# Resonant rigidity for Schrödinger operators (in even dimensions)

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For $\operatorname{Im}\lambda > 0$, let $R_V(\lambda) = (-\Delta + V - \lambda^2)^{-1}$. This is bounded on $L^2(\mathbb{R}^d)$ for $\operatorname{Im}\lambda > 0$, with the possible exception of a finite number of points corresponding to eigenvalues.

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This explicit representation shows us that for any function $\chi \in L^\infty_c(\mathbb{R}^3)$,

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Similar (but more complicated looking) things happen in any dimension; the space to which the Schwartz kernel continues is dimension-dependent.

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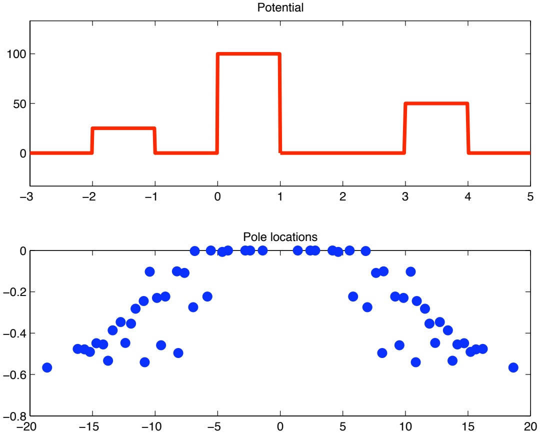
The poles of $\chi R_V(\lambda)\chi$ are called *resonances*.

# An example on $\mathbb{R}$.

Thanks to M. Zworski for the figures.




Computed using `squarepot.m`

`http://www.cims.nyu.edu/~dbindel/resonant1d/`

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- Rigidity? What do resonances say about the potential?

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- $d$ even, $m \in \mathbb{Z}$:

  $$n_{m,V}(r) = \{\lambda_j \in \mathcal{R}es(V) : |\lambda_j| \leq r, \ m\pi < \arg \lambda_j < (m+1)\pi\}$$

- If $d = 1$, as $r \to \infty$,

$$\#\{\lambda_j \in \mathcal{R}es(V) : |\lambda_j| \leq r\} = \frac{1}{\pi}(\text{length convex hull } V) r + o(r)$$

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- If $d$ even,

$$n_{m,V}(r) \leq Cr^d, \text{ as } r\to\infty.$$

  Vodev
- optimal, in a sense

Lower bounds?

- For *d odd*: $V \in C_c^\infty(\mathbb{R}^d;\mathbb{R})$, $V \not\equiv 0$,

$$\limsup_{r\to\infty} \frac{n_{odd,V}(r)}{r} > 0$$

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- Better lower bounds for specific classes examples; fixed sign, generically

### Theorem

*(d odd due to Smith-Zworski, d even C-) Let* $V \in L^\infty_c(\mathbb{R}^d;\mathbb{R})$, $V \not\equiv 0$. *Then*

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- *If* $d \geq 5$ *is odd,* $\mathcal{R}es(V)$ *is nonempty.*

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- *If* $d = 3$, $\mathcal{R}es(V)$ *has infinitely many elements.*
- *If* $d \geq 5$ *is odd,* $\mathcal{R}es(V)$ *is nonempty.*
- *If* $d$ *is even, and* $d \neq 4$, $\mathcal{R}es(V)$ *contains infinitely many elements, with a quantitative lower bound. If* $d = 4$ *and* $0$ *is not a resonance, the same is true.*

Resonant rigidity:

- If $\mathcal{R}es(V_1) = \mathcal{R}es(V_2)$, $V_1, V_2 \in L^\infty_c(\mathbb{R}^d;\mathbb{R})$, and $k \in \mathbb{N}$, then
$$V_1 \in H^k(\mathbb{R}^d) \Leftrightarrow V_2 \in H^k(\mathbb{R}^d)$$
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- If *d* is even, $V \in C^\infty_c(\mathbb{R}^d;\mathbb{R})$, then $\mathcal{R}es(V)$ determines all the heat coefficients of $V$. (C-) (*d* odd: miss up to two?)

Resonant rigidity:
Given $V_0 \in L^\infty_c(\mathbb{R}^d;\mathbb{R})$ with support in $\overline{B}(R)$, set

$$Iso(V_0, R) = \{V \in L^\infty_c(\mathbb{R}^d;\mathbb{R}) : \mathcal{R}es(V) = \mathcal{R}es(V_0), \\ \text{supp}\, V \subset \overline{B}(R)\}$$

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*($d = 1,\ 3$: Hislop-Wolf; $d$ even: C-) Let $V_0 \in C^\infty_c(\mathbb{R}^d; \mathbb{R})$ have support in $\overline{B}(R)$. Then $Iso(V_0, R)$ is compact in the $C^\infty$ topology if $d \leq 3$. If $d \geq 4$ is even, then a weaker statement holds.*

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cf. results of Brüning, Donnelly: Schrödinger operators on compact manifolds

Resonant rigidity:

**Theorem**

*(C-) If $d$ is even, and $V_1, V_2 \in L^\infty_c(\mathbb{R}^d;\mathbb{R})$, then $\mathcal{R}es(V_1)$ and $\mathcal{R}es(V_2)$ cannot differ by a nonzero number of nonzero elements*

Contrast: $d = 1$, Korotyaev: within class of potentials $L^1_c(\mathbb{R};\mathbb{R})$, can "move" resonances (with restrictions)

Important ingredients:
(1) Birman-Krein trace formula: for $t > 0$

$$\operatorname{tr}(e^{t(\Delta - V)} - e^{t\Delta}) = \frac{1}{2\pi i}\int_0^\infty e^{-t\lambda^2} \frac{\frac{d}{d\lambda}\det S(\lambda)}{\det S(\lambda)} d\lambda + \sum_{k=1}^{K} e^{t\mu_k^2} + \beta(V, d)$$

Here $S$ is the scattering matrix and $-\mu_1^2 \leq ... \leq -\mu_K^2 \leq 0$ are the eigenvalues of $-\Delta + V$.

Important ingredients:
(2)

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*(weaker version; C-) Let* $V_1,\ V_2 \in L^\infty_c(\mathbb{R}^d;\mathbb{R})$*, with scattering matrices* $S_1,\ S_2$*. If* $d = 4$*, assume* either *that* 0 *is not a resonance* or $V_1, V_2 \in C^\infty_c$*. Set*

$$F(z) = \frac{\det S_1(e^z)}{\det S_2(e^z)}.$$

*Suppose* $F$ *has finitely many poles.*

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$$F(z) = \frac{\det S_1(e^z)}{\det S_2(e^z)}.$$

*Suppose* $F$ *has finitely many poles. Then* $F(z) \equiv 1$*; that is,* $\det S_1(\lambda) = \det S_2(\lambda)$ *for all* $\lambda$*.*

Earlier version: $V_2 \equiv 0, V_1 \in C^\infty_c$: Sá Barreto ($d \geq 4$), L-H Chen ($d = 2$)

Important ingredients:
(3)

### Theorem

*(Smith-Zworski) For $V \in L^\infty(\mathbb{R}^d; \mathbb{R})$, $k \in \mathbb{N}$, if $V \in H^k$, then there are constants $c_1, ...., c_{k+1}$, a function $r_{k+2}$ so that*

$$\mathrm{tr}(e^{-t(-\Delta+V)} - e^{t\Delta}) =$$

$$(4\pi t)^{-d/2}(c_1 t + c_2 t^2 + ... + c_{k+1} t^{k+1} + r_{k+2}(t) t^{k+2}) \text{ when } t \downarrow 0$$

*with $|r_{k+2}(t)| \leq C$ for $0 \leq t \leq 1$. Conversely, if such an expansion holds, $V \in H^k$.*