T)u\| + \|f_n(T)[g_n(T)u - g(T)u]\| \\ &\lesssim \sup_n \|f_n(T)\|\,\epsilon\end{aligned}$$

for all $n > N$. Hence, $(fg)(T)u = \lim_n f_n(T)g(T)u = f(T)g(T)u$ for all $u \in X$.

It remains to prove (1) and (3), since (2) holds by the coincidence of (2.1) and (3.3). If $\phi_n(z) = (1 + \frac{i}{rn}z)^{-1}$ for all $z \in S^o_{\theta\cup r}$ and $n \in \mathbb{N}$, then by Lemma 3.2 we have

$$\mathbf{1}(T)u = \lim_n \phi_n(T)u = \lim_n (I + \tfrac{i}{rn}T)^{-1}u = u$$

for all $u \in X$. The final part of the theorem follows from the algebra homomorphism property, as in the proof of Lemma 3.5. □

3.2. **Local Quadratic Estimates.** Fix $\omega \in [0, \pi/2)$ and $R \geq 0$, and let T be an operator of type $S_{\omega\cup R}$ on X. The Φ-class of holomorphic functions is introduced below to develop a local version of the McIntosh approximation technique. This essential tool is used to prove the equivalence of local quadratic norms.

Definition 3.8. Given $\theta \in (0, \pi/2)$, $r \geq 0$ and $\beta > 0$, define $\Phi^\beta(S^o_{\theta,r})$ to be the set of all $\phi \in \Theta^\beta(S^o_{\theta,r})$ with the following properties:

(1) For all $z \in S^o_{\theta,r}$, $\phi(z) \neq 0$;
(2) $\inf_{z\in D^o_r} |\phi(z)| \neq 0$;
(3) $\sup_{t\geq 1} |\phi_t(z)| \lesssim |\phi(z)|$ for all $z \in S^o_{\theta,r} \setminus D_r$,

where $S^o_{\theta,r}$ denotes either $S^o_{\theta\cup r}$ or $S^o_{\theta\setminus r}$. Note that (2) is obviated in the case of $S^o_{\theta\setminus r}$.

The following result is the local version of an exercise in Lecture 3 of [1].

Lemma 3.9 (McIntosh approximation)**.** Let $\theta \in (\omega, \pi/2)$ and $r > R$. Given nondegenerate $\psi \in \Psi(S^o_{\theta\cup r})$ and $\phi \in \Phi(S^o_{\theta\cup r})$, there exist $\eta \in \Psi(S^o_{\theta\cup r})$ and $\varphi \in \Theta(S^o_{\theta\cup r})$ such that

$$\int_0^1 \eta_t(z)\psi_t(z)\ \frac{\mathrm{d}t}{t} + \varphi(z)\phi(z) = 1 \tag{3.4}$$

for all $z \in S^o_{\theta\cup r}$. Given $0 < \alpha < \beta \leq 1$ and $f \in \Theta(S^o_{\theta\cup r})$, if

$$\Psi_{\alpha,\beta}(z) = f(z) \int_\alpha^\beta \eta_t(z)\psi_t(z)\ \frac{\mathrm{d}t}{t} \quad \text{and} \quad \Phi(z) = f(z)\varphi(z)\phi(z)$$

for all $z \in S^o_{\theta\cup r}$, then

$$\lim_{\alpha\to 0} \|(\Psi_{\alpha,1}(T) + \Phi(T))u - f(T)u\| = 0 \tag{3.5}$$

for all $u \in X$. Moreover, if T has a bounded $H^\infty(S^o_{\theta\cup r})$ functional calculus, then this holds for any $f \in H^\infty(S^o_{\theta\cup r})$.

Proof. Given $f \in H^\infty(S^o_{\theta\cup r})$, let $f_-(z) = f(-z)$ and $f^*(z) = \overline{f(\bar{z})}$ for all $z \in S^o_{\theta\cup r}$. Let $c = \int_0^\infty |\psi(t)\psi(-t)\phi(t)\phi(-t)|^2\ \frac{\mathrm{d}t}{t}$ and define the functions

$$\eta = c^{-1}\psi^*\psi_-\psi^*_-\phi\phi^*\phi_-\phi^*_- \quad \text{and} \quad \varphi = \frac{1}{\phi}\left(1 - \int_0^1 \eta_t\psi_t \frac{\mathrm{d}t}{t}\right),$$

in which case (3.4) is immediate and $\eta \in \Psi(S^o_{\theta\cup r})$. The function φ is holomorphic on $S^o_{\theta\cup r}$ by Morera's Theorem, since $\phi(z) \neq 0$ for all $z \in S^o_{\theta\cup r}$, and bounded on D^o_r, since $\inf_{z\in D^o_r} |\phi(z)| \neq 0$. A change of variable shows that $\int_0^\infty \eta_t(x)\psi_t(x)\frac{\mathrm{d}t}{t} = 1$ for all $x \in \mathbb{R} \setminus \{0\}$, and since $z \mapsto \int_0^\infty \eta_t(z)\psi_t(z)\frac{\mathrm{d}t}{t}$ is holomorphic on S^o_θ, we must have $\int_0^\infty \eta_t(z)\psi_t(z)\frac{\mathrm{d}t}{t} = 1$ for all $z \in S^o_\theta$. It then follows from property (3) in Definition 3.8 that

$$|\varphi(z)| = \frac{1}{|\phi(z)|}\int_1^\infty |\eta_t(z)\psi_t(z)|\frac{\mathrm{d}t}{t} \lesssim \frac{\sup_{t\geq 1}|\phi_t(z)|}{|\phi(z)|}\int_1^\infty (t|z|)^{-\delta}\frac{\mathrm{d}t}{t} \lesssim |z|^{-\delta}$$

for all $z \in S^o_\theta$ and some $\delta > 0$, so $\varphi \in \Theta(S^o_{\theta\cup r})$.

To prove (3.5), let $f \in \Theta(S^o_{\theta\cup r})$ and note that there exists $\delta > 0$ such that

$$\begin{aligned} |\Psi_{\alpha,1}(z)| &\lesssim |f(z)| \int_0^1 \min(|tz|^\delta, |tz|^{-\delta})\ \frac{\mathrm{d}t}{t} \\ &= \min(\|f\|_\infty, |z|^{-\delta}) \left(|z|^\delta \int_0^{1/|z|} t^\delta\ \frac{\mathrm{d}t}{t} + |z|^{-\delta}\int_{1/|z|}^\infty t^{-\delta}\ \frac{\mathrm{d}t}{t}\right) \\ &\lesssim \min(\|f\|_\infty, |z|^{-\delta}) \end{aligned} \tag{3.6}$$

for all $\alpha \in (0,1)$ and $z \in S^o_{\theta\cup r}$, where the constants associated with each instance of $\lesssim$ do not depend on α. This shows that $\Psi_{\alpha,1} + \Phi$ is in $\Theta(S^o_{\theta\cup r})$ for all $\alpha \in (0,1)$ with $\sup_{\alpha\in(0,1)} |\Psi_{\alpha,1}(z) + \Phi(z)| \leq c|z|^{-\delta}$ for some $c > 0$. Also, given a compact set $K \subset S^o_{\theta\cup r}$, it follows from (3.4) that there exists $c_K > 0$ such that

$$|\Psi_{\alpha,1}(z) + \Phi(z) - f(z)| \leq \|f\|_\infty \int_0^\alpha |\eta_t(z)\psi_t(z)|\ \frac{\mathrm{d}t}{t} \lesssim |\alpha z|^\delta \leq c_K\alpha^\delta$$

for all $\alpha \in (0,1)$ and $z \in K$. Therefore, the sequence $(\Psi_{1/n,1} + \Phi)_n$ converges to f uniformly on compact subsets of $S^o_{\theta\cup r}$, and (3.5) follows from the version of the convergence lemma in Proposition 3.4.

Now let $f \in H^\infty(S^o_{\theta\cup r})$ and suppose that T has a bounded $H^\infty(S^o_{\theta\cup r})$ functional calculus. It follows as in (3.6) that $\sup_{\alpha\in(0,1)} \|\Psi_{\alpha,1} + \Phi\|_\infty \lesssim \|f\|_\infty < \infty$. Also, there exists $\delta > 0$ such that

$$\begin{aligned} |\Psi_{\alpha,1}(z)| &\lesssim \|f\|_\infty \int_\alpha^1 \min(|tz|^\delta, |tz|^{-\delta})\ \frac{\mathrm{d}t}{t} \\ &= \|f\|_\infty \min\left(|z|^\delta \int_\alpha^1 t^\delta\ \frac{\mathrm{d}t}{t}, |z|^{-\delta}\int_\alpha^1 t^{-\delta}\ \frac{\mathrm{d}t}{t}\right) \\ &\lesssim \min(|z|^\delta, |\alpha z|^{-\delta}) \\ &\leq \alpha^{-\delta}|z|^{-\delta} \end{aligned}$$

for all $\alpha \in (0,1)$ and $z \in S^o_{\theta\cup r}$. This shows that $\Psi_{\alpha,1} + \Phi$ is in $\Theta(S^o_{\theta\cup r})$ for all $\alpha \in (0,1)$, and since T has a bounded $H^\infty(S^o_{\theta\cup r})$ functional calculus, the result follows by Theorem 3.7. $\square$

We now introduce local quadratic norms on X adapted to the operator T and define the notion of local quadratic estimates.

Definition 3.10. Let $\theta \in (\omega, \pi/2)$ and $r > R$. Given $\psi \in \Psi(S^o_{\theta\cup r})$ and $\phi \in \Phi(S^o_{\theta\cup r})$, define the *local quadratic norm* $\|\cdot\|_{T,\psi,\phi}$ by

$$\|u\|_{T,\psi,\phi} = \left(\int_0^1 \|\psi_t(T)u\|^2\ \frac{\mathrm{d}t}{t} + \|\phi(T)u\|^2\right)^{\frac{1}{2}}$$

for all $u \in X$. The operator T satisfies (ψ,ϕ) *quadratic estimates* if there exists $c > 0$ such that $\|u\|_{T,\psi,\phi} \leq c\|u\|$ for all $u \in X$, and *reverse* (ψ,ϕ) *quadratic estimates* if there exists $c > 0$ such that $\|u\| \leq c\|u\|_{T,\psi,\phi}$ for all $u \in X$ satisfying $\|u\|_{T,\psi,\phi} < \infty$.

Given nondegenerate $\psi \in \Psi(S^o_{\theta\cup r})$ and $\phi \in \Phi(S^o_{\theta\cup r})$ in Definition 3.10, if T has a bounded $H^\infty(S^o_{\theta\cup r})$ functional calculus, then Lemma 3.9 implies that the local quadratic norm $\|\cdot\|_{T,\psi,\phi}$ is indeed a norm on X. We use the next two lemmas to prove that families of local quadratic norms are equivalent for operators that have a bounded $H^\infty(S^o_{\theta\cup r})$ functional calculus. These are local analogues of results in [1].

Lemma 3.11. Let $\theta \in (\omega, \pi/2)$ and $r > R$. Given $\psi, \tilde{\psi} \in \Psi(S^o_{\theta\cup r})$ and $\phi \in \Theta(S^o_{\theta\cup r})$, there exists $c > 0$ and $\delta > 0$ such that the following hold:

(1) $\|(f\psi_t)(T)\| \le c\|f\|_\infty$;

(2) $\|(f\phi)(T)\| \le c\|f\|_\infty$;

(3) $\|(f\phi\psi_t)(T)\| \le c\|f\|_\infty t^\delta(1 + \log(1/t))$;

(4) $\|(f\psi_t\tilde{\psi}_s)(T)\| \le c\|f\|_\infty \times \begin{cases} (s/t)^\delta(1+\log(t/s)) & \text{if} \quad s \in (0,t]; \\ (t/s)^\delta(1+\log(s/t)) & \text{if} \quad s \in (t,1] \end{cases}$

for all $t \in (0,1]$ and $f \in H^\infty(S^0_{\theta\cup r})$.

Proof. Fix $\tilde{\theta} \in (\omega, \theta)$ and $\tilde{r} \in (R, r)$. Let γ denote the boundary of $S_{\tilde{\theta}\cup\tilde{r}}$ oriented clockwise. Choose $\delta > 0$ so that $\psi, \tilde{\psi} \in \Psi^\delta_\delta(S^o_{\theta\cup r})$ and $\phi \in \Theta^\delta(S^o_{\theta\cup r})$. The resolvent bounds then imply that

$$\begin{aligned}
\frac{\|(f\psi_t)(T)\|}{\|f\|_\infty} &\lesssim \frac{1}{\|f\|_\infty}\int_\gamma |f(z)\psi_t(z)|\|R_T(z)\|\ |\mathrm{d}z| \\
&\lesssim \int_\gamma \min(|tz|^\delta, |tz|^{-\delta})\ \frac{|\mathrm{d}z|}{|z|} \\
&\lesssim t^\delta \int_{|z|=\tilde{r}} |z|^{\delta-1}|\mathrm{d}z| + t^\delta \int_{\tilde{r}}^{\tilde{r}/t} |z|^{\delta-1}\ \mathrm{d}|z| + t^{-\delta}\int_{\tilde{r}/t}^\infty |z|^{-\delta-1}\ \mathrm{d}|z| \\
&\lesssim 1
\end{aligned}$$

for all $t \in (0,1]$. Similarly, we obtain

$$\frac{\|(f\phi)(T)\|}{\|f\|_\infty} \lesssim \int_{|z|=\tilde{r}} |z|^{-1}\ |\mathrm{d}z| + \int_{\tilde{r}}^\infty |z|^{-\delta-1}\ \mathrm{d}|z| \lesssim 1$$

and

$$\begin{aligned}
\frac{\|(f\phi\psi_t)(T)\|}{\|f\|_\infty} &\lesssim \int_\gamma \min(1, |z|^{-\delta})\min(|tz|^\delta, |tz|^{-\delta})\ \frac{|\mathrm{d}z|}{|z|} \\
&\lesssim t^\delta \int_{|z|=\tilde{r}} |z|^{\delta-1}\ |\mathrm{d}z| + t^\delta \int_{\tilde{r}}^{\tilde{r}/t} |z|^{-1}\ \mathrm{d}|z| + t^{-\delta}\int_{\tilde{r}/t}^\infty |z|^{-2\delta-1}\ \mathrm{d}|z| \\
&\lesssim t^\delta + t^\delta\log(1/t) + t^{-\delta}(1/t)^{-2\delta} \\
&\lesssim t^\delta(1+\log(1/t)).
\end{aligned}$$

for all $t \in (0,1]$. Also, if $0 < s \le t \le 1$, then

$$\begin{aligned}
\frac{\|(f\psi_t\tilde{\psi}_s)(T)\|}{\|f\|_\infty} &\lesssim \int_\gamma \min(|tz|^\delta, |tz|^{-\delta})\min(|sz|^\delta, |sz|^{-\delta})\ \frac{|\mathrm{d}z|}{|z|} \\
&\lesssim (s/t)^\delta \int_{|z|=\tilde{r}} |z|^{-1}\ |\mathrm{d}z| + (st)^\delta \int_{\tilde{r}}^{\tilde{r}/t} |z|^{2\delta-1}\ \mathrm{d}|z| \\
&\qquad + (s/t)^\delta \int_{\tilde{r}/t}^{\tilde{r}/s} |z|^{-1}\ \mathrm{d}|z| + (st)^{-\delta}\int_{\tilde{r}/s}^\infty |z|^{-2\delta}\ \mathrm{d}|z| \\
&\lesssim (s/t)^\delta + (st)^\delta(1/t)^{2\delta} + (s/t)^\delta\log(t/s) + (st)^{-\delta}(1/s)^{-2\delta} \\
&\lesssim (s/t)^\delta(1+\log(t/s)).
\end{aligned}$$

The same argument applied in the case $0 < t < s \le 1$ completes the proof. $\square$

Lemma 3.12. Let $\theta \in (\omega, \pi/2)$ and $r > R$. Let $\psi \in \Psi(S^o_{\theta\cup r})$ and $\phi \in \Phi(S^o_{\theta\cup r})$. If $(u_n)_n$ is sequence in X and there exists $u \in X$ such that the following hold:

(1) $\|u_n\|_{T,\psi,\phi} < \infty$ for all $n \in \mathbb{N}$;
(2) $(u_n)_n$ is a Cauchy sequence under the local quadratic norm $\|\cdot\|_{T,\psi,\phi}$;
(3) $\lim_{n\to\infty} \|u_n - u\| = 0$,

then $\|u\|_{T,\psi,\phi} < \infty$ and $\lim_{n\to\infty} \|u_n - u\|_{T,\psi,\phi} = 0$.

Proof. For each $\alpha \in (0, 1)$, choose $N(\alpha) \in \mathbb{N}$ so that $\|u_{N(\alpha)} - u\|^2 < 1/(1 - \log \alpha)$. Lemma 3.11 then implies that

$$\begin{aligned}
&\int_\alpha^1 \|\psi_t(T)u\|^2 \,\frac{\mathrm{d}t}{t} + \|\phi(T)u\|^2 \\
&\le \int_\alpha^1 \|\psi_t(T)(u_{N(\alpha)} - u)\|^2 \,\frac{\mathrm{d}t}{t} + \|\phi(T)(u_{N(\alpha)} - u)\|^2 + \sup_n \|u_n\|^2_{T,\psi,\phi} \\
&\lesssim (1 - \log \alpha)\|u_{N(\alpha)} - u\|^2 + \sup_n \|u_n\|^2_{T,\psi,\phi} \\
&\lesssim \sup_n \|u_n\|^2_{T,\psi,\phi}
\end{aligned}$$

for all $\alpha \in (0, 1)$. The Cauchy condition guarantees that $\sup_n \|u_n\|_{T,\psi,\phi} < \infty$, so we must have $\|u\|_{T,\psi,\phi} < \infty$.

For each $\epsilon > 0$, conditions (2) and (3) combined with the result just proved guarantee that there exists $\alpha_0 \in (0, 1)$ and $N \in \mathbb{N}$ such that

$$\sup_{n>N} \int_0^{\alpha_0} \|\psi_t(T)u_n\|^2 \,\frac{\mathrm{d}t}{t} < \epsilon, \quad \sup_{n>N} \|u_n - u\| < \epsilon \quad \text{and} \quad \int_0^{\alpha_0} \|\psi_t(T)u\|^2 \,\frac{\mathrm{d}t}{t} < \epsilon.$$

Lemma 3.11 then implies that

$$\|u_n - u\|^2_{T,\psi,\phi} \le \left(\int_0^{\alpha_0} + \int_{\alpha_0}^1\right) \|\psi_t(T)(u_n - u)\|^2 \,\frac{\mathrm{d}t}{t} + \|\phi(T)(u_n - u)\|^2 L \lesssim \epsilon$$

for all $n > N$, as required. □

The following result is essential for establishing the connection between bounded holomorphic functional calculi and quadratic estimates. This is a local analogue of Proposition E in [1].

Proposition 3.13. Let $\theta \in (\omega, \pi/2)$ and $r > R$. Given nondegenerate functions ψ, $\tilde{\psi} \in \Psi(S^o_{\theta\cup r})$ and ϕ, $\tilde{\phi} \in \Phi(S^o_{\theta\cup r})$, there exists $c > 0$ such that

$$\|f(T)u\|_{T,\tilde{\psi},\tilde{\phi}} \le c\|f\|_\infty \|u\|_{T,\psi,\phi}$$

for all $f \in \Theta(S^o_{\theta\cup r})$ and $u \in X$ satisfying $\|u\|_{T,\psi,\phi} < \infty$. Moreover, if T has a bounded $H^\infty(S^o_{\theta\cup r})$ functional calculus, then there exists $c > 0$ such that

$$\|f(T)u\|_{T,\tilde{\psi},\tilde{\phi}} \le c\|f\|_\infty \|u\|_{T,\psi,\phi}$$

for all $f \in H^\infty(S^o_{\theta\cup r})$ and $u \in X$ satisfying $\|u\|_{T,\psi,\phi} < \infty$.

Proof. Let $f \in \Theta(S^o_{\theta\cup r})$ and let $u \in X$ satisfying $\|u\|_{T,\psi,\phi} < \infty$. Lemma 3.9 gives $\eta \in \Psi(S^o_{\theta\cup r})$ and $\varphi \in \Theta(S^o_{\theta\cup r})$ such that

$$\int_0^1 \eta_t(z)\psi_t(z)\psi_t(z) \,\frac{\mathrm{d}t}{t} + \varphi(z)\phi(z) = 1$$

for all $z \in S^o_{\theta\cup r}$. Given $0 < \alpha < \beta \le 1$, define

$$\Psi_{\alpha,\beta}(z) = f(z) \int_\alpha^\beta \eta_t(z)\psi_t(z)\psi_t(z) \,\frac{\mathrm{d}t}{t} \quad \text{and} \quad \Phi(z) = f(z)\varphi(z)\phi(z)$$

for all $z \in S^o_{\theta\cup r}$, so $\lim_{\alpha\to 0} \|(\Psi_{\alpha,1}(T) + \Phi(T))u - f(T)u\| = 0$. Now write

$$\begin{aligned}\|\Psi_{\alpha,\beta}(T)u + \Phi(T)u\|^2_{T,\tilde{\psi},\tilde{\phi}} &\le \int_0^1 \|\tilde{\psi}_t(T)\Psi_{\alpha,\beta}(T)u\|^2 \ \frac{\mathrm{d}t}{t} + \int_0^1 \|\tilde{\psi}_t(T)\Phi(T)u\|^2 \ \frac{\mathrm{d}t}{t} \\ &\quad + \|\tilde{\phi}(T)\Psi_{\alpha,\beta}(T)u\|^2 + \|\tilde{\phi}(T)\Phi(T)u\|^2 \\ &= I_1 + I_2 + I_3 + I_4.\end{aligned}$$

We use Lemma 3.11 to obtain the following Schur-type estimates:

Estimate for I_1:

$$\begin{aligned} I_1 &= \int_0^1 \left\| \int_\alpha^\beta (\tilde{\psi}_t\psi_s)(T)(f\eta_s\psi_s)(T)u \ \frac{\mathrm{d}s}{s} \right\|^2 \ \frac{\mathrm{d}t}{t} \\ &\le \int_0^1 \left(\int_\alpha^\beta \|(\tilde{\psi}_t\psi_s)(T)\| \|(f\eta_s\psi_s)(T)u\| \ \frac{\mathrm{d}s}{s} \right)^2 \ \frac{\mathrm{d}t}{t} \\ &\le \int_0^1 \left(\int_\alpha^\beta \|(\tilde{\psi}_t\psi_s)(T)u\| \ \frac{\mathrm{d}s}{s} \right) \left(\int_\alpha^\beta \|(\tilde{\psi}_t\psi_s)(T)\| \|(f\eta_s\psi_s)(T)u\|^2 \ \frac{\mathrm{d}s}{s} \right) \ \frac{\mathrm{d}t}{t} \\ &\le \sup_{t\in(0,1]} \left(\int_\alpha^\beta \|(\tilde{\psi}_t\psi_s)(T)u\| \ \frac{\mathrm{d}s}{s} \right) \int_0^1 \int_\alpha^\beta \|(\tilde{\psi}_t\psi_s)(T)\| \|(f\eta_s\psi_s)(T)u\|^2 \ \frac{\mathrm{d}s}{s} \ \frac{\mathrm{d}t}{t} \\ &\lesssim \sup_{s\in(0,1]} \left(\int_0^1 \|(\tilde{\psi}_t\psi_s)(T)\| \ \frac{\mathrm{d}t}{t} \right) \int_\alpha^\beta \|(f\eta_s)(T)\psi_s(T)u\|^2 \ \frac{\mathrm{d}s}{s} \\ &\lesssim \|f\|^2_\infty \int_\alpha^\beta \|\psi_t(T)u\|^2 \ \frac{\mathrm{d}t}{t}; \end{aligned}$$

Estimate for I_2:

$$\begin{aligned} I_2 &= \int_0^1 \|(f\varphi\tilde{\psi}_t)(T)\phi(T)u\|^2 \ \frac{\mathrm{d}t}{t} \\ &\lesssim \|f\|^2_\infty \int_0^1 t^{2\eta}(1+\log(1/t))^2 \ \frac{\mathrm{d}t}{t} \|\phi(T)u\|^2 \\ &\lesssim \|f\|^2_\infty \|\phi(T)u\|^2; \end{aligned}$$

Estimate for I_3:

$$\begin{aligned} I_3 &= \left\| \int_\alpha^\beta (f\tilde{\phi}\eta_s\psi_s)(T)\psi_s(T)u \ \frac{\mathrm{d}s}{s} \right\|^2 \\ &\le \int_\alpha^\beta \|(f\tilde{\phi}\eta_s\psi_s)(T)\|^2 \ \frac{\mathrm{d}s}{s} \int_\alpha^\beta \|\psi_s(T)u\|^2 \ \frac{\mathrm{d}s}{s} \\ &\lesssim \|f\|^2_\infty \int_\alpha^\beta t^{2\eta}(1+\log(1/t))^2 \ \frac{\mathrm{d}t}{t} \int_\alpha^\beta \|\psi_t(T)u\|^2 \ \frac{\mathrm{d}t}{t} \\ &\lesssim \|f\|^2_\infty \int_\alpha^\beta \|\psi_t(T)u\|^2 \ \frac{\mathrm{d}t}{t}; \end{aligned}$$

Estimate for I_4:

$$I_4 = \|(f\tilde{\phi}\varphi)(T)\phi(T)u\|^2 \lesssim \|f\|^2_\infty \|\phi(T)u\|^2.$$

Therefore, we have

$$\|\Psi_{\alpha,1}(T)u + \Phi(T)u\|_{T,\tilde{\psi},\tilde{\phi}} \lesssim \|f\|_\infty \|u\|_{T,\psi,\phi}$$

for all $\alpha \in (0,1)$, and

$$\|\Psi_{\alpha,\beta}(T)u\|_{T,\tilde{\psi},\tilde{\phi}}^2 \le I_1 + I_3 \lesssim \|f\|_\infty^2 \int_\alpha^\beta \|\psi_t(T)u\|^2 \,\frac{\mathrm{d}t}{t}$$

for all $0 < \alpha < \beta \le 1$. Now, since $\|u\|_{T,\psi,\phi} < \infty$, for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\int_{\frac{1}{n}}^{\frac{1}{m}} \|\psi_t(T)u\|^2 \,\frac{\mathrm{d}t}{t} < \epsilon$$

for all $n > m > N$, which implies that

$$\begin{aligned}\|(\Psi_{1/n,1}(T) + \Phi(T))u - (\Psi_{1/m,1}(T) + \Phi(T))u\|_{T,\tilde{\psi},\tilde{\phi}} &= \|\Psi_{1/n,1/m}(T)u\|_{T,\tilde{\psi},\tilde{\phi}} \\ &\lesssim \|f\|_\infty \epsilon\end{aligned}$$

for all $n > m > N$. This shows that $(\Psi_{1/n,1}(T)u + \Phi(T)u)_n$ is a Cauchy sequence under the local quadratic norm $\|\cdot\|_{T,\tilde{\psi},\tilde{\phi}}$, so by Lemma 3.12 we have

$$\lim_{\alpha\to 0} \|(\Psi_{\alpha,1}(T) + \Phi(T))u - f(T)u\|_{T,\tilde{\psi},\tilde{\phi}} = 0$$

and $\|f(T)u\|_{T,\tilde{\psi},\tilde{\phi}} \lesssim \|f\|_\infty \|u\|_{T,\psi,\phi}$, as required.

Finally, if T has a bounded $H^\infty(S^o_{\theta\cup r})$ functional calculus, then the proof above holds for $f \in H^\infty(S^o_{\theta\cup r})$ by Lemma 3.9. □

3.3. **The Main Equivalence.** We connect the theory from the previous two sections. The first result is an immediate consequence of Proposition 3.13.

Proposition 3.14. Let $\omega \in [0,\pi/2)$ and $R \ge 0$. Let T be an operator of type $S_{\omega\cup R}$ on X. If there exists $\theta_0 \in (\omega,\pi/2)$, $r_0 > R$, nondegenerate $\psi,\tilde{\psi} \in \Psi(S^o_{\theta_0\cup r_0})$ and nondegenerate $\phi,\tilde{\phi} \in \Phi(S^o_{\theta_0\cup r_0})$ such that T satisfies (ψ,ϕ) quadratic estimates and reverse $(\tilde{\psi},\tilde{\phi})$ quadratic estimates, then T has a bounded $H^\infty(S^o_{\theta\cup r})$ functional calculus for all $\theta \in (\omega,\pi/2)$ and $r > R$.

Proof. Let $\theta \in (\omega,\pi/2)$ and $r > R$. Given $g \in H^\infty(S^o_{\theta_0\cup r_0})$, let g_0 denote the restriction of g to $S^o_{\min\{\theta,\theta_0\}\cup\min\{r,r_0\}}$. Using the properties of the $\Theta(S^o_{\theta\cup r})$ functional calculus, Proposition 3.13 implies that there exists $c > 0$ such that

$$\|f(T)u\|_{T,\tilde{\psi},\tilde{\phi}} = \|f_0(T)u\|_{T,\tilde{\psi}_0,\tilde{\phi}_0} \le c\|f_0\|_\infty \|u\|_{T,\psi_0,\phi_0} \le c\|f\|_\infty \|u\|_{T,\psi,\phi}$$

for all $f \in \Theta(S^o_{\theta\cup r})$ and $u \in X$ satisfying $\|u\|_{T,\psi,\phi} < \infty$. The quadratic estimates then imply that there exists $\tilde{c} > 0$ such that

$$\|f(T)u\| \le \tilde{c}\|f\|_\infty \|u\|$$

for all $f \in \Theta(S^o_{\theta\cup r})$ and $u \in X$, as required. □

A converse of the above result holds for dual pairs of operators.

Definition 3.15. A *dual pair of Banach spaces* $\langle X, X'\rangle$ is a pair of complex Banach spaces (X, X') associated with a sesquilinear form $\langle\cdot,\cdot\rangle$ on $X \times X'$ that satisfies the following properties:

(1) $|\langle u,v\rangle| \le C_0 \|u\|_X \|v\|_{X'}$ for all $u \in X$ and $v \in X'$;

(2) $\|u\|_X \le C_1 \sup_{v\in X'} \frac{|\langle u,v\rangle|}{\|v\|_{X'}}$ for all $u \in X$;

(3) $\|v\|_{X'} \le C_2 \sup_{u\in X} \frac{|\langle u,v\rangle|}{\|u\|_X}$ for all $v \in X'$,

for some constants C_0, C_1 and $C_2 > 0$.

Definition 3.16. Given a dual pair of Banach spaces $\langle X, X'\rangle$, a *dual pair of operators* $\langle T, T'\rangle$ consists of an operator T on X and an operator T' on X' such that

$$\langle Tu, v\rangle = \langle u, T'v\rangle$$

for all $u \in \mathsf{D}(T)$ and $v \in \mathsf{D}(T')$.

If T is an operator of type $S_{\omega\cup R}$ on a Hilbert space, then the adjoint operator T^* provides a dual pair of operators $\langle T, T^*\rangle$ of type $S_{\omega\cup R}$ under the inner-product. We use the next lemma to prove the equivalence of bounded holomorphic functional calculi and quadratic estimates.

Lemma 3.17. Let $\omega \in [0, \pi/2)$ and $R \geq 0$. Let $\langle T, T'\rangle$ be a dual pair of operators of type $S_{\omega\cup R}$. If $\theta \in (\omega, \pi/2)$ and $r > R$, then T has a bounded $H^\infty(S^o_{\theta\cup r})$ functional calculus if and only if T' has a bounded $H^\infty(S^o_{\theta\cup r})$ functional calculus. Moreover, if T has a bounded $H^\infty(S^o_{\theta\cup r})$ functional calculus, then

$$\langle f(T)u, v\rangle = \langle u, f^*(T')v\rangle$$

for all $u \in X$, $v \in X'$ and $f \in H^\infty(S^o_{\theta\cup r})$, where f^* is given by Definition 2.3.

Proof. Let $\theta \in (\omega, \pi/2)$ and $r > R$. If $z \in \mathbb{C} \setminus S_{\omega\cup R}$, then

$$\begin{aligned}\langle R_T(z)u, v\rangle &= \langle R_T(z)u, (\bar{z}I - T')R_{T'}(\bar{z})v\rangle \\ &= \langle zR_T(z)u, R_{T'}(\bar{z})v\rangle - \langle TR_T(z)u, R_{T'}(\bar{z})v\rangle \\ &= \langle u, R_{T'}(\bar{z})v\rangle\end{aligned}$$

for all $u \in X$ and $v \in X'$, since $\mathsf{R}(R_T(z)) \subseteq \mathsf{D}(T)$ and $\mathsf{R}(R_{T'}(\bar{z})) \subseteq \mathsf{D}(T')$. This shows that, for an appropriate contour γ in $\mathbb{C}$, we have

$$\begin{aligned}\langle \phi(T)u, v\rangle &= \frac{1}{2\pi i}\int_\gamma \phi(z)\langle R_T(z)u, v\rangle \frac{\mathrm{d}z}{z} \\ &= \frac{1}{2\pi i}\int_\gamma \phi(z)\langle u, R_{T'}(\bar{z})v\rangle \frac{\mathrm{d}z}{z} \\ &= \langle u, \phi^*(T')v\rangle\end{aligned}$$

for all $u \in X$, $v \in X'$ and $\phi \in \Theta(S^o_{\theta\cup r})$. Therefore, we have

$$\frac{\|\phi(T)u\|_X}{\|u\|_X} \lesssim \sup_{v\in X'} \frac{|\langle \phi(T)u, v\rangle|}{\|u\|_X\|v\|_{X'}} = \sup_{v\in X'} \frac{|\langle u, \phi^*(T')v\rangle|}{\|u\|_X\|v\|_{X'}} \lesssim \sup_{v\in X'} \frac{\|\phi^*(T')v\|_{X'}}{\|v\|_{X'}}$$

for all $u \in X$ and $\phi \in \Theta(S^o_{\theta\cup r})$. The dual version of this inequality holds by the same reasoning. Therefore, there exists $c > 0$ such that $\frac{1}{c}\|\phi(T)\| \leq \|\phi^*(T')\| \leq c\|\phi(T)\|$ for all $\phi \in \Theta(S^o_{\theta\cup r})$, which proves that T has a bounded $H^\infty(S^o_{\theta\cup r})$ functional calculus if and only if T' has a bounded $H^\infty(S^o_{\theta\cup r})$ functional calculus.

Now suppose that T has a bounded $H^\infty(S^o_{\theta\cup r})$ functional calculus. Let $(\phi_n)_n$ be a sequence of functions satisfying the requirements of Definition 3.6 so that

$$f(T)u = \lim_n (f\phi_n)(T)u$$

for all $u \in X$ and $f \in H^\infty(S^o_{\theta\cup r})$. For each $\epsilon > 0$, $u \in X$ and $v \in X'$, there exists $N \in \mathbb{N}$ such that

$$|\langle (f\phi_n)(T)u, v\rangle - \langle f(T)u, v\rangle| \lesssim \|(f\phi_n)(T)u - f(T)u\|\|v\| < \epsilon$$

for all $n > N$. The dual version of this statement also holds, so we have

$$\langle f(T)u, v\rangle = \lim_{n\to\infty} \langle (f\phi_n)(T)u, v\rangle = \lim_{n\to\infty} \langle u, (f^*\phi_n^*)(T')v\rangle = \langle u, f^*(T')v\rangle$$

for all $u \in X$ and $v \in X'$, as required. □

This brings us to the principal result of this section. The proof is based on the proof of Theorem 7 in [11] and Theorem F in [1].

Theorem 3.18. *Let $\omega \in [0,\pi/2)$ and $R \geq 0$. Let $\langle T, T'\rangle$ be a dual pair of operators of type $S_{\omega\cup R}$ on $\langle X, X'\rangle$. The following statements are equivalent:*

(1) *The operators T and T' satisfy (ψ,ϕ) quadratic estimates for all ψ in $\Psi(S^o_{\theta\cup r})$ and ϕ in $\Phi(S^o_{\theta\cup r})$ and all θ in $(\omega,\pi/2)$ and $r > R$;*
(2) *There exists θ in $(\omega,\pi/2)$, $r > R$ and nondegenerate $\psi,\tilde{\psi}$ in $\Psi(S^o_{\theta\cup r})$ and nondegenerate $\phi,\tilde{\phi}$ in $\Phi(S^o_{\theta\cup r})$ such that T satisfies (ψ,ϕ) quadratic estimates and T' satisfies $(\tilde{\psi},\tilde{\phi})$ quadratic estimates;*
(3) *The operator T has a bounded $H^\infty(S^o_{\theta\cup r})$ functional calculus for all θ in $(\omega,\pi/2)$ and $r > R$;*
(4) *There exists θ in $(\omega,\pi/2)$ and $r > R$ such that T has a bounded $H^\infty(S^o_{\theta\cup r})$ functional calculus.*

Proof. It suffices to prove that (2) implies (3) and that (4) implies (1). First, suppose that (2) holds. Fix $\theta_0 \in (\omega,\pi/2)$, $r_0 > R$, nondegenerate $\psi,\tilde{\psi} \in \Psi(S^o_{\theta_0\cup r_0})$ and nondegenerate $\phi,\tilde{\phi} \in \Phi(S^o_{\theta_0\cup r_0})$ such that T satisfies (ψ,ϕ) quadratic estimates and T' satisfies $(\tilde{\psi},\tilde{\phi})$ quadratic estimates. Let $\theta \in (\omega,\pi/2)$ and $r > R$. Lemma 3.9 gives $\eta \in \Psi(S^o_{\theta_0\cup r_0})$ and $\varphi \in \Theta(S^o_{\theta_0\cup r_0})$ such that

$$\int_0^1 \eta_t(z)\tilde{\psi}_t^*(z)\psi_t(z)\,\frac{\mathrm{d}t}{t} + \varphi(z)\tilde{\phi}^*(z)\phi(z) = 1$$

for all $z \in S^o_{\theta_0 r_0}$. Given $\alpha \in (0,1)$ and $f \in \Theta(S^o_{\theta\cup r})$, if

$$\Psi_{\alpha,1}(z) = f(z)\int_\alpha^1 \eta_t(z)\tilde{\psi}_t^*(z)\psi_t(z)\,\frac{\mathrm{d}t}{t} \quad \text{and} \quad \Phi(z) = f(z)\varphi(z)\tilde{\phi}^*(z)\phi(z)$$

for all $z \in S^o_{\min\{\theta,\theta_0\}\cup\min\{r,r_0\}}$, then

$$\lim_{\alpha\to 0}\|(\Psi_{\alpha,1}(T)+\Phi(T))u - f(T)u\|_X = \lim_{\alpha\to 0}\|(\Psi_{\alpha,1}(T)+\Phi(T))u - f_0(T)u\|_X = 0$$

for all $u \in X$, where f_0 denotes the restriction of f to $S^o_{\min\{\theta,\theta_0\}\cup\min\{r,r_0\}}$. The dual pairing and Lemma 3.11 imply that

$$\begin{aligned}
&|\langle \Psi_{\alpha,1}(T)u + \Phi(T)u, v\rangle| \\
&\quad\leq \int_\alpha^1 |\langle (f\eta_t)(T)\psi_t(T)u, \psi_t(T')v\rangle|\,\frac{\mathrm{d}t}{t} + |\langle (f\varphi)(T)\phi(T)u, \phi(T')v\rangle| \\
&\quad\lesssim \int_\alpha^1 \|(f\eta_t)(T)\|\|\psi_t(T)u\|_X\|\psi_t(T')v\|_{X'}\,\frac{\mathrm{d}t}{t} + \|(f\varphi)(T)\|\|\phi(T)u\|_X\|\phi(T')v\|_{X'} \\
&\quad\lesssim \|f\|_\infty\|u\|_{T,\psi,\phi}\|v\|_{T',\psi,\phi}
\end{aligned}$$

for all $u \in X$, $v \in X'$, $\alpha \in (0,1)$ and $f \in \Theta(S^o_{\theta\cup r})$. The quadratic estimates then imply that

$$|\langle f(T)u, v\rangle| \lesssim \|f\|_\infty\|u\|_{T,\psi,\phi}\|v\|_{T',\psi,\phi} \lesssim \|f\|_\infty\|u\|_X\|v\|_{X'}$$

for all $u \in X$, $v \in X'$ and $f \in \Theta(S^o_{\theta\cup r})$, which implies (3).

Now, suppose that (4) holds. Fix $\theta_0 \in (\omega,\pi/2)$ and $r_0 > R$ such that T has a bounded $H^\infty(S^o_{\theta_0\cup r_0})$ functional calculus, and choose nondegenerate $\tilde{\psi} \in \Psi(S^o_{\theta_0\cup r_0})$ and nondegenerate $\tilde{\phi} \in \Phi(S^o_{\theta_0\cup r_0})$. Let $\theta \in (\omega,\pi/2)$, $r > R$, $\psi \in \Psi(S^o_{\theta\cup r})$ be nondegenerate and $\phi \in \Phi(S^o_{\theta\cup r})$ be nondegenerate. Given $g \in H^\infty(S^o_{\theta\cup r})$, let g_0 denote the restriction of g to $S^o_{\min\{\theta,\theta_0\}\cup\min\{r,r_0\}}$. A discrete version of Proposition 3.13 shows that

$$\|f(T)u\|_{T,\psi,\phi} = \|f_0(T)u\|_{T,\psi_0,\phi_0} \lesssim \|f\|_\infty\left(\sum_{k=0}^\infty \|\tilde{\psi}_{2^{-k}}(T)u\|_X^2 + \|\tilde{\phi}(T)u\|_X^2\right)^{\frac{1}{2}}$$

for all $f \in H^\infty(S^o_{\theta\cup r})$ and $u \in X$ for which the right-hand-side is finite. In particular, since we can take f to be a constant function, this shows that

$$\|u\|^2_{T,\psi,\phi} \lesssim \sum_{k=0}^{\infty} \|\tilde{\psi}_{2^{-k}}(T)u\|^2_X + \|\tilde{\phi}(T)u\|^2_X$$

for all $u \in X$ for which the right-hand-side is finite. Choose $w \in X'$ such that $\|w\|_{X'} = 1$ and $\sup\{|\langle\tilde{\psi}_{2^{-k}}(T)u, v\rangle| : v \in X',\ \|v\|_{X'} = 1\} \leq 2|\langle\tilde{\psi}_{2^{-k}}(T)u, w\rangle|$. The dual pairing and Lemma 3.11 then imply that

$$\begin{aligned}
&\sum_{k=0}^{n} \|\tilde{\psi}_{2^{-k}}(T)u\|^2_X + \|\tilde{\phi}(T)u\|^2_X \\
&\qquad \lesssim \sum_{k=0}^{n} |\langle\tilde{\psi}_{2^{-k}}(T)u, w\rangle| \|u\|_X + \|u\|^2_X \\
&\qquad = \sum_{k=0}^{n} |\langle u, \tilde{\psi}^*_{2^{-k}}(T')w\rangle| \|u\|_X + \|u\|^2_X \\
&\qquad = \sum_{k=0}^{n} \operatorname{sgn}\left(\langle u, \tilde{\psi}^*_{2^{-k}}(T')w\rangle\right) \langle u, \tilde{\psi}^*_{2^{-k}}(T')w\rangle \|u\|_X + \|u\|^2_X \\
&\qquad \leq \sup_{r_k \in \{-1,1\}} \langle u, \textstyle\sum_{k=0}^{n} r_k \tilde{\psi}^*_{2^{-k}}(T')w\rangle \|u\|_X + \|u\|^2_X \\
&\qquad \leq \sup_{r_k \in \{-1,1\}} \|\left(\textstyle\sum_{k=0}^{n} r_k \tilde{\psi}^*_{2^{-k}}\right)(T')\| \|w\|_{X'} \|u\|^2_X \\
&\qquad \lesssim \|u\|^2_X
\end{aligned}$$

for all $u \in X$ and $n \in \mathbb{N}$, where the final inequality holds because Lemma 3.17 implies that T' has a bounded $H^\infty(S^o_{\theta_0\cup r_0})$ functional calculus, and because

$$\sum_{k=0}^{n} r_k \tilde{\psi}^*_{2^{-k}}$$

is in $\Psi(S^o_{\theta_0\cup r_0})$ for any sequence $(r_k)_k$ taking values in $\{-1, 1\}$ and all $n \in \mathbb{N}$. This shows that T satisfies (ψ, ϕ) quadratic estimates. The same reasoning shows that T' satisfies (ψ, ϕ) quadratic estimates, which implies (1). $\square$

4. Operators of Type $S_{\omega\setminus R}$

We develop an analogous theory for the following class of operators, where X denotes a nontrivial complex reflexive Banach space.

Definition 4.1. Let $\omega \in [0, \pi/2)$ and $R > 0$. An operator T on X is of *type* $S_{\omega\setminus R}$ if $\sigma(T) \subseteq S_{\omega\setminus R} \cup \{0\}$, and for each $\theta \in (\omega, \pi/2)$ and $r \in [0, R)$, there exists $C_{\theta,r} > 0$ such that

$$\|R_T(z)\| \leq \frac{C_{\theta,r}}{|z|}$$

for all $z \in \mathbb{C} \setminus (S_{\theta\setminus r} \cup \{0\})$.

The theory of type $S_{\omega\setminus R}$ operators is similar to that of type $S_{\omega\cup R}$ operators. The main difference arises for operators with a nontrivial null space, which means that 0 is in the spectrum. The following specialization of Lemma 3.2 allows us to deal with this possibility. The proof is omitted since it is essentially the same as the proof of Theorem 3.8 in [5].

Lemma 4.2. Let $\omega \in [0, \pi/2)$ and $R > 0$. Let T be an operator of type $S_{\omega \backslash R}$ on X. If $r \in (0, R)$, then the following hold

$$\overline{\mathsf{D}(T)} = \{u \in X : \lim_{n\to\infty}(I + \tfrac{i}{rn}T)^{-1}u = u\};$$
$$\overline{\mathsf{R}(T)} = \{u \in X : \lim_{n\to\infty}(I + \tfrac{in}{r}T)^{-1}u = 0\};$$
$$\mathsf{N}(T) = \{u \in X : \lim_{n\to\infty}(I + \tfrac{in}{r}T)^{-1}u = u\},$$

and $\overline{\mathsf{D}(T)} = \overline{\mathsf{R}(T)} \oplus \mathsf{N}(T) = X$.

For the remainder of this section, fix $\omega \in [0, \pi/2)$ and $R > 0$, and let T be an operator of type $S_{\omega \backslash R}$ on X. Also, let $\mathsf{P}_{\overline{\mathsf{R}(T)}}$ and $\mathsf{P}_{\mathsf{N}(T)}$ denote the projections from X onto $\overline{\mathsf{R}(T)}$ and $\mathsf{N}(T)$, as given by Lemma 4.2. We introduce an analogue of Definition 3.3.

Definition 4.3 ($\Theta(S^o_{\theta\backslash r})$ functional calculus)**.** Given $\theta \in (\omega, \pi/2)$, $r \in [0, R)$ and $\phi \in \Theta(S^o_{\theta\backslash r})$, define $\phi(T_{\overline{\mathsf{R}}}) \in \mathcal{L}(X)$ by

$$\phi(T_{\overline{\mathsf{R}}})u = \frac{1}{2\pi i}\int_{+\partial S^o_{\tilde\theta\backslash\tilde r}} \phi(z)R_T(z)u \,\mathrm{d}z := \lim_{\rho\to\infty}\frac{1}{2\pi i}\int_{+\partial S^o_{\tilde\theta\backslash\tilde r}\cap D_\rho}\phi(z)R_T(z)u\,\mathrm{d}z \tag{4.1}$$

for all $u \in X$, where $\tilde\theta \in (\omega, \theta)$, $\tilde r \in (r, R)$ and $+\partial S^o_{\tilde\theta\backslash\tilde r}$ denotes the boundary of $S^o_{\tilde\theta\backslash\tilde r}$ oriented clockwise.

A standard calculation shows that the mapping $\Theta(S^o_{\theta\backslash r}) \mapsto \mathcal{L}(X)$ given by (4.1) is an algebra homomorphism. The reason for the notation $\phi(T_{\overline{\mathsf{R}}})$ will become apparent in Lemma 4.5. This requires the following convergence lemma for the $\Theta(S^o_{\theta\backslash r})$ functional calculus, which is proved in essentially the same way as Proposition 3.4.

Proposition 4.4. Let $\theta \in (\omega, \pi/2)$ and $r \in [0, R)$. If $(\phi_n)_n$ is a sequence in $\Theta(S^o_{\theta\backslash r})$ and there exists $c, \delta > 0$ and $\phi \in \Theta(S^o_{\theta\backslash r})$ such that the following hold:

(1) $\sup_n |\phi_n(z)| \le c|z|^{-\delta}$ for all $z \in S^o_{\theta\backslash r}$;
(2) ϕ_n converges to ϕ uniformly on compacts subsets of $S^o_{\theta\backslash r}$,

then $\phi_n(T_{\overline{\mathsf{R}}})$ converges to $\phi(T_{\overline{\mathsf{R}}})$ in $\mathcal{L}(X)$.

We now establish the connection between the operators defined by (2.1) and (4.1).

Lemma 4.5. Let $\theta \in (\omega, \pi/2)$ and $r \in [0, R)$. If Ω is an open set in $\mathbb{C}_\infty$ that contains $S^o_{\theta\backslash r} \cup \{0, \infty\}$ and $\phi \in \Theta(S^o_{\theta\backslash r}) \cap H(\Omega)$, then

$$\phi(T)u = \phi(T_{\overline{\mathsf{R}}})\mathsf{P}_{\overline{\mathsf{R}(T)}}u + \phi(0)\mathsf{P}_{N(T)}u$$

for all $u \in X$. If $\phi \in \Theta(S^o_{\theta\backslash r})$, then

$$\phi(T_{\overline{\mathsf{R}}})u = \phi(T_{\overline{\mathsf{R}}})\mathsf{P}_{\overline{\mathsf{R}(T)}}u = \mathsf{P}_{\overline{\mathsf{R}(T)}}\phi(T_{\overline{\mathsf{R}}})\mathsf{P}_{\overline{\mathsf{R}(T)}}u$$

for all $u \in X$.

Proof. Let Ω be an open set in $\mathbb{C}_\infty$ containing $S^o_{\theta\backslash r} \cup \{0, \infty\}$. Suppose that $\phi \in \Theta(S^o_{\theta\backslash r}) \cap H(\Omega)$. If γ is a contour satisfying the requirements of (2.1), then Cauchy's Theorem, the resolvent bounds in Definition 4.1 and the Θ-class decay imply that

$$\begin{aligned}\phi(T)u &= \phi(\infty)u + \frac{1}{2\pi i}\int_\gamma \phi(z)R_T(z)u \,\mathrm{d}z \\ &= \frac{1}{2\pi i}\left(\int_{+\partial S^o_{\tilde\theta\setminus\tilde r}} + \int_{+\partial D_\delta}\right)\phi(z)R_T(z)u \,\mathrm{d}z\end{aligned}$$

for all $u \in X$, $\tilde\theta \in (\omega, \theta)$, $\tilde r \in (r, R)$ and $\delta \in (0, r)$ satisfying $D_\delta \subset \Omega$.

If $u \in \mathsf{N}(T)$, then $R_T(z)u = \frac{1}{z}u$ for all $z \in \rho(T)$. The function $z \mapsto \frac{1}{z}\phi(z)$ is holomorphic in $S^o_{\tilde\theta\setminus\tilde r}$ and in a neighbourhood of infinity. Therefore, Cauchy's Theorem and the Θ-class decay imply that

$$\int_{+\partial S^o_{\tilde\theta\setminus\tilde r}} \phi(z)R_T(z)u \,\mathrm{d}z = \int_{+\partial S^o_{\tilde\theta\setminus\tilde r}} \frac{\phi(z)}{z}u \,\mathrm{d}z = 0 \tag{4.2}$$

for all $u \in \mathsf{N}(T)$. Also, Cauchy's integral formula implies that

$$\int_{+\partial D_\delta} \phi(z)R_T(z)u \,\mathrm{d}z = \int_{+\partial D_\delta} \frac{\phi(z)}{z-0}u \,\mathrm{d}z = 2\pi i\,\phi(0)u$$

for all $u \in \mathsf{N}(T)$.

If $u \in \mathsf{R}(T)$, then there exists $v \in X$ such that $u = Tv$, in which case

$$\|zR_T(z)u\| = \|zR_T(z)Tv\| = \|z(zR_T(z) - I)v\| \le |z|(C_{\theta,r} + 1)\|v\|$$

for all $z \in D_\delta \setminus \{0\}$ and $\delta \in (0, r)$. A limiting argument then shows that for each $\epsilon > 0$ and $u \in \overline{\mathsf{R}(T)}$, there exists $\eta \in (0, r)$ such that $\|zR_T(z)u\| < \epsilon$ for all $z \in D_\eta \setminus \{0\}$, in which case

$$\left\|\int_{+\partial D_\eta} \phi(z)R_T(z)u \,\mathrm{d}z\right\| \le \|\phi\|_\infty \int_{|z|=\eta} \|zR_T(z)u\| \,\frac{|\mathrm{d}z|}{|z|} < 2\pi\|\phi\|_\infty\epsilon.$$

Another application of Cauchy's Theorem allows us to conclude that

$$\int_{+\partial D_\delta} \phi(z)R_T(z)u \,\mathrm{d}z = 0$$

for all $u \in \overline{\mathsf{R}(T)}$, which completes the proof of the first part of the theorem.

Now let $\phi \in \Theta(S^o_{\theta\setminus r})$. To complete the proof, it suffices to show that $\phi(T_{\overline{R}})u$ is in $\overline{\mathsf{R}(T)}$ for all $u \in \overline{\mathsf{R}(T)}$, since (4.2) implies that $\phi(T_{\overline{R}})u = \phi(T_{\overline{R}})\mathsf{P}_{\overline{R(T)}}u$ for all $u \in X$. For each $n \in \mathbb{N}$, define

$$\psi_n(z) = \frac{1}{1 - \frac{i}{rn}z} - \frac{1}{1 - \frac{rn}{i}z} = \frac{-(\frac{i}{rn} + \frac{rn}{i})z}{1 - (\frac{i}{rn} + \frac{rn}{i})z + z^2}$$

for all $z \in \mathbb{C} \setminus \{\frac{rn}{i}, \frac{i}{rn}\}$. The sequence $(\phi\psi_n)_n$ in $\Theta(S^o_{\theta\setminus r})$ converges to ϕ uniformly on compact subsets of $S^o_{\theta\setminus r}$ and there exists $c, \delta > 0$ such that

$$\sup_n |\phi(z)\psi_n(z)| \le \sup_n \|\psi_n\|_{L^\infty(S^o_{\theta\setminus r})}|\phi(z)| \le c|z|^{-\delta}$$

for all $z \in S^o_{\theta\setminus r}$, so Proposition 4.4 implies that $\lim_n \|(\phi\psi_n)(T_{\overline{R}})u - \phi(T_{\overline{R}})u\| = 0$ for all $u \in X$. The first part of this lemma then shows that

$$\begin{aligned}(\phi\psi_n)_n(T_{\overline{R}})u &= \psi_n(T_{\overline{R}})\phi(T_{\overline{R}})u \\ &= \psi_n(T)\mathsf{P}_{\overline{\mathsf{R}(T)}}\phi(T_{\overline{R}})u \\ &= [(I - \tfrac{i}{rn}T)^{-1} - (I - \tfrac{rn}{i}T)^{-1}]\mathsf{P}_{\overline{\mathsf{R}(T)}}\phi(T_{\overline{R}})u \\ &= TR_T(\tfrac{rn}{i})R_T(\tfrac{i}{rn})\mathsf{P}_{\overline{\mathsf{R}(T)}}\phi(T_{\overline{R}})u\end{aligned}$$

for all $u \in X$ and $n \in \mathbb{N}$, which completes the proof. □

We use the following class of functions to incorporate the null space of T in a holomorphic functional calculus.

Definition 4.6. Given $\theta \in [0, \pi/2)$ and $r \geq 0$, define $H^\infty(S^o_{\theta\backslash r}, \{0\})$ to be the algebra of functions that are defined on $S^o_{\theta\backslash r} \cup \{0\}$ and holomorphic on $S^o_{\theta\backslash r}$.

The next lemma, which is proved in the same way as Lemma 3.5, allows us to derive an $H^\infty(S^o_{\theta\backslash r}, \{0\})$ functional calculus from the $\Theta(S^o_{\theta\backslash r})$ functional calculus.

Lemma 4.7. Let $\theta \in (\omega, \pi/2)$ and $r \in [0, R)$. If $(\phi_n)_n$ is a sequence in $\Theta(S^o_{\theta\backslash r})$ and there exists $f \in H^\infty(S^o_{\theta\backslash r})$ such that the following hold:

(1) $\sup_n \|\phi_n\|_\infty < \infty$;
(2) $\sup_n \|\phi_n(T_{\overline{R}})\| < \infty$;
(3) ϕ_n converges to f uniformly on compacts subsets of $S^o_{\theta\backslash r}$,

then $\lim_n \phi_n(T_{\overline{R}})u$ exists in X for all $u \in X$. Moreover, if $f \in \Theta(S^o_{\theta\backslash r})$, then $\lim_n \phi_n(T_{\overline{R}})u = f(T_{\overline{R}})u$ for all $u \in X$.

This suggests the following definition.

Definition 4.8 ($H^\infty(S^o_{\theta\backslash r}, \{0\})$ functional calculus)**.** Given both $\theta \in (\omega, \pi/2)$ and $r \in [0, R)$, the operator T has a *bounded* $H^\infty(S^o_{\theta\backslash r}, \{0\})$ *functional calculus* if there exists $c > 0$ such that

$$\|\phi(T_{\overline{\mathsf{R}}})\| \leq c\|\phi\|_\infty$$

for all $\phi \in \Theta(S^o_{\theta\backslash r})$. If T has a bounded $H^\infty(S^o_{\theta\backslash r}, \{0\})$ functional calculus and $f \in H^\infty(S^o_{\theta\backslash r}, \{0\})$, then define $f(T) \in \mathcal{L}(X)$ by

$$f(T)u = \lim_n (f\phi_n)(T_{\overline{\mathsf{R}}})\mathsf{P}_{\overline{\mathsf{R}(T)}}u + f(0)\mathsf{P}_{N(T)}u \tag{4.3}$$

for all $u \in X$, where $(\phi_n)_n$ is a uniformly bounded sequence in $\Theta(S^o_{\theta\backslash r})$ that converges to 1 uniformly on compact subsets of $S^o_{\theta\backslash r}$.

The operator in (4.3) is well-defined by Lemma 4.7. The requirement that T has a bounded $H^\infty(S^o_{\theta\backslash r}, \{0\})$ functional calculus implies that

$$\|f(T)\| \leq \sup_n \|(f\phi_n)(T_{\overline{\mathsf{R}}})\| + |f(0)| \leq c \sup_n \|f\phi_n\|_{L^\infty(S^o_{\theta\backslash r})} + |f(0)| \leq c\|f\|_\infty$$

for all $f \in H^\infty(S^o_{\theta\backslash r}, \{0\})$, where c is the constant from Definition 4.8.

Lemma 4.7 also shows that the operators defined by (4.1) and (4.3) coincide on $\overline{\mathsf{R}(T)}$ for functions in $\Theta(S^o_{\theta\backslash r}) \cap H^\infty(S^o_{\theta\backslash r}, \{0\})$. Furthermore, if Ω is an open set in $\mathbb{C}_\infty$ that contains $(S^o_{\theta\backslash r}) \cup \{0, \infty\}$, then the operators defined by (2.1) and (4.3) coincide on X for functions in $H^\infty(S^o_{\theta\backslash r}, \{0\}) \cap H(\Omega)$ by Theorem 2.2 and Lemma 4.5. There is also the following analogue of Theorem 3.7.

Theorem 4.9. *The mapping given by* (4.3) *is an algebra homomorphism from* $H^\infty(S^o_{\theta\backslash r}, \{0\})$ *into* $\mathcal{L}(X)$ *with following properties:*

(1) *If* $\mathbf{1}(z) = 1$ *for all* $z \in S^o_{\theta\backslash r} \cup \{0\}$, *then* $\mathbf{1}(T) = I$ *on* X;
(2) *If* $\lambda \in \mathbb{C} \setminus (S_{\omega\backslash R} \cup \{0\})$ *and* $f(z) = (\lambda - z)^{-1}$ *for all* $z \in S^o_{\theta\backslash r} \cup \{0\}$, *then* $f(T) = R_T(\lambda)$;
(3) *If* $(f_n)_n$ *is a sequence in* $H^\infty(S^o_{\theta\backslash r}, \{0\})$ *and there exists* $f \in H^\infty(S^o_{\theta\backslash r}, \{0\})$ *such that the following hold:*
   (i) $\sup_n \|f_n\|_\infty < \infty$;
   (ii) $\sup_n \|f_n(T)\| < \infty$;
   (iii) f_n *converges to* f *uniformly on compacts subsets of* $S^o_{\theta\backslash r} \cup \{0\}$,

   then $\|f(T)\| \leq \sup_n \|f_n(T)\|$ *and* $\lim_n f_n(T)u = f(T)u$ *for all* $u \in X$.

Proof. Let $f, g \in H^\infty(S^o_{\theta\backslash r}, \{0\})$. If $u \in \overline{\mathsf{R}(T)}$, then using Lemma 4.5 and following the proof of Theorem 3.7, we obtain $(fg)(T)u = f(T)g(T)u$. If $u \in \mathsf{N}(T)$, then

$$(fg)(T)u = f(0)g(0)u = f(0)g(T)u = f(T)g(T)u.$$

It remains to prove (1) and (3), since (2) holds by the coincidence of (2.1) and (4.3). If $\phi_n(z) = (1 + \frac{i}{rn}z)^{-1}$ for all $z \in S^o_{\theta\backslash r}$ and $n \in \mathbb{N}$, then Lemmas 4.2 and 4.5 imply that

$$\mathbf{1}(T)u = \lim_n \phi_n(T_{\overline{\mathsf{R}}})\mathsf{P}_{\overline{\mathsf{R}(T)}}u + \mathsf{P}_{\mathsf{N}(T)}u = \lim_n (I + \tfrac{i}{rn}T)^{-1}\mathsf{P}_{\overline{\mathsf{R}(T)}}u + \mathsf{P}_{\mathsf{N}(T)}u = u$$

for all $u \in X$. Now let $(f_n)_n$ be a sequence in $H^\infty(S^o_{\theta\backslash r}, \{0\})$ with the properties listed in the theorem. If $u \in \overline{\mathsf{R}(T)}$, then using Lemma 4.7 and following the proof of Theorem 3.7, we obtain $\lim_n f_n(T)u = f(T)u$. If $u \in \mathsf{N}(T)$, then

$$\lim_n f_n(T)u = \lim_n f_n(0)u = f(0)u = f(T)u,$$

which completes the proof. □

All of the results in Section 3.2 have a natural analogue for type $S_{\omega\backslash R}$ operators with restrictions to $\overline{\mathsf{R}(T)}$ where required. The proofs are essentially the same. In particular, the McIntosh approximation technique goes over directly. Local quadratic estimates are then restricted to $\overline{\mathsf{R}(T)}$, as below.

Definition 4.10. Let $\theta \in (\omega, \pi/2)$ and $r \in [0, R)$. Given both $\psi \in \Psi(S^o_{\theta\backslash r})$ and $\phi \in \Phi(S^o_{\theta\backslash r})$, define the *local quadratic norm* $\|\cdot\|_{T_{\overline{\mathsf{R}}},\psi,\phi}$ by

$$\|u\|_{T_{\overline{\mathsf{R}}},\psi,\phi} = \left(\int_0^1 \|\psi_t(T_{\overline{\mathsf{R}}})u\|^2 \ \frac{\mathrm{d}t}{t} + \|\phi(T_{\overline{\mathsf{R}}})u\|^2\right)^{\frac{1}{2}}$$

for all $u \in X$. The operator T satisfies (ψ, ϕ) *quadratic estimates on* $\overline{\mathsf{R}(T)}$ if there exists $c > 0$ such that $\|u\|_{T_{\overline{\mathsf{R}}},\psi,\phi} \leq c\|u\|$ for all $u \in \overline{\mathsf{R}(T)}$, and *reverse* (ψ, ϕ) *quadratic estimates on* $\overline{\mathsf{R}(T)}$ if there exists $c > 0$ such that $\|u\| \leq c\|u\|_{T_{\overline{\mathsf{R}}},\psi,\phi}$ for all $u \in \overline{\mathsf{R}(T)}$ satisfying $\|u\|_{T_{\overline{\mathsf{R}}},\psi,\phi} < \infty$.

The next result is an immediate consequence of the analogue of Proposition 3.13 for type $S_{\omega\backslash R}$ operators.

Proposition 4.11. Let $\omega \in [0, \pi/2)$ and $R > 0$. Let T be an operator of type $S_{\omega\backslash R}$ on X. If there exists $\theta_0 \in (\omega, \pi/2)$, $r_0 \in [0, R)$, nondegenerate $\psi, \tilde{\psi} \in \Psi(S^o_{\theta_0\backslash r_0})$ and nondegenerate $\phi, \tilde{\phi} \in \Phi(S^o_{\theta_0\backslash r_0})$ such that T satisfies (ψ, ϕ) quadratic estimates on $\overline{\mathsf{R}(T)}$ and reverse $(\tilde{\psi}, \tilde{\phi})$ quadratic estimates on $\overline{\mathsf{R}(T)}$, then T has a bounded $H^\infty(S^o_{\theta\backslash r}, \{0\})$ functional calculus for all $\theta \in (\omega, \pi/2)$ and $r \in [0, R)$.

The full equivalence also holds for dual pairs of operators of type $S_{\omega\backslash R}$.

Theorem 4.12. *Let $\omega \in [0, \pi/2)$ and $R > 0$. Let $\langle T, T'\rangle$ be a dual pair of operators of type $S_{\omega\backslash R}$ on $\langle X, X'\rangle$. The following statements are equivalent:*

1. *The operators T and T' satisfy (ψ, ϕ) quadratic estimates on $\overline{\mathsf{R}(T)}$ and $\overline{\mathsf{R}(T')}$ for all ψ in $\Psi(S^o_{\theta\backslash r})$ and ϕ in $\Phi(S^o_{\theta\backslash r})$ and all θ in $(\omega, \pi/2)$ and r in $[0, R)$;*
2. *There exists θ in $(\omega, \pi/2)$, r in $[0, R)$, nondegenerate $\psi, \tilde{\psi}$ in $\Psi(S^o_{\theta\backslash r})$ and nondegenerate $\phi, \tilde{\phi}$ in $\Phi(S^o_{\theta\backslash r})$ such that T satisfies (ψ, ϕ) quadratic estimates on $\overline{\mathsf{R}(T)}$ and T' satisfies $(\tilde{\psi}, \tilde{\phi})$ quadratic estimates on $\overline{\mathsf{R}(T')}$;*
3. *The operator T has a bounded $H^\infty(S^o_{\theta\backslash r}, \{0\})$ functional calculus for all θ in $(\omega, \pi/2)$ and r in $[0, R)$;*

(4) *There exists* θ *in* $(\omega, \pi/2)$ *and* r *in* $[0, R)$ *such that the operator* T *has a bounded* $H^\infty(S^o_{\theta\backslash r}, \{0\})$ *functional calculus.*

A dual pair $\langle T, T'\rangle$ of operators of type $S_{\omega\backslash R}$ is also a dual pair of operators of type S_ω, as defined in [5]. Therefore, we conclude that Theorem 4.12 and the standard equivalence for operators of type S_ω, as in Theorem 2.4 of [5], show that local quadratic estimates are equivalent to standard quadratic estimates for operators of type $S_{\omega\backslash R}$.

Acknowledgements

This work was conducted at the Centre for Mathematics and its Applications at the Australian National University. I was supported by this institution and by the Australian Government through an Australian Postgraduate Award.

A part of this work was conducted during my attendance at the AMSI-ANU Workshop on Spectral Theory and Harmonic Analysis, 13-17 July 2009, Canberra, Australia. I would like to thank the organizers and attendees for providing an open and encouraging environment. In particular, I would like to thank Alan McIntosh, whose work inspired this paper, for his helpful suggestions and interest in this work.

References

[1] David Albrecht, Xuan Duong, and Alan McIntosh, *Operator theory and harmonic analysis*, Instructional Workshop on Analysis and Geometry, Part III (Canberra, 1995), Proc. Centre Math. Appl. Austral. Nat. Univ., vol. 34, Austral. Nat. Univ., Canberra, 1996, pp. 77–136.

[2] Pascal Auscher, Steve Hofmann, Michael Lacey, Alan McIntosh, and Ph. Tchamitchian, *The solution of the Kato square root problem for second order elliptic operators on* $\mathbb{R}^n$, Ann. of Math. (2) **156** (2002), no. 2, 633–654.

[3] Andreas Axelsson, Stephen Keith, and Alan McIntosh, *Quadratic estimates and functional calculi of perturbed Dirac operators*, Invent. Math. **163** (2006), no. 3, 455–497.

[4] Andrea Carbonaro, Alan McIntosh, and Andrew Morris, *Local Hardy spaces of differential forms on Riemannian manifolds*, arXiv:1004.0018 (2010).

[5] Michael Cowling, Ian Doust, Alan McIntosh, and Atsushi Yagi, *Banach space operators with a bounded* H^∞ *functional calculus*, J. Austral. Math. Soc. Ser. A **60** (1996), no. 1, 51–89.

[6] Nelson Dunford and Jacob T. Schwartz, *Linear Operators. I. General Theory*, With the assistance of W. G. Bade and R. G. Bartle. Pure and Applied Mathematics, Vol. 7, Interscience Publishers, Inc., New York, 1958.

[7] Edwin Franks and Alan McIntosh, *Discrete quadratic estimates and holomorphic functional calculi in Banach spaces*, Bull. Austral. Math. Soc. **58** (1998), no. 2, 271–290.

[8] David Goldberg, *A local version of real Hardy spaces*, Duke Math. J. **46** (1979), no. 1, 27–42.

[9] Steve Hofmann, Michael Lacey, and Alan McIntosh, *The solution of the Kato problem for divergence form elliptic operators with Gaussian heat kernel bounds*, Ann. of Math. (2) **156** (2002), no. 2, 623–631.

[10] Tuomas Hytönen, Alan McIntosh, and Pierre Portal, *Kato's square root problem in Banach spaces*, J. Funct. Anal. **254** (2008), no. 3, 675–726.

[11] Alan McIntosh, *Operators which have an* H_∞ *functional calculus*, Miniconference on operator theory and partial differential equations (North Ryde, 1986), Proc. Centre Math. Anal. Austral. Nat. Univ., vol. 14, Austral. Nat. Univ., Canberra, 1986, pp. 210–231.

[12] Angus E. Taylor, *Spectral theory of closed distributive operators*, Acta Math. **84** (1951), 189–224.

Andrew J. Morris, Centre for Mathematics and its Applications, Mathematical Sciences Institute, Australian National University, Canberra, ACT 0200, Australia

E-mail address: `Andrew.Morris@anu.edu.au`